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REFLEXIVE PULL-BACKS AND BASE EXTENSION

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Abstract

We prove that Viehweg's moduli functor of stable surfaces is locally closed.

1. Introduction

The moduli theory of curves has been studied extensively in the past few decades. A very important and useful feature of the theory is that the moduli space of smooth projective curves of genus g admits a geometrically meaningful compactification as the moduli space of stable curves of genus g. The success of moduli theory of curves leads naturally to a desire for a similar theory for higher dimensional varieties.

In recent years there has been great progress in the moduli theory of surfaces and higher dimensional varieties by Alexeev, Kollár, Shepherd-Barron, and Viehweg [1]; [11]; [13]; [14]. According to their work, moduli spaces exist for many moduli problems, in particular for smooth canonically polarized varieties. More generally it is established that if a moduli problem satisfies certain properties, then a corresponding (coarse) moduli space exists. The most important of these properties are *separatedness, boundedness* and *local closedness*. According to the above authors' work the former two of these hold for the moduli problem of canonically polarized stable surfaces – the candidate for a geometrically meaningful compactification of the moduli space of smooth canonically polarized surfaces. Local closedness, however, has presented a very stubborn problem.

In fact, one of the main problems is that it is not entirely clear what the "right" definition of the moduli functor should be. This is a very delicate

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problem as one would like to make the functor large enough to obtain a compact moduli space, but enlargening the class too much could lead to a loss of separatedness and/or boundedness.

In addition, not only the admissible models have to be decided, but also the admissible families of those models. Experts generally agree on what models should be allowed (the *semi log canonical models*). However, the right notion of admissible families is still to be decided.

Both Kollár and Viehweg suggest reasonable definitions, but local closedness has yet to be established for either of their moduli functors. At this time it is not even clear whether their definitions differ. However, we should point out that Kollár's moduli functor is known to satisfy a weak form of local closedness. Precisely, after passage to a formal or étale local ring, local closedness holds provided we restrict to base change morphisms arising from local ring homomorphisms [12], §14.

The goal of this paper is to prove that Viehweg's moduli functor of canonically polarized varieties is locally closed.

Definitions and notation. Every scheme is considered to be of finite type over an algebraically closed field k unless specifically noted otherwise.

Let $f : X \to S$ be a morphism. Then X_s denotes the fibre of f over the point $s \in S$ and f_s denotes the restriction of f to X_s. More generally, for a morphism $\alpha : T \to S$, let $f_T : X_T = X \times_S T \to T$. In particular one has the following commutative diagram:

$$\begin{array}{ccc} X_T = X \times_S T & \xrightarrow{\alpha_X} & X \\ {\scriptstyle f_T}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow{\alpha} & S \end{array}$$

For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{F}_T$ will denote $\alpha_X^*\mathcal{F}$ on X_T. Tensor products of $\mathcal{O}_{X_T}$-modules are over $\mathcal{O}_{X_T}$. These conventions will be used through the entire article.

We will write $\mathcal{F}^*$ for the dual $\mathcal{O}_X$-module $\mathcal{H}om_X(\mathcal{F}, \mathcal{O}_X)$ when there is no risk of confusion. The double dual $\mathcal{F}^{**}$ is called the *reflexive hull* of $\mathcal{F}$ and there is a natural $\mathcal{O}_X$-module homomorphism

$$\mathcal{F} \to \mathcal{F}^{**};$$

$\mathcal{F}$ is said to be *reflexive* if this is an isomorphism. We shall also consider reflexive powers

$$\mathcal{F}^{[m]} := (\mathcal{F}^{\otimes m})^{**} \quad \mathcal{F}^{[-m]} := (\mathcal{F}^{\otimes m})^{*}$$

for $m > 0$. In general, there exist natural maps

$$(\mathcal{F}^{**})_T \to (\mathcal{F}_T)^{**} \quad \text{and} \quad (\mathcal{F}^{[m]})_T \to (\mathcal{F}_T)^{[m]}$$

which need not be isomorphisms, even when $\mathcal{F}$ is reflexive. Of course, these maps are isomorphisms when $\mathcal{F}$ is locally free.
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2. Moduli functors

Definition 2.1. Fix a base scheme B. The *moduli functor of polarized proper schemes* is the contravariant functor

$$\mathfrak{MP} : B\text{-schemes} \to \text{Sets}$$

given by

$$\mathfrak{MP}(S/B) := \left\{ \begin{matrix} \text{pairs } (f : X \to S, \mathcal{L}), \text{ where} \\ f \text{ is a flat and proper morphism,} \\ \mathcal{L} \text{ is an } f\text{-ample line bundle on } X \end{matrix} \right\} \Big/ \sim$$

where two families $(f_1 : X_1 \to S, \mathcal{L}_1)$ and $(f_2 : X_2 \to S, \mathcal{L}_2)$ are *equivalent* $[(f_1 : X_1 \to S, \mathcal{L}_1) \sim (f_2 : X_2 \to S, \mathcal{L}_2)]$ iff there is an isomorphism $h : X_1 \to X_2$ such that $f_1 = f_2 \circ h$ and there is a line bundle $\mathcal{M}$ on S such that $\mathcal{L}_1 \cong h^*\mathcal{L}_2 \otimes f_1^*\mathcal{M}$. For any morphism of B-schemes, $\alpha : T \to S$, we have

$$\mathfrak{MP}(\alpha)(X \to S, \mathcal{L}) = (X_T \to T, \alpha_X^*\mathcal{L}).$$

In this article we will restrict to the case of $B = k$ an algebraically closed field. $\mathcal{S}ch_k$ will denote the category of k-schemes.

