Stable log surfaces and limits of quartic plane curves

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Abstract

Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d \geq 4$. We regard pairs $(\mathbb{P}^2, C)$ as stable log surfaces, higher-dimensional analogs to pointed stable curves. Using the log minimal model program, Kollár, Shepherd-Barron, and Alexeev have constructed projective moduli spaces for stable log surfaces. Unfortunately, few explicit examples of these moduli spaces are known. The purpose of this paper is to give a concrete description of these spaces for plane curves of small degree. In particular, we show that the moduli space of stable log surfaces corresponding to quartic plane curves coincides with the moduli space of stable curves of genus three.

1 Introduction

Stable log surfaces are pairs (S, C) consisting of a surface S and a curve $C \subset S$ satisfying the following properties:

1. (S, C) has semi-log canonical singularities;
2. $K_S + C$ is ample.

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The notion of an *allowable family* $(\mathcal{S}, \mathcal{C}) \to B$ of stable log surfaces is a bit subtle and will be explained in section 3.1. In this paper, we generally consider *smoothable* stable log surfaces, i.e. those possessing allowable deformations to smooth pairs.

The construction of projective moduli spaces for stable surfaces may be found in the book [25] and the papers [20], [16], and [1]; the construction for stable log surfaces is given in [2]. Unfortunately, even in the simplest cases we have no good description for the stable log surfaces arising in a given compactification. Indeed, there are very few examples of moduli spaces where the boundary components have been worked out explicitly (however, there is work of Alexeev and Nakamura on degenerations of polarized abelian varieties [4] [3].)

In this paper, we apply these ideas to degenerations of plane curves of degree $d \geq 4$. Let $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^3))$ be the linear system of degree d plane curves and U the open subset parametrizing smooth curves. The orbit space $\mathcal{P}_d = U/\mathrm{PGL}_3$ is a quasi-projective variety (by Geometric Invariant Theory), and is a coarse moduli space for smooth plane curves of degree d. We have a morphism $j : \mathcal{P}_d \to \mathcal{M}_{g(d)}$, where $\mathcal{M}_{g(d)}$ is the moduli space of genus $g(d) = \frac{1}{2}(d-1)(d-2)$ curves. Elements of the closure of $j(\mathcal{P}_d)$ in $\overline{\mathcal{M}}_{g(d)}$ are called *limiting plane curves.* Determining which stable curves are limiting plane curves seems to be a difficult open problem (but see [10], [11], [6], and [13].)

If C is a smooth plane curve of degree $d \geq 4$, then $(\mathbb{P}^2, C)$ is a stable log surface. Let $\mathcal{M}$ be the corresponding connected moduli scheme of smoothable stable log surfaces. We use $\overline{\mathcal{P}}_d$ to denote the closure of $\mathcal{P}_d$ in this moduli space. There exists a forgetting morphism $j : \mathcal{M} \to \overline{\mathcal{M}}_{g(d)}$ extending the morphism defined above. Essentially, j exists because C is nodal, $K_S + C$ is ample, and $(K_S + C)|C = K_C$ in our situation.

Thus each limiting plane curve comes imbedded in some limiting plane with semi-log canonical singularities. We shall see in the final section that this representation is not always unique. Nevertheless, understanding the boundary components of the moduli space of stable log surfaces should shed light on the geometry of limiting plane curves. Some systematic results of this type may be found in [13]. Here we focus on the case of quartic plane curves, where we can get a clear picture of all the boundary components. Our main result is:

Theorem 1.1 (Main Theorem) $j : \overline{\mathcal{P}}_4 \to \overline{\mathcal{M}}_3$ *is an isomorphism. Fur-*

thermore, the space $\mathcal{M}$ coincides with $\overline{\mathcal{P}}_4$.

In particular, every stable curve of genus three sits naturally in a unique limiting plane. Does this limiting plane have a natural, uniform, synthetic construction?

This paper is organized as follows. In section two, we prove some general results about plane curves and stable log surfaces. In the next section, we recall some basic results from deformation theory and some facts about surface singularities. In the fourth section, we enumerate the limiting planes containing various stable curves of genus three. Much of this is a case-by-case analysis, but a conceptual explanation is given at the end of section four. In section five, we give a brief application of our results to degenerations of degree-two Del Pezzo surfaces. The final section includes some observations on higher degree plane curves.

Throughout this paper, we work over $\mathbb{C}$. Alessio Corti has informed me that he and his students independently discovered some of the results discussed here. Pyung-Lyun Kang has independently computed stable limits arising from certain smoothings of singular plane quartics [14] [15].

2 General results on plane curves

Our first task is to understand j over the stable curves that can be represented as nodal plane curves. The following result generalizes the classical fact that smooth plane curves are abstractly isomorphic iff they are projectively equivalent [5] pp. 56.

Proposition 2.1 *Any nodal plane curve C of degree $d \geq 4$ is stable and the corresponding pair $(\mathbb{P}^2, C)$ is a stable log surface. If C_1 and C_2 are nodal plane curves of degree $d \geq 4$, then $C_1 \cong C_2$ as stable curves iff C_1 and C_2 are projectively equivalent. Thus j maps the locus of nodal plane curves bijectively onto its image.*

Proof. Let $C \subset \mathbb{P}^2$ be a nodal plane curve of degree $d \geq 4$. The pair $(\mathbb{P}^2, C)$ is log canonical and $K_{\mathbb{P}^2} + C$ is ample, so $(\mathbb{P}^2, C)$ is a stable log surface. The adjunction formula $\omega_C = \mathcal{O}_C(K_{\mathbb{P}^2} + C)$ implies that ω_C is ample, thus C is stable.

Now assume that C has two distinct planar representations C_1 and C_2. An easy cohomology computation shows that any linear series imbedding C

as a plane curve is complete, so it suffices to prove that the corresponding imbedding line bundles are equal. Let $p_1+p_2+\ldots+p_d$ be a generic hyperplane section of C_1, regarded as a Cartier divisor on $C_2 \subset \mathbb{P}^2$. By Serre duality, $p_1, \ldots, p_d$ impose just $d-2$ linearly independent conditions on the linear series $|\omega_{C_2}| = |\mathcal{O}_{C_2}(d-3)|$. This forces the points to be collinear on C_2, so that $\mathcal{O}_{C_2}(1) = \mathcal{O}_{C_2}(p_1 + \ldots + p_d) = \mathcal{O}_{C_1}(1)$. $\square$

The next proposition shows that $\mathcal{M}$ is well-behaved over the nodal plane curves.

Proposition 2.2 *Let $C \in \overline{\mathcal{M}}_{g(d)}$ be a curve imbedded as a nodal plane curve of degree $d \geq 4$. Then $\mathcal{M}$ is smooth (as a stack) at $(\mathbb{P}^2, C)$ and the derivative of j is injective.*

Proof. Represent $C \subset \mathbb{P}^2$ as a nodal plane curve; this representation is unique by the last proposition. First, since $\mathbb{P}^2$ is rigid, the tangent space $T_{(\mathbb{P}^2,C)}\mathcal{M}$ equals the cokernel of $r : H^0(T_{\mathbb{P}^2}) \to H^0(\mathcal{O}_C(C))$ (see section 3.1 for exact sequences computing these tangent spaces). We have $H^0(T_{\mathbb{P}^2}(-C)) = 0$ and $\mathrm{Hom}_C(\Omega^1_C, \mathcal{O}_C) = 0$, so r is injective. Since $H^1(\mathcal{O}_C(C)) = 0$ the cokernel of r has dimension $\frac{1}{2}(d+2)(d+1) - 9$. This is the dimension of $\mathcal{M}$ at $(\mathbb{P}^2, C)$, so $\mathcal{M}$ is smooth.