Any subfunctor of this moduli functor will be called *a* moduli functor of polarized proper schemes.

Definition 2.2. A subfunctor $\mathfrak{F} \subset \mathfrak{MP}$ is called *locally closed* iff the following condition is satisfied:

For every $(f : X \to S, \mathcal{L}) \in \mathfrak{MP}(S)$ there is a locally closed subscheme $j : S^u \hookrightarrow S$ such that if $\alpha : T \to S$ is any morphism then

$$(f_T : X_T \to T, \alpha_X^*\mathcal{L}) \in \mathfrak{F}(T) \quad \text{iff there is a factorization} \quad T \xrightarrow{\alpha} S^u \xrightarrow{j} S.$$

We say that $\mathfrak{F} \subset \mathfrak{MP}$ is *open* iff $S^u \subset S$ is open for every S.

For the definition of *bounded, separated,* and *complete* moduli functors the reader is referred to [14], 1.15.

Definition 2.3. Fix a polynomial $h \in \mathbb{Q}[t]$ be such that $h(\mathbb{Z}) \subseteq \mathbb{Z}$. The *moduli functor of polarized schemes with Hilbert polynomial h* is the subfunctor $\mathfrak{MP}_h$ of $\mathfrak{MP}$ given by:

$$\mathfrak{MP}_h(S) = \{(f : X \to S, \mathcal{L}) \in \mathfrak{MP}(S) \mid \chi(\mathcal{L}^{\nu}_{X_s}) = h(\nu) \text{ for all } \nu \in \mathbb{Z} \text{ and } s \in S\}.$$

This is an open and closed subfunctor.

Definition 2.4. A subfunctor $\mathfrak{M}^{[N]} \subset \mathfrak{MP}$ is called a *moduli functor of canonically polarized* $\mathbb{Q}$*-Gorenstein schemes of index* N if, for each $(f : X \to S, \mathcal{L}) \in \mathfrak{M}^{[N]}(S)$,

(2.4.1) X_s is connected, Cohen-Macaulay, and Gorenstein outside a closed subscheme of codimension at least two for each $s \in S$;
(2.4.2) f is equivalent to one of the form $(f : X \to S, \omega^{[N]}_{X/S})$.

Remark 2.5. Assumption 2.4.1 implies the fibers are equidimensional projective schemes. One can show that the relative Cohen-Macaulay condition is open (see [5] EGAIV$_3$, 12.2.1), as is the locus where the relative dualizing sheaf is locally free (the relative Gorenstein locus). Since the complement to the relative Gorenstein locus intersects the fibers in codimension ≤ 2 over an open subset of the base, it follows that Assumption 2.4.1 is open. Note that the singularity assumptions may also be expressed as a condition on the *morphism* f: its relative dualizing complex is supported in degree $-\dim(X/S)$ and the resulting dualizing sheaf is locally free over an open subset whose complement meets each fiber in codimension two. We refer the reader to [3] for a recent account of relative duality.

We emphasize that for families of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index N, $\omega^{[N]}_{X/S}$ is invertible *by definition.* This is a condition on the morphism, not just a condition on the fibers. Indeed, $\omega^{[N]}_{X/S}$ may fail to be invertible even when $\omega^{[N]}_{X_s}$ is invertible for each $s \in S$ (see [13]). Also, it is not entirely obvious that Assumption 2.4.2 actually yields a subfunctor, i.e., that families of canonically polarized schemes pull back to families of canonically polarized schemes. This is proved in the following lemma:

Lemma 2.6. *Given a family of canonically polarized* $\mathbb{Q}$*-Gorenstein schemes of index* N*,* $f : X \to S$*, and a morphism* $\alpha : T \to S$*, we have*

$$\alpha^*_X \omega^{[N]}_{X/S} \simeq \omega^{[N]}_{X_T/T}.$$

Proof. Let $U \subset X$ be the relative Gorenstein locus of f, i.e., the largest open subset U of X such that U_s is Gorenstein for all $s \in S$ or equivalently the largest open subset U of X such that $\omega_{X/S}|_U$ is a line bundle. Then

$\omega_{X/S}^{[N]}|_U \simeq \omega_{U/S}^N$ and hence

$$\alpha_X^*\omega_{X/S}^{[N]}|_{\alpha_X^{-1}U} \simeq \alpha_X^*\omega_{U/S}^N \simeq \omega_{\alpha_X^{-1}U/T}^N \simeq \omega_{X_T/T}^{[N]}|_{\alpha_X^{-1}U}.$$

Now $\operatorname{codim}(U_s, X_s) \geq 2$ for all $s \in S$, so $\operatorname{codim}((\alpha_X^{-1}U)_t, (X_T)_t) \geq 2$ for all $t \in T$. Finally $\alpha_X^*\omega_{X/S}^{[N]}$ and $\omega_{X_T/T}^{[N]}$ are reflexive, so since they are isomorphic on $\alpha_X^{-1}U$, they are isomorphic on X_T (cf. 3.6.2). $\square$

If $\mathfrak{M}^{[N]}$ is a functor of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index N then we can consider $\mathfrak{M}_h^{[N]}$ as well. An argument using 3.6 (and very similar to Lemma 2.6) implies

$$\begin{aligned} \mathfrak{M}_h^{[N]}(S) &= \{(f: X \to S) \in \mathfrak{M}^{[N]}(S) \mid \chi(\omega_{X_s}^{[\nu\cdot N]}) = h(\nu) \\ &\qquad \text{for all } \nu \in \mathbb{Z} \text{ and } s \in S.\}. \end{aligned}$$

Remark 2.7. Note that we speak of "a" moduli functor and not "the" moduli functor. The reason is that in order to obtain a relatively nice moduli space one has to restrict to a smaller class than all the canonically polarized $\mathbb{Q}$-Gorenstein schemes of index N. On the other hand one could consider "the" moduli space of smooth varieties, but in that case one would not obtain a compact moduli space. The "right" class of schemes will be somewhere between these two and part of the difficulty is to identify that class.

Assumptions 2.8. Assume the following:

(2.8.1) $\mathfrak{M}_h^{[N]}$ is locally closed;
(2.8.2) $\mathfrak{M}_h^{[N]}$ is bounded;
(2.8.3) $\mathfrak{M}_h^{[N]}$ is separated;
(2.8.4) $\mathfrak{M}_h^{[N]}$ is complete;
(2.8.5) for all smooth projective curves S, and for all $(f: X \to S) \in \mathfrak{M}_h^{[N]}(S)$, the sheaf $f_*\omega_{X/S}^{[\nu\cdot N]}$ is semi-positive for all ν sufficiently large and divisible.

Theorem 2.9. [10], *4.2.1;* [13], *5.7;* [11], *5.6;* [14], *9.23, 9.30 Assume that k has characteristic zero. We retain the notation introduced above and assume that $\mathfrak{M}_h^{[N]}$ satifies the conditions of 2.8. Let $\nu > 0$ be a fixed integer such that $\omega_X^{[\nu\cdot N]}$ is very ample and without higher cohomology for all $X \in \mathfrak{M}_h^{[N]}(k)$.*

Then there exists a coarse algebraic moduli space $\mathcal{M}_h^{[N]}$ for $\mathfrak{M}_h^{[N]}$ which is a projective scheme and for all $\mu \gg 0$ there exist a $p > 0$ and an ample invertible sheaf $\lambda_{\mu\cdot\nu\cdot N}^{(p)}$ on $\mathcal{M}_h^{[N]}$, such that for all $(f: X \to S) \in \mathfrak{M}_h^{[N]}(S)$ and for the induced morphism $\phi: S \to \mathcal{M}_h^{[N]}$ one has $\phi^*\lambda_{\mu\cdot\nu\cdot N}^{(p)} = \left(\det f_*\omega_{X/S}^{[\mu\cdot\nu\cdot N]}\right)^p$.

Viehweg's Functor. Property $\mathbf{V}^{[N]}$ Consider a family of polarized varieties, $f : X \to S$, satisfying Assumption 2.4.1. It will be said that f satisfies property $\mathbf{V}^{[N]}$ if $\omega_{X/S}^{[N]}$ is invertible.

Note that families of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index N automatically satisfy property $\mathbf{V}^{[N]}$ (by Assumption 2.4.2).

Definition 2.10. Let $\mathfrak{V}_h^{[N],d}$ be the moduli functor of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index N and Hilbert polynomial h satisfying the following:

(2.10.1) for each $s \in S$, X_s is a reduced scheme of dimension d and has semi log canonical singularities.

We emphasize that we are retaining Assumptions 2.4.1 and 2.4.2. Note that each fiber X_s automatically has index N.

Let $N' = mN$ be a positive multiple of N and $h'(t) = h(mt)$. There is a natural transformation,

$$\mathfrak{V}_h^{[N],d} \to \mathfrak{V}_{h'}^{[N'],d},$$

induced by taking the mth power of the canonical polarization.

Kollár's Functor. Property K Consider a family of polarized varieties, $f : X \to S$, satisfying Assumption 2.4.1. It will be said that f satisfies property **K** if

$$\alpha_X^* \omega_{X/S}^{[j]} \simeq \omega_{X_T/T}^{[j]}$$

for any morphism, $\alpha : T \to S$, and each $j \in \mathbb{Z}$.

For canonically polarized $\mathbb{Q}$-Gorenstein schemes of index N, it suffices to verify this for $j = 1, \ldots, N-1$. Indeed, 3.6 yields

$$\omega_{X/S}^{[j+\nu N]} = \omega_{X/S}^{[j]} \otimes (\omega_{X/S}^{[N]})^{\nu}.$$

Definition 2.11. Let $\mathfrak{K}_h^{[N],d}$ be the moduli functor of canonically polarized $\mathbb{Q}$- Gorenstein schemes of index N and Hilbert polynomial h satisfying the following:

(2.11.1) for each $s \in S$, X_s is a reduced scheme of dimension d and has semi log canonical singularities;

(2.11.2) each family $(f : X \to S, \omega_{X/S}^{[N]}) \in \mathfrak{K}_h^{[N],d}(S)$ satisfies property **K**.

If a family satisfies property **K**, then the family is in $\mathfrak{K}_h^{[N],d}$ if the indices of the fibers all divide N. Let $N' = mN$ be a positive multiple of N and $h'(t) = h(mt)$. Then the natural transformation,

$$\mathfrak{K}_h^{[N],d} \to \mathfrak{K}_{h'}^{[N'],d},$$

induced by taking the mth power of the canonical polarization, is an open immersion.

These conditions are stronger than those of Viehweg's functor, so there is a natural transformation of moduli functors,

$$\mathfrak{K}_h^{[N],d} \to \mathfrak{V}_h^{[N],d},$$

inducing a bijection between $\mathfrak{K}_h^{[N],d}(k)$ and $\mathfrak{V}_h^{[N],d}(k)$.

Moduli of Surfaces: Smoothability and Boundedness.