The map of tangent spaces $dj : T_{(\mathbb{P}^2,C)}\mathcal{M} \to T_C\overline{\mathcal{M}}_{g(d)}$ is induced by the connecting homomorphism $H^0(\mathcal{O}_C(C)) \to \mathrm{Ext}^1_C(\Omega^1_C, \mathcal{O}_C)$, which has kernel $\mathrm{Hom}_C(\Omega^1_{\mathbb{P}^2}|C, \mathcal{O}_C)$. However, since $H^1(T_{\mathbb{P}^2}(-C)) = 0$ the restriction map

$$H^0(T_{\mathbb{P}^2}) \to H^0(T_{\mathbb{P}^2}|C) = \mathrm{Hom}(\Omega^1_{\mathbb{P}^2}|C, \mathcal{O}_C)$$

is surjective, so dj is injective. $\square$

By definition, the *dual graph* of a stable curve is the graph with vertices corresponding to its irreducible components and edges corresponding to intersections of these components. A graph is *n-connected* if n edges must be removed to disconnect it. The following general fact was pointed out by Joe Harris:

Proposition 2.3 *Let C be a stable curve of arithmetic genus g. Then $|\omega_C|$ is basepoint free (resp. very ample) if and only if the dual graph of C is two-connected (resp. three-connected and C is not in the closure of the hyperelliptic locus). Furthermore, $|\omega_C^{\otimes 2}|$ is very ample if and only if $g > 2$ and C has no irreducible component of arithmetic genus one intersecting the other components in one point.*

Proof. This is clear if C is smooth. If C is singular, we reinterpret sections of ω_C as meromorphic differentials on the normalization satisfying certain compatibility relations.

In this paper, we apply Proposition 2.3 only to stable curves of genus three. It yields a stratification of $\overline{\mathcal{M}}_3$ and provides a framework for our computations. In particular, using the characterization of the curves with planar representations, Proposition 2.2, and the fact that $\overline{\mathcal{M}}_3$ is smooth (as a stack) of dimension six, we obtain:

Corollary 2.4 *Let C be a stable curve of genus three. Assume that C is three-connected and not contained in the closure of the hyperelliptic locus. Then C may be canonically imbedded as a plane quartic so that the log surface $(\mathbb{P}^2, C)$ is stable. Furthermore, j is an isomorphism at these points.*

3 Deformation theory and singularities

3.1 Deformations of pairs

First we consider infinitesimal deformations of pairs (S, C), using the formalism of Ran [23] for deformations of maps $f : C \to S$. The corresponding cohomology groups $T^i_{S,C}$ sit in the following exact sequence

$$\begin{aligned} 0 \to \quad & T^0_{S,C} \to T^0_C \oplus T^0_S \to \mathrm{Hom}_f(\Omega^1_S, \mathcal{O}_C) \to T^1_{S,C} \\ \to \quad & T^1_S \oplus T^1_C \to \mathrm{Ext}^1_f(\Omega^1_S, \mathcal{O}_C) \to T^2_{S,C} \to T^2_S \oplus T^2_C \to \mathrm{Ext}^2_f(\Omega^1_S, \mathcal{O}_C) \end{aligned}$$

where $\mathrm{Ext}^i_f(\Omega^1_S, \mathcal{O}_C)$ is computed by the spectral sequences

$$\mathrm{Ext}^p_C(\mathbb{L}^q f^*\Omega^1_S, \mathcal{O}_C) \quad \text{and} \quad \mathrm{Ext}^p_S(\Omega^1_S, \mathbb{R}^q f_*\mathcal{O}_C).$$

Proposition 3.1 *If $C \subset S$ then the second spectral sequence degenerates, i.e. $\mathrm{Ext}^i_f(\Omega^1_S, \mathcal{O}_C) = \mathrm{Ext}^i_S(\Omega^1_S, \mathcal{O}_C)$. We obtain the long exact sequence*

$$\begin{aligned} 0 \quad \to \quad & \mathrm{Hom}_S(\Omega^1_S, \mathcal{O}_S(-C)) \to T^0_{S,C} \to T^0_C \to \mathrm{Ext}^1_S(\Omega^1_S, \mathcal{O}_S(-C)) \to T^1_{S,C} \\ \to \quad & T^1_C \to \mathrm{Ext}^2_S(\Omega^1_S, \mathcal{O}_S(-C)) \to T^2_{S,C} \to T^2_C \end{aligned}$$

where $\mathcal{O}_S(-C)$ denotes the ideal sheaf of C.

If C is Cartier and Ω^1_S has no torsion with support in C then the first spectral sequence also degenerates, i.e. $\mathrm{Ext}^i_f(\Omega^1_S, \mathcal{O}_C) = \mathrm{Ext}^i_C(f^*\Omega^1_S, \mathcal{O}_C)$. *We obtain the long exact sequence*

$$\begin{aligned} 0 &\to T^0_{S,C} \to T^0_S \to H^0(C, \mathcal{O}_C(C)) \to T^1_{S,C} \to T^1_S \to H^1(C, \mathcal{O}_C(C)) \\ &\to T^2_{S,C} \to T^2_S. \end{aligned}$$

Proof. To prove the first statement, we observe that f is a closed imbedding and thus a finite morphism. Hence the higher direct images $\mathbb{R}^i f_* \mathcal{O}_C$ vanish. The long exact sequence comes from applying $\mathrm{Ext}^i_S(\Omega^1_S, -)$ to

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0.$$

As for the second statement, since C is Cartier this short exact sequence is a resolution of $\mathcal{O}_C$ by invertible sheaves. This implies that $\mathcal{T}or^S_i(\Omega^1_S, \mathcal{O}_C) = \mathbb{L}^i f^*\Omega^1_S = 0$ for $i > 1$. On the other hand, $\mathbb{L}^1 f^*\Omega^1_S$ is simply the kernel of

$$\Omega^1_S \otimes \mathcal{O}_S(-C) \to \Omega^1_S$$

which coincides with the sections of Ω^1_S with support along C. $\square$

Next, we recall the definition of allowable families of semi-log canonical log surfaces. Let $\pi : (\mathcal{S}, \mathcal{C}) \to B$ be a family of such surfaces; this means that π and $\pi|\mathcal{C}$ are flat and the fibers are semi-log canonical. We say π is *allowable* if each reflexive power of $\omega_\pi(\mathcal{C})$ commutes with base extensions and some such power is locally free (see [16] and [18] for details). It follows that $(K_S + C)^2$ is locally constant in allowable families. Geometrically, allowable deformations of (S, C) are precisely those lifting locally to index-one covers.

Remark: For a semi-log canonical pair (S, C) with C Cartier, any deformation of the pair restricting to an allowable deformation of S is also an allowable deformation of (S, C). In particular, when C is Cartier and S is Gorenstein any deformation of (S, C) is allowable.