Definition 2.12. Let $\mathfrak{V}_{h,\mathrm{sm}}^{[N],2}(k)$ denote the following subset of $\mathfrak{V}_h^{[N],2}(k)$:

$$\begin{aligned}\mathfrak{V}_{h,\mathrm{sm}}^{[N],2}(k) \quad = \quad & \{X \mid X \in \mathfrak{V}_h^{[N],2}(k), \text{ and } \exists (g: Y \to C) \in \mathfrak{V}_h^{[N],2}(C), \text{ such that} \\ & C \text{ is an irreducible curve, } X \simeq X_c \text{ for some } c \in C \text{, and} \\ & X_{\mathrm{gen}} \text{ is a normal surface with at most rational double points.}\}\end{aligned}$$

We define $\mathfrak{K}_{h,\mathrm{sm}}^{[N],2}(k)$ analogously.

Once we construct the moduli schemes $\mathcal{V}_h^{[N],2}$ and $\mathcal{K}_h^{[N],2}$, we may realize $\mathfrak{V}_{h,\mathrm{sm}}^{[N],2}(k)$ and $\mathfrak{K}_{h,\mathrm{sm}}^{[N],2}(k)$ as the closed points of certain subvarieties. The points satisfying the smoothability condition form a union of irreducible components, and this union forms a closed subvariety. However, this subvariety does not necessarily admit a natural scheme structure.

Remark 2.13. Assume that k has characteristic zero for the remainder of this subsection. [1], 5.11 implies that there exists an $N \in \mathbb{N}$ such that

$$\mathfrak{V}_h(k) := \bigcup_{m \in \mathbb{N}} \mathfrak{V}_{h(mt)}^{[m],2}(k) = \mathfrak{V}_{h(Nt)}^{[N],2}(k)$$

and

$$\mathfrak{V}_{h,\mathrm{sm}}(k) := \bigcup_{m \in \mathbb{N}} \mathfrak{V}_{h(mt),\mathrm{sm}}^{[m],2}(k) = \mathfrak{V}_{h(Nt),\mathrm{sm}}^{[N],2}(k).$$

In order to construct moduli spaces for $\mathfrak{V}_h^{[N],d}$ and $\mathfrak{K}_h^{[N],d}$, one has to verify the assumptions of 2.8. All the properties listed in 2.8, except 2.8.1, are the same for both $\mathfrak{V}_h^{[N],d}$ and $\mathfrak{K}_h^{[N],d}$.

- [10], 2.1.2 implies 2.8.2.
- [13], 5.1 implies 2.8.3 and 2.8.4, at least for the irreducible components satisying the smoothability condition. For the general case, one has to construct a unique stable limit for a one-parameter family of *nonnormal* stable surfaces. Consider its normalization as a family of stable log surfaces with boundary equal to the conductor. Apply the log minimal model program and the results of [7] to obtain a unique limiting stable log surface. We glue back together along the conductor to recover the stable limit of our original family.
- [11], 4.12 implies 2.8.5.

That leaves us to verify 2.8.1, and in the rest of the article we will concentrate on this property.

Proof of Local Closedness. To prove that $\mathfrak{V}_h^{[N],d}$ is locally closed, one would naturally list the properties of the functor and prove one by one that all of them are locally closed. However, this requires special attention. A potentially tricky part is that the order of this procedure matters. For instance, the requirement that $\omega_{X/S}^{[N]}$ be invertible should not be considered until only open properties are left, because it may very well happen that $\omega_{X/S}^{[N]}$ is not invertible along an admissible fiber X_s, but $\omega_{X_T/T}^{[N]}$ becomes invertible after restricting to some locally closed $T \subset S$ containing s. In particular the locus where $\omega_{X/S}^{[N]}$ is invertible does not coincide with the locus where $\omega_{X_s}^{[N]}$ is invertible. The key problem is: *taking reflexive powers does not generally commute with base extension.*

The next theorem is the main result of this article. Here we reduce local closedness to a rather technical statement which will be proved in the next section.

Theorem 2.14. *The moduli functor of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index N is locally closed.*

Proof. In proving local closedness, we address the conditions imposed on the *fibers* of $f : X \to S$ separately from the conditions imposed on the *morphism* f itself. We have already observed in 2.5 that condition 2.4.1 is open. Now we turn to condition 2.4.2, i.e., $\omega_{X/S}^{[N]}$ is locally free. Suppose we are given an arbitrary family of polarized varieties $(f : X \to S, \mathcal{L})$ with fibers satisfying 2.4.1. We apply 3.11 with $\mathcal{F} = \omega_{X/S}^{\otimes N}$. This sheaf may be terribly singular, perhaps even with torsion along certain fibers. However, $\omega_{X/S}^{\otimes N}$ does have one salient property: it commutes with arbitrary base extensions $\alpha : T \to S$, i.e.,

$$\alpha_X^* \omega_{X/S}^{\otimes N} = \omega_{X_T/T}^{\otimes N}.$$

By 3.11 there exists a locally closed subscheme $S^u \subset S$ with the following universal property. Given a morphism $\alpha : T \to S$, there exists an invertible sheaf $\mathcal{N}$ on T and an isomorphism

$$(\omega_{X_T/T}^{\otimes N})^{**} \xrightarrow{\simeq} \mathcal{L}_T \otimes f_T^* \mathcal{N}$$

if and only if α factors through S^u. By definition we have

$$\omega_{X_T/T}^{[N]} = (\omega_{X_T/T}^{\otimes N})^{**},$$

so the proof that condition 2.4.2 is closed is complete. $\square$

Theorem 2.15. *If k is a field of characteristic zero then $\mathfrak{V}_h^{[N],2}$ is a locally closed moduli functor. In particular, $\mathfrak{V}_{h,sm}^{[N],2}(k)$ is locally closed.*

Proof. It remains to verify that condition 2.10.1 is locally closed once conditions 2.4.1 and 2.4.2 are imposed. In particular, we may assume that we have families of canonically polarized $\mathbb{Q}$-Gorenstein varieties of index N.

The condition that the geometric fibers X_s are reduced is open by [5] EGAIV$_3$, 12.2.1. The locus where the fibers have semi log canonical singularities is open by [8], 2.6 (see also [13],§5). □

Remark 2.16. (characteristic zero) If one assumes the existence of minimal models in dimension $d+1$, the results of [8] imply that having semi log canonical singularities is an open condition for families of canonically polarized $\mathbb{Q}$-Gorenstein varieties of index N. It follows that $\mathfrak{V}_h^{[N],d}$ is locally closed.

3. Local closedness of reflexive pull-backs

We first recall the following result from [5] EGAIV, §6.3:

Proposition 3.1. *Let A and B be noetherian local rings, k the residue field of A, $\phi : A \to B$ a local homomorphism, M an A-module of finite type, and N a B-module of finite type. If N is a flat and nonzero as an A-module then*

$$\operatorname{depth}_B(M \otimes_A N) = \operatorname{depth}_A(M) + \operatorname{depth}_{B\otimes_A k}(N \otimes_A k).$$

We shall assume that all schemes are noetherian.

Let $f : X \to S$ be a flat morphism of schemes and $\mathcal{E}$ a coherent $\mathcal{O}_X$-module flat over S. We shall say $\mathcal{E}$ is S_r relative to f if the following holds: for each $x \in X, s = f(x), F = X_s$ we have

$$\operatorname{depth}_{\mathcal{O}_{F,x}}(\mathcal{E}|_F) \geq \min(r, \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s}).$$

In other words, the restriction of $\mathcal{E}$ to each fiber is S_r. By 3.1, this is equivalent to

$$\operatorname{depth}_{\mathcal{O}_{X,x}}(\mathcal{E}) \geq \operatorname{depth}_{\mathcal{O}_{S,s}}(\mathcal{O}_{S,s}) + \min(r, \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s}).$$

Two special cases deserve further attention. If $\mathcal{E} = \mathcal{O}_X$ then our definition coincides with the ordinary definition of an S_r morphism (see [5] EGAIV, §6). If $S = \operatorname{Spec}(K)$, where K is a field, then we recover a notion of an S_r *sheaf.* For example, a sheaf is S_1 provided it has no imbedded points. We can translate the definition of S_r-sheaves using the cohomological interpretation of depth (see [6], §III.3 Ex. 4).

Proposition 3.2. *Let $f : X \to S$ be a flat morphism of schemes and $\mathcal{E}$ a coherent $\mathcal{O}_X$-module flat over S. Then $\mathcal{E}$ is S_r relative to f if and only if, for* *each $x \in X, s = f(x)$, we have*

$$\min\{i : H^i_x(\mathcal{E}) \neq 0\} \geq \operatorname{depth}_{\mathcal{O}_{S,s}}(\mathcal{O}_{S,s}) + \min(r, \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s}).$$

Here $H^i_x(\mathcal{E})$ denotes cohomology on $\operatorname{Spec}(\mathcal{O}_{X,x})$ with support at the closed point and coefficients in $\mathcal{E}_x$. If $Z \subset X$ is closed, we use $\mathcal{H}^i_Z$ to denote the local cohomology *sheaf* associated to cohomology on X with support along Z.

Proposition 3.3. *Let $f : X \to S$ be a flat morphism of schemes and $\mathcal{E}$ a coherent $\mathcal{O}_X$-module flat over S and S_r relative to f. Let $Z \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I}_Z$. Assume that* $\operatorname{codim}(Z_s, X_s) \geq r$ *for each $s \in S$. Then we have* $\operatorname{depth}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{E}) \geq r$*, or equivalently,*

$$\mathcal{H}^k_Z(\mathcal{E}) = 0 \textit{ for } k = 0, \dots, r-1$$

Proof. The cohomological interpretation of depth gives the equivalence of the two conclusions.

For each point $x \in Z$ we have

$$H^0_x = H^0_x \circ \mathcal{H}^0_Z$$

which induces a spectral sequence

$$E_2^{p,q} := H^p_x \circ \mathcal{H}^q_Z \Longrightarrow H^{p+q}_x.$$

The proof is by induction on k, starting with $k = 0$. Assume that $\mathcal{H}^0_Z(\mathcal{E}) \neq 0$ and its support contains a point $x \in Z$. We have $H^0_x(\mathcal{H}^0_Z(\mathcal{E})) \neq 0$ and thus $H^0_x(\mathcal{E}) \neq 0$. Writing $s = f(x)$, we obtain a contradiction of 3.2. Now assume that $\mathcal{H}^i_Z(\mathcal{E}) = 0$ for $i = 0, \dots, k-1$ where $k < r$, but that $\mathcal{H}^k_Z(\mathcal{E}) \neq 0$ and its support contains $x \in Z$. It follows that $H^k_x(\mathcal{E}) = H^0_x(\mathcal{H}^k_Z(\mathcal{E})) \neq 0$, which again contradicts 3.2. □

Corollary 3.4. *Let $f : X \to S$ be a flat S_r morphism, and $Z \subset X$ a subscheme such that* $\operatorname{codim}(Z_s, X_s) \geq r$ *for each $s \in S$. Then* $\operatorname{grade}(\mathcal{I}_Z) \geq r$.