We now develop results on infinitesimal allowable deformations. First assume that $C = \emptyset$. Consider the cohomology groups $T^i_S = \mathrm{Ext}^i_S(\Omega^1_S, \mathcal{O}_S)$ and the analytic sheaves $\mathcal{T}^i_S = \mathcal{E}xt^i{}_S(\Omega^1_S, \mathcal{O}_S)$ for $i = 0, 1, 2$. These are related by the local-global spectral sequence

$$H^p(\mathcal{E}xt^q_S(\Omega^1_S, \mathcal{O}_S))$$

abutting to $\mathrm{Ext}^{p+q}_S(\Omega^1_S, \mathcal{O}_S)$. We introduce certain subsheaves $\tilde{\mathcal{T}}^i_S \subset \mathcal{T}^i_S$ capturing the infinitesimal properties of allowable deformations. Let $U \subset S$ be an analytic neighborhood with index-one cover V; U is a cyclic quotient of V. We define $\tilde{\mathcal{T}}^i_S(U)$ as the invariant part of $\mathcal{T}^i_V$. Evidently $\tilde{\mathcal{T}}^1_S \subset \mathcal{T}^1_S$ and we claim that $\tilde{\mathcal{T}}^0_S = \mathcal{T}^0_S$. Elements of $\mathcal{T}^0_S(U)$ and $\mathcal{T}^0_V$ are derivations on U and V, so it suffices to check that each derivation on U arises from an invariant one on V. Indeed, every function on V can be written as a sum of terms f_i with $f_i^r \in \mathcal{O}_U$, where r is the degree of the cover. Given a derivation $D \in \mathcal{T}^0_S(U)$, we set

$$Df_i = \frac{Df_i^r}{rf_i^{r-1}}$$

to obtain an invariant derivation on V. By definition $\tilde{\mathcal{T}}^1_S \subset \mathcal{T}^1_S$. Using the local-global spectral sequence, we define $\tilde{T}^0_S = T^0_S$ and $\tilde{T}^1_S$ as the elements of T^1_S mapped to $H^0(\tilde{\mathcal{T}}^1_S)$.

Now assume that C is a nonempty Cartier divisor on S. We define $\tilde{T}^0_{S,C} = T^0_{S,C}$. Using the remark and Proposition 3.1, we define $\tilde{T}^1_{S,C}$ as the elements of $T^1_{S,C}$ mapping to $\tilde{T}^1_S$. This clearly parametrizes first order allowable deformations of (S, C). We obtain the following corollary from Proposition 3.1:

Corollary 3.2 *Let (S, C) be a semi-log canonical log surface with C Cartier. First order allowable deformations are computed by the exact sequence*

$$0 \to \tilde{T}^0_{S,C} \to \tilde{T}^0_S \to H^0(C, \mathcal{O}_C(C)) \to \tilde{T}^1_{S,C} \to \tilde{T}^1_S \to H^1(C, \mathcal{O}_C(C)).$$

We also establish the following smoothness criterion:

Proposition 3.3 *Let (S, C) be a semi-log canonical log surface such that C is Cartier. Assume that $H^1(\mathcal{O}_C(C)) = 0$ and the space of allowable deformations of S is smooth. Then the space of allowable deformations of (S, C) is smooth as well.*

Proof. Assume we are given a tangent vector $v \in \tilde{T}^1_{S,C}$. We know that the corresponding element in $\tilde{T}^1_S$ comes from some allowable family $\mathcal{S} \to \mathrm{Spec}\mathbb{C}[[t]]$. C is Cartier, so if it deforms to $\mathcal{C} \subset \mathcal{S}$ then the resulting $(\mathcal{S}, \mathcal{C})$ is allowable. The obstruction to extending a Cartier divisor from order n to order $n + 1$ lies in $H^1(\mathcal{O}_C(C))$, which is zero by hypothesis. □

We write

$$\widetilde{\mathrm{Ext}}^0_S(\Omega^1_S, \mathcal{O}_S(-C)) = \mathrm{Ext}^0_S(\Omega^1_S, \mathcal{O}_S(-C))$$

and $\widetilde{\mathrm{Ext}}^1_S(\Omega^1_S, \mathcal{O}_S(-C))$ for the elements of $\mathrm{Ext}^1_S(\Omega^1_S, \mathcal{O}_S(-C))$ mapped to $H^0(\tilde{\mathcal{T}}^1_S)$. Again applying Proposition 3.1 we obtain

Corollary 3.4 *Let (S, C) be a semi-log canonical log surface with C Cartier. First order allowable deformations are computed by the exact sequence*

$$0 \to \mathrm{Hom}_S(\Omega^1_S, \mathcal{O}_S(-C)) \to \tilde{T}^0_{S,C} \to T^0_C \to \widetilde{\mathrm{Ext}}^1_S(\Omega^1_S, \mathcal{O}_S(-C)) \to \tilde{T}^1_{S,C} \to T^1_C.$$

Assume further that S is Gorenstein along C, i.e. $\tilde{\mathcal{T}}^1_S$ and $\mathcal{T}^1_S$ coincide along C. Then $\tilde{\mathcal{T}}^1_S(-C) = \tilde{\mathcal{T}}^1_S \cap \mathcal{T}^1_S(-C)$ and

$$\begin{array}{ccc} H^0(\tilde{\mathcal{T}}^1_S(-C)) & \to & H^0(\tilde{\mathcal{T}}^1_S) \\ \downarrow & & \downarrow \\ H^0(\mathcal{T}^1_S(-C)) & \to & H^0(\mathcal{T}^1_S) \end{array}$$

is a pull-back diagram. It follows that

$$\widetilde{\mathrm{Ext}}^1_S(\Omega^1_S, \mathcal{O}_S(-C)) \subset \mathrm{Ext}^1_S(\Omega^1_S, \mathcal{O}_S(-C))$$

consists of the elements mapped to $H^0(\tilde{\mathcal{T}}^1_S(-C)) \subset H^0(\mathcal{T}^1_S)$. The local-global spectral sequence gives the exact sequence

$$\begin{aligned} 0 \;\to\; & H^1(\mathrm{Hom}_S(\Omega^1_S, \mathcal{O}_S(-C))) \to \widetilde{\mathrm{Ext}}^1_S(\Omega^1_S, \mathcal{O}_S(-C)) \\ \to\; & \ker[H^0(\tilde{\mathcal{T}}^1_S(-C)) \to H^2(\mathcal{T}^0_S(-C))] \to 0. \end{aligned}$$

Applying Proposition 3.1 in this context, we obtain

Proposition 3.5 *Let (S, C) be a semi-log canonical log surface such that C is Cartier and S is Gorenstein near C. Assume that $H^1(\mathcal{T}^0_S(-C)) = 0$ and $H^0(\tilde{\mathcal{T}}^1_S(-C)) = 0$ or injects into $H^2(\mathcal{T}^0_S(-C))$. Then $\tilde{T}^1_{S,C}$ injects into T^1_C.*

3.2 Some surface singularities

Consider the singularity arising from the group action

$$(x, z) \to (\epsilon^a x, \epsilon z)$$

where ϵ is a primitive rth root of unity, $1 \leq a < r$, and $(a, r) = 1$. Its minimal resolution can be described quite explicitly. It consists of a chain of rational curves $E_1, E_2, \ldots, E_m$ with self-intersections $E_i^2 = -b_i$

$$\overset{-b_1}{\circ} - \cdots - \overset{-b_m}{\circ}\ .$$

The b_i are computed from the continued fraction representation

$$\frac{r}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}}.$$

The proper transforms of $x = 0$ and $z = 0$ meet the first and last exceptional curves of this chain. (See [9] §2.6 for a good exposition of this subject.) We shall label these singularities $\frac{1}{r}(a, 1)$. For instance, the singularity A_g corresponds to $\frac{1}{g+1}(g, 1)$.