Proposition 3.5. *Let $f : X \to S$ be a flat morphism of schemes, $\mathcal{E}$ a coherent sheaf flat over S and S_2 relative to f, and $j : U \hookrightarrow X$ an open subscheme with complement Z. Assume that* $\operatorname{codim}(Z_s, X_s) \geq 2$ *for each $s \in S$. Then the natural map*

$$\mathcal{E} \to j_*(\mathcal{E}|_U)$$

is an isomorphism.

Proof. The long exact sequence

$$0 \to \mathcal{H}^0_Z(\mathcal{E}) \to \mathcal{E} \to j_*(\mathcal{E}|_U) \to \mathcal{H}^1_Z(\mathcal{E}) \to 0 \tag{3.5.1}$$

yields the isomorphism. □

Proposition 3.6. *Let $f : X \to S$ be a flat S_2 morphism, $\mathcal{F}$ a reflexive coherent $\mathcal{O}_X$-module. Let $Z \subset X$ be a closed subscheme so that* $\operatorname{codim}(Z_s, X_s) \geq 2$ *for each $s \in S$, and let U be the complement of Z.*

(3.6.1) *Then* $\mathcal{H}^k_Z(\mathcal{F}) = 0$ *for* $k = 0, 1$ *and the natural map* $\mathcal{F} \to j_*(\mathcal{F}|_U)$ *is an isomorphism.*

(3.6.2) *Let* $\mathcal{F}'$ *be another coherent* $\mathcal{O}_X$*-module which is either* S_2 *relative to* f *or reflexive. If* $\mathcal{F}|_U \simeq \mathcal{F}'|_U$ *then* $\mathcal{F} \simeq \mathcal{F}'$.

Proof. Consider a presentation of $\mathcal{F}^*$ by locally free sheaves

$$\mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{F}^* \to 0.$$

On dualizing we obtain

$$0 \to \mathcal{F} \to \mathcal{E}_1^* \to \mathcal{E}_2^*.$$

Since the $\mathcal{E}_i^*$ are locally free and f is S_2, 3.3 yields

$$\mathcal{H}^k_Z(\mathcal{E}_i) = 0 \text{ for } k = 0, 1.$$

Taking the associated long exact sequences, we obtain the desired vanishing for $\mathcal{F}$. The long exact sequence in local cohomology (cf.3.5.1) yields the first isomorphism. The isomorphism between $\mathcal{F}$ and $\mathcal{F}'$ is obtained by pushing forward. □

We obtain a criterion for when push-forwards of reflexive sheaves are reflexive:

Corollary 3.7. *Let* $f : X \to S$ *be a flat* S_2 *morphism and* $j : U \hookrightarrow X$ *an open subscheme with complement* Z. *Assume that* $\operatorname{codim}(Z_s, X_s) \geq 2$ *for each* $s \in S$. *If* $\mathcal{G}$ *is a reflexive coherent sheaf on* U *then* $j_*\mathcal{G}$ *is also reflexive and coherent.*

Proof. Choose a coherent subsheaf $\mathcal{E} \subset j_*\mathcal{G}$ so the induced map $\mathcal{E}|_U \to \mathcal{G}$ is an isomorphism [6], §II.5 Ex.15. The reflexive hull $\mathcal{E}^{**}$ is also coherent and we have $\mathcal{E}^{**}|_U \simeq \mathcal{G}$. An application of 3.6.2 implies

$$\mathcal{E}^{**} \to j_*(\mathcal{E}^{**}|_U) \simeq j_*\mathcal{G}$$

is an isomorphism. □

We also obtain the following (cf. [2], 1.4.1):

Corollary 3.8. *Let* $f : X \to S$ *be a flat* S_2 *morphism,* $\mathcal{E}$ *a coherent sheaf flat over* S *and* S_2 *relative to* f, *and* $j : U \hookrightarrow X$ *an open subscheme with complement* Z. *Assume that* $\operatorname{codim}(Z_s, X_s) \geq 2$ *for each* $s \in S$ *and* $\mathcal{E}|_U$ *is reflexive. Then* $\mathcal{E}$ *is reflexive, and furthermore, for each* $\alpha : T \to S$ *the pull-back* $\mathcal{E}_T$ *is reflexive.*

Proof. We apply 3.6.2 to show that the natural map $\mathcal{E} \to \mathcal{E}^{**}$ is an isomorphism. Our hypotheses are preserved under base extension, so $\mathcal{E}_T$ is reflexive for each $\alpha : T \to S$. □

Suppose that $f : X \to S$ is a flat projective Cohen-Macaulay morphism of relative dimension d. Then the relative dualizing sheaf $\omega_{X/S}$ exists, commutes with base extension, and is S_d relative to f [9], §9,21; [3], 3.6.1; [4], 21.8. In light of our previous results, it is natural to compare the relative dualizing sheaf with the reflexive hull of a coherent sheaf.

Theorem 3.9. *Let $f : X \to S$ be a flat projective Cohen-Macaulay morphism of relative dimension d with geometrically connected fibers. Let $\mathcal{G}$ be a coherent sheaf on X, and $U \hookrightarrow X$ an open subset with complement Z so that $\operatorname{codim}(Z_s, X_s) \geq 2$ for each $s \in S$. Assume that $\omega_{X/S}$ and $\mathcal{G}$ are locally free on U. Then there exists a locally closed subset $S^u \subset S$ with the following property. Given a morphism $\alpha : T \to S$, there exists an invertible sheaf $\mathcal{N}$ on T and an isomorphism*

$$(\mathcal{G}_T)^{**} \xrightarrow{\simeq} \omega_{X_T/T} \otimes f_T^*\mathcal{N} \tag{3.9.1}$$

if and only if α factors through S^u.