Basic results on semi-log canonical singularities may be found in [20] §3-5 and [19] chapters 3 and 12. The simplest non-normal semi-log canonical singularity is the quotient of $xy = 0$ by the cyclic group action

$$(x, y, z) \to (\epsilon^a x, \epsilon^{r-a} y, \epsilon z)$$

where ϵ is a primitive rth root of unity, $1 \leq a < r$, and $(a, r) = 1$. This will be denoted $\Sigma_{r,a}$; it is the union of two cyclic quotient singularities of types $\frac{1}{r}(a, 1)$ and $\frac{1}{r}(r - a, 1)$. The index-one cover of $\Sigma_{r,a}$ consists of two smooth surfaces meeting in normal crossings. By definition, allowable infinitesimal deformations of $\Sigma_{r,a}$ correspond to deformations of its index-one cover that are fixed under the cyclic group action.

To analyze these deformations, we generalize some of Friedman's results on deformations of normal crossings varieties [8]. Let $S = S_1 \cup_B S_2$ be the union of two smooth surfaces meeting in normal crossings along a smooth curve B. Then the sheaf $\mathcal{T}^1_S$ is equal to $\mathcal{O}_{S_1}(B)|B \otimes \mathcal{O}_{S_2}(B)|B$, the product of the corresponding normal bundles [8] 2.3. Each infinitesimal deformation of S yields a section in $H^0(\mathcal{T}^1_S)$, which is nonzero iff the deformation is topologically nontrivial along B (i.e. it smooths the singularities supported on B.) Furthermore, if $H^2(\mathcal{T}^0_S) = 0$ then each such section is realized by an infinitesimal deformation of S.

Analogous results apply to *allowable* infinitesimal deformations of surfaces with singularities of type $\Sigma_{r,a}$:

Proposition 3.6 *Let $S = S_1 \cup_B S_2$ be a surface with singularities of type $\Sigma_{r,a}$ along a curve B. Then the product*

$$\tilde{\mathcal{T}}^1_S = \mathcal{O}_{S_1}(B)|B \otimes \mathcal{O}_{S_2}(B)|B$$

is a well-defined integral Cartier divisor on B. Each allowable infinitesimal deformations of S yields a section in $H^0(\tilde{\mathcal{T}}^1_S)$, which is nonzero iff the deformation is topologically nontrivial along B.

Proof. The notation $\mathcal{O}_{S_i}(B)|B$ denotes the $\mathbb{Q}$-linear combination of Cartier divisors obtained as follows: pick a minimal resolution of S_i, compute the numerical pull-back of B to this resolution, and restrict it to the proper transform of B. In our situation, the fractional parts coming from S_1 and S_2 cancel each other, so the resulting product is integral.

So let $s \in S$ be a singularity of type $\Sigma_{r,a}$, and let U be an analytic open neighborhood of s in S. The index-one cover $V \to U$ is a cyclic cover of degree r ramified only at s. Let B' be the primage of B; $B' \to B$ is also ramified only at s. As we have seen, $V = V_1 \cup V_2$ has ordinary double points along B'. Using Friedman's results, we find that $\mathcal{T}^1_V = \mathcal{O}_{V_1}(B')|B' \otimes \mathcal{O}_{V_2}(B')|B'$. Furthermore, $\mathcal{O}_{S_1}(B)|B \otimes \mathcal{O}_{S_2}(B)|B$ pulls back to $\mathcal{T}^1_V$ on B', and its sections coincide with the invariant sections of $\mathcal{T}^1_V$. $\square$

4 Limiting plane quartic curves

We outline the strategy for completing the proof of the main theorem. Let C be a stable curve of genus three. Using the stratification arising from Proposition 2.3, we may assume C is either hyperelliptic and three-connected, two-connected but not three-connected, or just one-connected. The key steps are:

1. Describe a surface S containing C such that (S, C) is a stable log surface. It will turn out that C is Cartier and S is smooth or satisfies $xy = 0$ at points of C.

2. Show that (S, C) has an allowable deformation to a plane quartic.

3. Show that j is an isomorphism at (S, C), using two possible approaches. One approach is to apply Corollary 3.5 to prove that dj is injective at (S, C). The other is to prove that $\mathcal{M}$ is smooth (using Corollary 3.3) and j is injective.

4.1 Three-connected hyperelliptic curves

Let C be a smooth hyperelliptic curve of genus 3, with double cover $r : C \to \mathbb{P}^1$. We have that $r_*\mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4)$, so that we can regard C as a bisection of the rational ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(+4)) = \mathbb{F}_4$. C is disjoint from the zero section (i.e. the section with self-intersection -4). Blowing down this (-4)-curve, we obtain a surface S isomorphic to the weighted projective plane $\mathbb{P}(1,1,4)$ (see [9] pp. 35 for definitions of weighted projective spaces). S has a cyclic quotient singularity of type $\frac{1}{4}(1,1)$. The pair (S,C) is semi-log canonical, because S has log terminal singularities and C is smooth and disjoint from the singularities of S. Since K_S+C is ample, we conclude that (S,C) is a stable log surface.

We can picture $S \subset \mathbb{P}^5$ as the cone over a rational normal quartic curve in $\mathbb{P}^4$; C is a quadric hypersurface section of this cone. Indeed, this description applies to all three-connected hyperelliptic stable curves. We claim S has an allowable deformation to a Veronese surface isomorphic to $\mathbb{P}^2$. Indeed, the cone over a Veronese surface has a terminal singularity of index two at the vertex. A generic hyperplane section of this cone is a Veronese, whereas a generic hyperplane section through the vertex is the cone over a rational quartic curve.

Now we analyze the deformation spaces of the pairs (S,C). We claim the deformation space of S is smooth of dimension one. It suffices to show its tangent space is one-dimensional. We have that $H^1(\mathcal{H}om(\Omega^1_S,\mathcal{O}_S)) = H^2(\mathcal{H}om(\Omega^1_S,\mathcal{O}_S)) = 0$ by a straightforward cohomology computation (e.g. [7] §2.3). Local deformations of the vertex of S coincide with invariant deformations of its index-one cover, which has an A_1 singularity. Therefore, $\tilde{T}^1_S = H^0(\tilde{\mathcal{T}}^1_S) = \mathbb{C}$, which proves the claim. Since $\mathcal{O}_C(C) = \omega_C^{\otimes 4}$ has no higher cohomology, $\mathcal{M}$ is smooth of dimension six at (S,C) (Proposition 3.3).

Now we show that j is injective. The imbedding $C \hookrightarrow S$ is rigid (modulo automorphisms of S) because $H^1(T_S(-C)) = 0$ [7] §2.3. Hence the restriction of j to the locus where $S = \mathbb{P}(1,1,4)$ has injective derivative. Second, any nontrivial allowable deformation of $\mathbb{P}(1,1,4)$ is to a Veronese surface, and such deformations force C to deform to a nonhyperelliptic curve. Consequently, $j : \mathcal{M} \to \overline{\mathcal{M}}_3$ is injective (and thus an isomorphism) over the three-connected hyperelliptic curves.

4.2 Two-connected curves

Let C be a curve which is two-connected but not three-connected; it is the union of two curves C_1 and C_2 which are two-connected, meet in two points, and have arithmetic genus one. Each C_i is either a smooth elliptic curve, an irreducible nodal curve of arithmetic genus one, or the union of two rational curves meeting at two points (with one point of $C_1 \cap C_2$ on each component). Let L be the union of two distinct lines in $\mathbb{P}^2$. By Proposition 2.3, $|\omega_C|$ yields a double cover $r : C \to L$, with C_1 and C_2 dominating the components of L. This allows us to imbed C naturally into the union S' of two ruled surfaces $\mathbb{F}_2$, glued along a ruling B. C is disjoint from the zero sections of these surfaces. Let S be the surface obtained by blowing down these zero sections; its irreducible components S_1 and S_2 each have cyclic quotient singularities of type $\frac{1}{2}(1,1)$ and S has semi-log canonical singularities of type $\Sigma_{2,1}$. We can check that $K_S + C$ is ample, hence (S, C) is stable.

We can represent $S \subset \mathbb{P}^5$ as the cone over the union of two conic curves meeting in a single point but otherwise in linearly general position. The allowable deformations of S may be analyzed using Proposition 3.6. It has no topologically trivial deformations and $\tilde{\mathcal{T}}^1_S = \mathcal{O}_{\mathbb{P}^1}(\frac{1}{2} + \frac{1}{2})$ which has two sections. Thus $\dim \tilde{T}^1_S \leq 2$, and we have equality since S admits allowable deformations to the Veronese and $\mathbb{P}(1,1,4)$.

We apply Proposition 3.5 to show that dj is injective. First, we have that $H^0(\tilde{\mathcal{T}}^1_S(-C)) = H^0(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. On the other hand, $H^1(\mathcal{T}^0_S(-C))$ parametrizes first-order deformations of the imbedding $C \hookrightarrow S$ (or, equivalently, of $C \hookrightarrow S'$) modulo automorphisms. Setting $C_1 \cap C_2 = \{p_1, p_2\}$, this is the same as first order deformations of the pair of imbeddings

$$(C_i, p_1 + p_2) \hookrightarrow (\mathbb{F}_2, B_i)$$

modulo automorphisms of $\mathbb{F}_2$ stabilizing the ruling B_i. It is straightforward to check there are no such first order deformations.

4.3 Local stable reduction of cusps

The concept of *local stable reduction* for germs of cuspidal curves is developed systematically in [13]. Let

$$\mathcal{C} \subset \mathrm{Spec}\mathbb{C}[[x, y]] \times \Delta \to \Delta$$

be a family of plane curve germs, such that the central fiber C_0 is cuspidal and C_t is smooth for $t \neq 0$. Set $S_0 = \mathrm{Spec}\mathbb{C}[[x, y]]$ and $\mathcal{S} = S_0 \times \Delta$. We consider $(\mathcal{S}, \mathcal{C}) \to \Delta$ as a family of germs of log surfaces. Applying local stable reduction, we obtain a family of log surfaces with semi-log canonical singularities $(\mathcal{S}^c, \mathcal{C}^c) \to \tilde{\Delta}$ and a birational morphism $\phi : (\mathcal{S}^c, \mathcal{C}^c) \to (\mathcal{S} \times_\Delta \tilde{\Delta}, \mathcal{C} \times_\Delta \tilde{\Delta})$ such that $K_{\mathcal{S}^c} + \mathcal{C}^c$ is ample relative to ϕ. The following proposition details the local stable reductions for smoothings of a cusp C_0.

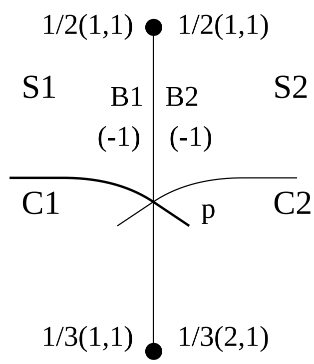

Figure 1: Local stable reduction of a cusp

Proposition 4.1 *Let $C_0 \subset \mathrm{Spec}\mathbb{C}[[x,y]]$ be a cusp with local analytic equation $y^2 = x^3$, $\mathcal{C} \to \Delta$ a smoothing of C_0. Then the local stable limits (S_0^c, C_0^c) are precisely the following:*

1. *S_1 is the toriodal blow-up of S_0 along the ideal $\{y^2, x^3, yx^2\}$. It has two singularities of types $\frac{1}{2}(1,1)$ and $\frac{1}{3}(1,1)$ along the exceptional divisor B_1. C_1 is the proper transform of C_0.*

2. *S_2 is the weighted projective space $\mathbb{P}(1,2,3)$ and B_2 is the effective curve generating its divisor class group. It has two singularities of types $\frac{1}{2}(1,1)$ and $\frac{1}{3}(2,1)$ along B_2. C_2 is a nodal curve generating the Picard group of S_2.*

3. *S_0^c is obtained by gluing S_1 and S_2 along B_1 and B_2 so that it has singularities of types $\Sigma_{2,1}$ and $\Sigma_{3,1}$, and $C_0^c = C_1 \cup_p C_2$ is connected.*

These stable limits are naturally parametrized by $\overline{\mathcal{M}}_{1,1}$.

Proof. See figure 1 for a schematic picture using our notational conventions for quotient singularities; the numbers in parentheses are self-intersections of the curves in the minimal resolution. See figure 2 for the imbedded resolution of a cusp.

Much of this is contained in Main Theorem 1 of [13], which implies that all of these surfaces do actually occur as local stable limits. (See also [22] §12 and 13 for a proof that most of these stable limits appear.) It remains to show that each stable limit has the form given in the proposition. By the Main Component Theorem of [13], it suffices to show that the moduli space containing the (non-smoothable) stable log surfaces $(S_2 = \mathbb{P}(1,2,3), B_2 + C_2)$ is isomorphic to $\overline{\mathcal{M}}_{1,1}$. We claim first that each $(C_2, p) \subset \overline{\mathcal{M}}_{1,1}$ sits as a Cartier divisor on $\mathbb{P}(1,2,3)$ with $p = B_2 \cap C_2$. Indeed, the linear series $\mathcal{O}_{C_2}(2p)$ yields a natural imbedding $C_2 \hookrightarrow \mathbb{F}_2$ with a ruling R tangent at p. Blowing up twice to separate this tangency and then blowing down all the (-2)-curves, we obtain an imbedding $C_2 \hookrightarrow \mathbb{P}(1,2,3)$ with the desired property.

We next claim $(\mathbb{P}(1,2,3), B_2)$ has no nontrivial allowable deformations (S_2, B_2). Since

$$(K_{S_2} + B_2)^2 = (K_{\mathbb{P}(1,2,3)} + B_2)^2 = \frac{25}{6}$$

any such deformation still has quotient singularities of types $\frac{1}{2}(1,1)$ and $\frac{1}{3}(2,1)$. A cohomology computation on the minimal resolution of $\mathbb{P}(1,2,3)$ shows it has no topologically trivial deformations.

Each Weil divisor linearly equivalent to $B_2 + C_2$ contains B_2 [7] 1.4. Hence it suffices to show the forgetting map $(\mathbb{P}(1,2,3), B_2 + C_2) \to (C_2, p)$ has injective derivative. Since $H^1(T_{\mathbb{P}(1,2,3)}(-C_2)) = 0$ [7] 2.3, our assertion follows from Proposition 3.5. □

4.4 Generic one-connected curves

Let C be a generic one-connected curve, i.e. $C = C_1 \cup_p C_2$ with C_1 and C_2 irreducible of genus two and one respectively, and p not a Weierstrass point for C_1. The limiting plane S occurs as the local stable reduction for quartic plane curves acquiring a single cusp. Precisely, $S = S_1 \cup_B S_2$ where S_1 is the toriodal blow-up of $\mathbb{P}^2$ and $S_2 = \mathbb{P}(1,2,3)$. C_1 is the proper transform of a cuspidal quartic and C_2 is the elliptic tail. The resulting log surface (S, C) clearly has semi-log canonical singularities. It is stable because $K_{S_1} + C_1 + B_1$ is ample by Lemma 4.3; $K_{S_2} + B_2 + C_2$ is ample because the Picard group of $\mathbb{P}(1,2,3)$ is cyclic. To show that every generic one-connected curve actually sits in such a surface, it suffices to check that an irreducible genus-two curve C_1 with distinguished non-Weierstrass point p can be mapped into $\mathbb{P}^2$ as a quartic curve with cusp at p. This follows from Lemma 4.4.