Proof. We produce a subset $S^u \subset S$ over which the isomorphism 3.9.1 exists. Let $W' \subset S$ denote the subscheme supporting $\mathbb{R}^d f_*\mathcal{G}$. Note that W' has a naturally defined scheme structure. Indeed, let $\mathcal{P}^\bullet \to \mathcal{G}$ be a resolution of $\mathcal{G}$ by locally free $\mathcal{O}_X$-modules. The proof of the cohomology and base change theorem [5] EGAIII, §7.7 produces a complex of locally free $\mathcal{O}_S$-modules,

$$0 \to \mathcal{E}_0 \to \mathcal{E}_1 \to \ldots \to \mathcal{E}_{d-1} \xrightarrow{\psi} \mathcal{E}_d \to 0$$

computing the direct image sheaves $\mathbb{R}^i(f_T)_*\mathcal{P}_T^\bullet$ for any base extension $T \to S$. Since the maximal fibre dimension of f is d, $R^j(f_T)_* = 0$ for all $j > d$. Furthermore, $\mathcal{G}$ is the highest non-zero cohomology sheaf of $\mathcal{P}^\bullet$, so

$$\mathbb{R}^d f_*\mathcal{P}^\bullet \simeq \mathbb{R}^d f_*\mathcal{G}.$$

In particular, $\mathbb{R}^d f_*\mathcal{G}$ is the cokernel of ψ and we define W' using the $\operatorname{rank}(\mathcal{E}_d)$-minors of ψ. It also follows that the formation of $\mathbb{R}^d f_*\mathcal{G}$ commutes with base extension, i.e.,

$$(\mathbb{R}^d f_*\mathcal{G})_s \to H^d(X_s, \mathcal{G}_s) \tag{3.9.2}$$

is an isomorphism for each $s \in S$.

Let $W \subset W'$ be the locally closed subset obtained by removing points where $\operatorname{rank}(\psi) < \operatorname{rank}(\mathcal{E}_d) - 1$ and $f_W : X_W \to W$ the corresponding morphism. Hence $\mathcal{M} := \mathbb{R}^d(f_W)_*\mathcal{G}_W = (\mathbb{R}^d f_*\mathcal{G})_W$ is locally free of rank one. The relative duality theorem of Kleiman [9], §10,21 gives an isomorphism of $\mathcal{O}_W$-modules

$$\mathcal{H}om_{f_W}(\mathcal{G}_W, \omega_{X_W/W} \otimes f_W^*\mathcal{M}) \xrightarrow{\simeq} \mathcal{H}om_W(\mathcal{M}, \mathcal{M}) = \mathcal{O}_W. \tag{3.9.3}$$

The identity $1 \in \mathcal{O}_W$ gives a natural homomorphism $\phi : \mathcal{G}_W \to \omega_{X_W/W} \otimes f_W^*\mathcal{M}$ which factors

$$\begin{array}{ccc} \mathcal{G}_W & \xrightarrow{\phi} & \omega_{X_W/W} \otimes f_W^*\mathcal{M} \\ \downarrow & & \| \\ (\mathcal{G}_W)^{**} & \xrightarrow{\phi^{**}} & \omega_{X_W/W} \otimes f_W^*\mathcal{M} \end{array}$$

because $\omega_{X_W/W}$ is reflexive.

Let $S^u \subset W$ be the open subset over which $\phi|_U$ is an isomorphism. The map ϕ^{**} is an isomorphism over S^u by 3.6. For any S^u-scheme T, the map

$$(\phi^{**})_T : ((\mathcal{G}_W)^{**})_T \longrightarrow (\omega_{X_W/W})_T \otimes f_T^*\mathcal{M}$$

induced by base extension is also an isomorphism. Since $(\omega_{X_W/W})_T = \omega_{X_T/T}$ is flat and reflexive, the same holds for $((\mathcal{G}_W)^{**})_T$. Hence 3.6 guarantees that the natural map $((\mathcal{G}_W)^{**})_T \to (\mathcal{G}_T)^{**}$ is an isomorphism.

It remains to show that S^u satisfies the universal property. Let T be an S-scheme, $\mathcal{N}$ an invertible sheaf on T, and

$$\rho^{**} : (\mathcal{G}_T)^{**} \xrightarrow{\simeq} \omega_{X_T/T} \otimes f_T^*\mathcal{N} \tag{3.9.4}$$

an isomorphism. For each $t \in T$, the natural map $\mathcal{G}_t \to ((\mathcal{G}_T)^{**})_t$ is an isomorphism over U_t, a subset with codimension ≥ 2 complement. We therefore obtain an isomorphism of cohomology groups

$$H^d(X_t, \mathcal{G}_t) \xrightarrow{\simeq} H^d(X_t, ((\mathcal{G}_T)^{**})_t)$$

and the base-change isomorphism 3.9.2 yields

$$\mathbb{R}^d(f_T)_*\mathcal{G}_T \xrightarrow{\simeq} \mathbb{R}^d(f_T)_*[(\mathcal{G}_T)^{**}].$$

The composed morphism

$$\rho : \mathcal{G}_T \to (\mathcal{G}_T)^{**} \to \omega_{X_T/T} \otimes f_T^*\mathcal{N}$$

induces

$$\mathbb{R}^d(f_T)_*\mathcal{G}_T \xrightarrow{\simeq} \mathbb{R}^d(f_T)_*[\omega_{X_T/T} \otimes f_T^*\mathcal{N}].$$