We have already seen how S arises as an allowable degeneration of $\mathbb{P}^2$. Since $H^1(\mathcal{O}_C(C)) = 0$ there are no obstructions to lifting a deformation of S to a deformation of (S, C), and C is a limiting plane curve.

Our toroidal blow-up of $\mathbb{P}^2$ is unique up to isomorphism, so S_1 has no topologically trivial deformations. The allowable deformation space of S is smooth of dimension one by Proposition 3.6. The vanishing of $H^1(\mathcal{O}_C(C))$ implies that $\mathcal{M}$ is smooth at (S, C) (by Proposition 3.3).

To show that j is injective over C, we first prove that the imbedding $C \hookrightarrow S$ is unique, up to automorphisms of S. By lemma 4.4, C_1 is represented as a cuspidal quartic in a unique way, and thus has a unique imbedding into S_1. We have already seen that C_2 has a unique imbedding into $S_2 = \mathbb{P}(1, 2, 3)$. Since every deformation of S forces C to deform to a three-connected curve, there are no nontrivial families of surfaces containing C.

4.5 Degenerate cases

We now account for the remaining cases. The corresponding stable curves C may all be disconnected by removing a point p_i. Then there is a unique genus-one irreducible component E_i meeting the rest of C in p_i. Let C_1 denote the closure of the complement of the union of these elliptic tails; note that C_1 is connected.

We describe the limiting plane S containing C. Our notation is explained in subsection 4.6. Using Lemma 4.4, we obtain a map $g : C_1 \to P$ so that the $g(p_i)$ are cusps on $C_0 = g(C_1)$. The limiting plane S is the surface arising when we apply local stable reduction to the cusps individually. It is the union of a toroidal blow-up $b : S_1 \to P$ and a number of copies of $\mathbb{P}(1, 2, 3)$, glued as specified in Proposition 4.1. In particular, there is one copy of $\mathbb{P}(1, 2, 3)$ for each elliptic tail of C (or for each cusp of C_0.) We obtain $C \subset S$ by imbedding each E_i into the corresponding weighted projective plane. (S, C) is a stable log surface because $K_{S_1} + C_1 + B_1$ is ample by Lemma 4.3.

The surface S is a degeneration of $\mathbb{P}^2$. Indeed, given a deformation $\mathcal{P} \to \Delta$ of P to $\mathbb{P}^2$, we blow up $\mathcal{P}$ toroidally at each of the points $g(p_i)$. (Such blow-ups are discussed in §5 of [13].) Hence for each $v \in H^0(\tilde{\mathcal{T}}^1_S)$ there exists a global deformation of S with tangent vector restricting to v.

For the rest of this section, we assume that C_0 is not a plane quartic with three cusps (or equivalently, that C_1 is not of genus zero). We return to the tricuspidal curves below. Under these assumptions, we analyze the allowable deformation space of S. By Lemma 4.2 S_1 has no equisingular deformations and we have already seen that $\mathbb{P}(1, 2, 3)$ has no equisingular deformations. Hence S also has no such deformations, $\tilde{T}^1_S = H^0(\tilde{\mathcal{T}}^1_S)$, and the space of allowable deformations is smooth. Looking component by component, it is not difficult to check that $\mathcal{O}_C(C)$ has no higher cohomology. It follows that C is a limiting plane quartic and $\mathcal{M}$ is smooth (Proposition 3.3).

To show that $C \hookrightarrow S$ is rigid (modulo automorphisms of S) we look at each irreducible component of S separately. The map of the (pointed curve) $(C_1, p_i)_{i=1,\dots,k}$ to S_1 is unique by Lemma 4.4. On the other hand, each genus-one tail has a unique imbedding into $\mathbb{P}(1,2,3)$ (up to automorphisms). Any topologically nontrivial deformation of S deforms either one of the components of B_1, or the singularities arising from P. Smoothing one of the components of B_1 entails smoothing one of the elliptic tails of C. Deforming the singularities arising from P entails deforming C to a curve that is no longer in the closure of the curves described in sections 4.1 or 4.2. Hence j is injective.

4.5.1 Tricuspidal curves

There is a unique plane quartic with three cusps, obtained by applying the standard Cremona transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ to a smooth conic C_1 tangent to the three distinguished lines. However, assume we are given three lines L_i $(i = 1, 2, 3)$ and points $s_i \in L_i$, all chosen generically. Then there is no smooth conic tangent to the L_i at the s_i. Hence the position and tangent directions of the cusps of C_0 are *not* in general position on $\mathbb{P}^2$. It follows that $S_1 \to \mathbb{P}^2$ is obtained by blowing up along three (nonreduced) points in special position, and S_1 has equisingular deformations.

We claim that the equisingular deformations of S_1 fail to extend to deformations of (S_1, C_1), even up to first order. Using the Cremona transformation, this translates as follows. Assume we are given lines L_i and points $s_i \in L_i$ such that there exists a conic tangent C_1 to the L_i at the s_i. Deform s_1 (in L_1) to first order, while fixing L_1, L_2, L_3, s_2, and s_3. Then there is no first order deformation of C_1 preserving the tangencies. This follows by a simple explicit computation, which we omit. Cohomologically, this translates into saying that

$$H^1(\mathcal{T}^0_{S_1}) \to H^1(\mathcal{O}_{C_1}(C_1))(= H^1(\mathcal{O}_{\mathbb{P}^1}(-2)))$$

and thus also

$$H^1(\mathcal{T}^0_S) \to H^1(\mathcal{O}_C(C))$$

is an isomorphism of one-dimensional spaces. As a consequence, $\tilde{T}^1_S \to H^1(\mathcal{O}_{C_1}(C_1))$ has one-dimensional image and three-dimensional kernel.

We complete the computation of $\tilde{T}^1_{S,C}$ using Corollary 3.2. We know that $\dim \tilde{T}^0_{S,C} = 0$ because (S, C) is stable, $\dim H^0(\mathcal{O}_C(C)) = 15$ by Riemann-Roch, and $\dim \tilde{T}^0_S = 3(\dim T^0_{\mathbb{P}(1,2,3)} - 1) = 12$. It follows that $\dim \tilde{T}^1_{S,C} =$

$3+15-12=6$. On the other hand, looking at each of the components individually we find that $H^1(\mathcal{T}^0_S(-C))=0$. Indeed, we have already seen that imbedding of the elliptic tails into the $\mathbb{P}(1,2,3)$ is rigid (see the proof of Proposition 4.1); $C_1 \hookrightarrow S_1$ is also rigid because $C_1^2=-2$. Proposition 3.6 implies that $H^0(\tilde{\mathcal{T}}^1_S(-C))=0$, so dj is injective by Proposition 3.5.

We have not shown directly that each (S,C) admits an allowable deformation; a priori this may only be the case for special choices of the elliptic tails. However, we have shown that the derivative of j is an isomorphism at each (S,C), which guarantees that j is an isomorphism as well.

This completes the proof of the Main Theorem.

4.6 Technical lemmas

Throughout this subsection $P=\mathbb{P}^2$, $\mathbb{P}(1,1,4)$, or the union of two copies of $\mathbb{P}(1,1,2)$, glued along rulings so that the vertices coincide. Each of these surfaces may be imbedded as a quartic surface in $\mathbb{P}^5$. Let $C_0 \subset P$ be a quadric hypersurface section. We assume that C_0 has only nodes and cusps and each branch of C_0 intersects the double curve of P transversally.

Lemma 4.2 *Each possible configuration of cusps is one of the following seven types:*

1. $P=\mathbb{P}^2$ *and* C_0 *has a single cusp;*
2. $P=\mathbb{P}(1,1,4)$ *and* C_0 *has one cusp;*
3. P *the union of two copies of* $\mathbb{P}(1,1,2)$ *and* C_0 *has one cusp;*
4. $P=\mathbb{P}^2$ *and* C_0 *has two cusps;*
5. $P=\mathbb{P}(1,1,4)$ *and* C_0 *has two cusps;*
6. P *the union of two copies of* $\mathbb{P}(1,1,2)$ *and* C_0 *has one cusp on each component;*
7. $P=\mathbb{P}^2$ *and* C_0 *has three cusps (obtained from the Cremona transformation described above).*

Except in the last case, there is a torus action on P *(with a dense orbit on each irreducible component) fixing the positions and tangencies of the cusps. In each case these positions and tangencies are unique up to automorphisms of* P.

Proof. We first check the list is complete. It is clear that C_0 cannot have more than three cusps. If $P = \mathbb{P}(1,1,4)$ then C_0 has at most two cusps; otherwise we would obtain a rational double cover of $\mathbb{P}^1$ ramified at three points. A similar argument applies if P is the union of two copies of $\mathbb{P}(1,1,2)$.

We construct the torus action case by case. If $P = \mathbb{P}^2$ and C_0 has two cusps, then the boundary of our torus in the union of the line joining the cusps and the lines tangent to them. These lines are in linearly general position because C_0 has degree four. If $P = \mathbb{P}(1,1,4)$ and C_0 has two cusps then there is a smooth hyperplane section tangent to both these cusps. The boundary is the union of the rulings meeting the cusps and this hyperplane. In the case where P has two components isomorphic to $\mathbb{P}(1,1,2)$, the boundary on each $\mathbb{P}(1,1,2)$ is the union of the distinguished ruling, the ruling meeting the cusp, and a smooth hyperplane section tangent to the cusp.

The last assertion follows from the existence of the torus action (except in the last case, where we use the Cremona transformation description). □

Lemma 4.3 *Retain the hypotheses of Lemma* 4.2. *Let* $b : S_1 \to P$ *be the toriodal blow-up of the cusps of* C_0, B_1 *the exceptional locus, and* C_1 *the proper transform of* C_0. *Then* $K_{S_1} + C_1 + B_1$ *is ample. Furthermore, if* C_0 *has at most two cusps then* S_1 *has no equisingular deformations.*

Proof. The last statement follows immediately from the previous result. We first prove the ampleness statement under the assumption that C_0 has at most two cusps, so that b is actually a toric blow-up. Then the boundary divisors for the torus action generate the effective cone of S_1, which allows us to check directly that $K_{S_1} + C_1 + B_1$ is ample.

The only remaining case is where $C_0 \subset \mathbb{P}^2$ has three cusps. Let $D = K_{S_1} + C_1 + B_1$ and $\sigma : P_m \to S_1$ an imbedded resolution of $(\mathbb{P}^2, C_0)$ (see figure 2). Let B_m and C_m be the partial transforms of the corresponding curves on S_1. Each irreducible component of B_m is a (-1)-curve on P_m. Let F and G be the union of the (-2) and (-3)-curves in the exceptional locus, and H the pull-back of the hyperplane class from $\mathbb{P}^2$. Let σ^*D be pull-back of D as a $\mathbb{Q}$-Cartier divisor:

$$\sigma^*D \equiv K_{P_m} + C_m + B_m + \frac{1}{2}F + \frac{2}{3}G = H - \frac{1}{2}F - \frac{1}{3}G - B.$$

$6\sigma^*D$ is integral and corresponds to the proper transforms of the linear series of sextics in $\mathbb{P}^2$ with cusps in the same positions as C_0. It is not hard to see

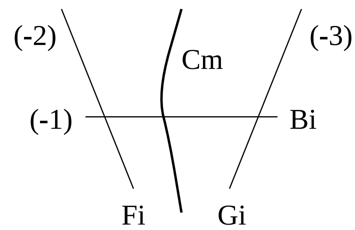

Figure 2: Minimal resolution of a cusp

that $b : S_1 \to \mathbb{P}^2$ blows up the base locus of this series and $6D$ is very ample on S_1. □

Lemma 4.4 *Let* $(C_1, p_i)_{i=1,\ldots,k}$ *be a pointed stable curve such that* $g(C_1) + k = 3$. *Then there exists a unique map* $g : C_1 \to P$ *with the following properties:*

1. g *is an isomorphism except at the points* p_i*;*

2. $g(p_i)$ *are cusps on* $C_0 := g(C_1)$ *and smooth points of* P.

Proof. This is proved using a case-by-case argument. If the p_i are not all fixed points of a hyperelliptic involution on C_1, then the image of g is $\mathbb{P}^2$. First, consider the image of C_1 under the linear system $|\omega_{C_1}(2\sum p_i)|$. If C_1 has genus two, then the image is clearly a quartic with a cusp at p_1. Otherwise the linear system yields an imbedding, but under our hypothesis there is a *unique* projection of C_1 to a quartic plane curve so that the p_i are mapped to cusps. Now assume the p_i are fixed points of a hyperelliptic involution on C_1. If C_1 is not the union of two rational curves meeting at two points, then we may represent C_1 as the ramified double cover of $L = \mathbb{P}^1$. Otherwise, C_1 is the double cover of a curve L isomorphic to two copies of $\mathbb{P}^1$ joined at a point. In either case, the map $r_1 : C_1 \to L$ ramifies at the p_i. Now consider the modified double cover $r_0 : C_0 \to L$, where the support of the branch divisor is as before, except that it has multiplicity three at the points $r_0(p_i)$. C_0 then has cusps over these points. Repeating the argument from the hyperelliptic case, we find that C_0 can be imbedded in either the weighted projective plane $\mathbb{P}(1, 1, 4)$, or the union of two quadric cones glued along two rulings. □

4.7 Explanation for the elliptic tails

We have seen that elliptic tails arise from applying local stable reduction to suitable cuspidal curves. This can be explained using the notion of *log canonical thresholds* [17] §8. Let C_0 be the germ of an isolated singularity in $S_0 = \mathrm{Spec}\mathbb{C}[[x, y]]$. The *log canonical threshold* of (S_0, C_0) is defined as

$$\sup\{a : K_{S_0} + aC_0 \text{ is log canonical}\}.$$

For instance, if C_0 is a cusp then $K_{S_0} + aC_0$ is log canonical for $a \leq \frac{5}{6}$ so the threshold equals $\frac{5}{6}$.

On the other hand, if C is a smooth quartic plane curve then $K_{\mathbb{P}^2} + aC$ is ample provided that $a > \frac{3}{4}$. This suggests that we try to construct a moduli spaces of log surfaces $\mathcal{M}(a)$ with boundary aC, where $\frac{3}{4} < a \leq \frac{5}{6}$. For this new moduli problem, the cuspidal curves are stable and the hyperelliptic and two-connected curves are precisely as described above. Presumably, the standard moduli space $\mathcal{M} = \mathcal{M}(1)$ is obtained by suitably blowing up the locus of cuspidal curves in $\mathcal{M}(a)$.

It is very natural to ask how what happens to these moduli spaces as we vary the parameter a. What are the maps between the various moduli spaces, and are there wall crossings like those found in Geometric Invariant Theory by Thaddeus [24]? Is there a natural stratification of the boudary components in terms of the log canonical thresholds of the corresponding plane curve singularities?

5 Limits of degree-two Del Pezzo surfaces

Here we give an application of our results on $\overline{\mathcal{P}}_4$ to degree-two Del Pezzo surfaces. This was suggested by János Kollár and aided by discussions with F. Gallego and B.P. Purnaprajna. Let T be a Del Pezzo surface of degree two. Then the anticanonical linear series $|-K_T|$ induces a double cover $\rho : T \to \mathbb{P}^2$ branched over a smooth quartic plane curve C. Conversely, given such a curve, the double cover of $\mathbb{P}^2$ branched over C is a Del Pezzo surface of degree two.

We shall use our description of limiting plane quartics to obtain a geometric compactification $\mathcal{Z}$ for the moduli space of degree-two Del Pezzo surfaces. This compactification has the following properties:

1. The points of $\mathcal{Z}$ parametrize semi-log canonical surfaces T equipped with a double cover $\rho : T \to S$.

2. The branch locus of ρ contains a curve $C \subset S$ such that $(S, C) \in \overline{\mathcal{P}}_4$. (It may also include singularities of S not contained in C.)

3. Each limiting plane quartic (S, C) has a unique such double cover and $\mathcal{Z} \cong \overline{\mathcal{P}}_4 \cong \overline{\mathcal{M}}_3$.

Given a limiting plane quartic (S, C), we construct a ramified double cover $\rho : T \to S$, which is a limit of the double covers of the plane described above. When C is two-connected the construction of T is clear. Here the class $[C]$ is twice a Cartier divisor on S, so we can construct ρ in the standard way. The resulting surface T occurs naturally as the limit of smooth Del Pezzos; it is easy to verify that T has semi-log canonical singularities. The construction is more subtle in the case where C is not two-connected. Here the class of $[C]$ is only twice a Weil divisor, so constructing ρ requires more care. Following ideas of Gallego and Purnaprajna, we find that ρ ramifies over C and over each of the $\Sigma_{2,1}$ singularities arising from elliptic tails of C (cf. Proposition 4.1). Furthermore, ρ is the limit of double covers of the plane branched over smooth quartics. The resulting surface T has semi-log canonical singularities: it has normal crossings where ρ ramifies, and it has essentially the same singularities as S where ρ is étale.

Using the modified spaces $\mathcal{M}(a)$ for $\frac{3}{4} < a \leq \frac{5}{6}$ yields an alternate compactification, which perhaps is more natural than $\mathcal{Z}$. As we have seen, cuspidal curves are stable for these spaces. The existence of cusps in the branch locus causes no serious difficulties; the resulting double cover has A_2 singularities. This approach has the added advantage that the branch curve $C \subset S$ is always twice a Cartier divisor, so constructing the double covers is technically easier.

6 Observations on curves of higher degree

The map $j : \overline{\mathcal{P}}_d \to \overline{\mathcal{M}}_{g(d)}$ is not generally an isomorphism, or even a bijection, onto its image. For instance, consider the case $d = 5$. The closure of the plane quintic curves contains all the hyperelliptic curves C of genus six ([12] 1.11 or [10].) The limiting g^2_d takes the form $2g^1_2 + p$ where p is any one of the 14 Weierstrass points of C. Each of these Weierstrauss points yields an element

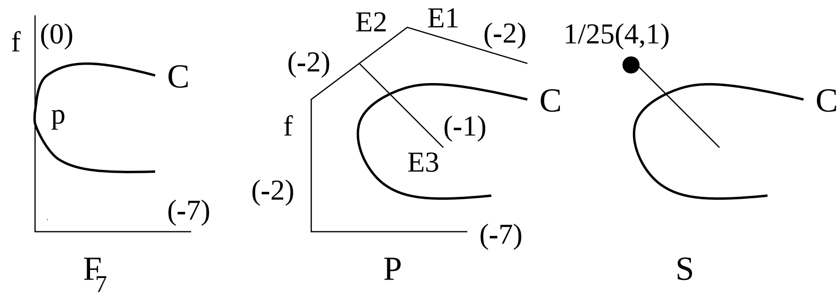

Figure 3: Limiting planes for genus six hyperelliptic curves

at the boundary of $\overline{\mathcal{P}}_5$. These are constructed by the following recipe. Let C be a hyperelliptic curve of genus 6, and represent C as a bisection of $\mathbb{F}_7$. Let f be a ruling of $\mathbb{F}_7$, tangent to C at one of the Weierstrauss points p. Let $\rho : P \to \mathbb{F}_7$ be obtained by taking the minimal imbedded resolution of $C \cup f$, and then blowing-up the intersection of the proper transfrom of C with the exceptional locus. P has three exceptional divisors E_1, E_2, and E_3 with self-intersections $-2, -2$, and -1 respectively. Let S be obtained by contracting E_1, E_2, the proper transform of f, and the zero section of $\mathbb{F}_7$ (see figure 3). One can show that S deforms to $\mathbb{P}^2$, with the plane quintics specializing to the hyperelliptic curve C (see [21] for more information on such degenerations).

The moduli space $\mathcal{M}$ generally has irreducible components besides $\overline{\mathcal{P}}_d$:

Proposition 6.1 *Let $d = 2n + 1$ with $n > 1$. Then $\mathcal{M}$ has at least one irreducible component besides $\mathcal{P}_d$, parametrizing curves of bidegree $(n+1, 2n)$ in $\mathbb{F}_0$.*

For instance, if $d = 5$ then $\mathcal{M}$ contains both the plane quintics and the trigonal curves of genus six. These components have dimensions 12 and 13 respectively. This example highlights another peculiarity of $\mathcal{M}$: its *expected dimension* is not well-defined on connected components, because the formula for the expected dimension is not invariant under allowable deformations.

Proof. We now prove the proposition. Let $C^\nu \subset \mathbb{F}_4$ be a smooth curve of class $n\xi + 2f$, where ξ is the section with $\xi^2 = 4$ and f is a ruling. The zero section of $\mathbb{F}_4$ intersects C^ν in two points p and q. Blow down the zero section to obtain $\mathbb{P}(1, 1, 4)$, which may be represented as the cone over a rational normal quartic curve. The image C_0 of C^ν has a node at the vertex of $\mathbb{P}(1, 1, 4)$, so the pair $(\mathbb{P}(1, 1, 4), C_0)$ has log canonical singularities. There

exist deformations $\mathcal{S} \to \Delta$ with $S_0 = \mathbb{P}(1,1,4)$ and the general fiber equal to either a Veronese surface or a rational quartic scroll isomorphic to $\mathbb{F}_0$. The curve C_0 deforms to either a plane curve of degree $2n+1$ or to a curve of bidegree $(n+1, 2n)$ in $\mathbb{F}_0$. In both cases, $K_{\mathcal{S}} + \mathcal{C}$ is $\mathbb{Q}$-Cartier so the deformation of the *pair* (S_0, C_0) is allowable. Note that for the deformation to $\mathbb{F}_0$, $K_{\mathcal{S}}$ and $\mathcal{C}$ are not individually $\mathbb{Q}$-Cartier. Hence the deformation of the surface alone is not allowable. □
We close with an open question:

Question 6.2 *Does* $j : \overline{\mathcal{P}}_d \to \overline{\mathcal{M}}_{g(d)}$ *ever have positive dimensional fibers?*

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