Since f_T is Cohen-Macaulay, relative duality yields an isomorphism

$$\mathbb{R}^d(f_T)_*\omega_{X_T/T} \simeq \mathcal{E}xt^d_{f_T}(\mathcal{O}_{X_T}, \omega_{X_T/T}) \xrightarrow{\simeq} \mathcal{H}om_T((f_T)_*\mathcal{O}_{X_T}, \mathcal{O}_T) \simeq \mathcal{O}_T,$$

where the last isomorphism follows from the fact that f_T has geometrically connected fibers. It follows that

$$\mathbb{R}^d(f_T)_*\mathcal{G}_T \simeq \mathbb{R}^d(f_T)_*[\omega_{X_T/T} \otimes f_T^*\mathcal{N}] \simeq \mathbb{R}^d(f_T)_*\omega_{X_T/T} \otimes \mathcal{N} \simeq \mathcal{N},$$

hence $T \to S$ factors as $T \to W \hookrightarrow S$ and $\mathcal{N} \simeq \mathcal{M}_T$. Applying duality again, we may regard ρ as an element of $\mathcal{H}om_T(\mathcal{N}, \mathcal{N}) \simeq (\mathcal{H}om_W(\mathcal{M}, \mathcal{M}))_T$ and compare ρ and ϕ_T over T. The identification in the isomorphism 3.9.3 is functorial, hence ϕ_T corresponds to $1 \in \mathcal{H}om_T(\mathcal{N}, \mathcal{N})$, ρ corresponds to some $r \in \mathcal{O}_T$, and $\rho = r\phi_T$.

To complete the proof it suffices to check that $r \in \mathcal{O}_T^*$. Consider the restriction of the isomorphism 3.9.4 to the open subset U_T where the sheaves are all locally free

$$\rho|_{U_T} = \rho^{**}|_{U_T} : (\mathcal{G}_T)|_{U_T} \xrightarrow{\simeq} (\omega_{X_T/T} \otimes f_T^*\mathcal{N})|_{U_T}.$$

If r were not invertible at some $t \in T$ then $\rho|_{U_T}$ would have nontrivial cokernel over t, a contradiction. □

Corollary 3.10. *Retain all the notation and hypotheses of 3.9. Assume in addition that $\mathcal{G}$ is S_2-relative to f. Then there exists a locally closed subset $S^u \subset S$ with the following property. Given a morphism $\alpha : T \to S$, there exists an invertible sheaf $\mathcal{N}$ on T and an isomorphism*

$$\mathcal{G}_T \xrightarrow{\simeq} \omega_{X_T/T} \otimes f_T^*\mathcal{N}$$

if and only if α factors through S^u.

Proof. This follows from 3.9 and 3.8. □

Theorem 3.11. *Let $f : X \to S$ be a flat projective Cohen-Macaulay morphism of relative dimension d with geometrically connected fibers. Let $\mathcal{L}$ be an invertible sheaf on X, $\mathcal{F}$ a coherent sheaf on X, and $U \hookrightarrow X$ an open subset with complement Z so that* $\operatorname{codim}(Z_s, X_s) \geq 2$ *for each $s \in S$. Assume that $\omega_{X/S}$ and $\mathcal{F}$ are locally free on U. Then there exists a locally closed subset $S^u \subset S$ with the following property. Given a morphism $\alpha : T \to S$, there exists an invertible sheaf $\mathcal{N}$ on T and an isomorphism*

$$(\mathcal{F}_T)^{**} \xrightarrow{\simeq} \mathcal{L}_T \otimes f_T^*\mathcal{N}$$

if and only if α factors through S^u.

Proof. Without loss of generality we may assume that $\mathcal{L}$ is trivial (replace $\mathcal{F}$ by $\mathcal{F} \otimes \mathcal{L}^{-1}$). Apply 3.9 with $\mathcal{G} = \mathcal{F} \otimes \omega_{X/S}$, so that $(\mathcal{G}_T)^{**} \simeq \omega_{X_T/T} \otimes f_T^*\mathcal{N}$ iff T factors through S^u. On the other hand, $(\mathcal{G}_T)^{**} \simeq \omega_{X_T/T} \otimes f_T^*\mathcal{N}$ if and only if $\mathcal{G}_T|_{U_T} \simeq (\omega_{X_T/T} \otimes f_T^*\mathcal{N})|_{U_T}$, which is the case exactly when $\mathcal{F}_T|_{U_T} \simeq f_T^*\mathcal{N}|_{U_T}$, or equivalently, when $(\mathcal{F}_T)^{**} \simeq f_T^*\mathcal{N}$ (by 3.6). □

Remark 3.12. It is natural to try to generalize this result for more general sheaves $\mathcal{L}$. The above argument is still valid provided $\mathcal{L}$ satisfies the following:

(3.12.1) $\mathcal{L}$ is S_2 relative to f;
(3.12.2) $\mathcal{L}|_U$ is invertible.

Proof. We apply 3.9, with $\mathcal{G} = \mathcal{F} \otimes \mathcal{L}^* \otimes \omega_{X/S}$. We obtain a locally closed subset $S^u \subset S$ such that $(\mathcal{G}_T)^{**} \simeq \omega_{X_T/T} \otimes f_T^*\mathcal{N}$ iff T factors through S^u. Again, this is the case if and only if

$$(\mathcal{F} \otimes \mathcal{L}^* \otimes \omega_{X_T/T})_T|_{U_T} \simeq f_T^*\mathcal{N}|_{U_T},$$

which by Assumption 3.12.2 is equivalent to

$$\mathcal{F}|_{U_T} \simeq \mathcal{L} \otimes f_T^* \mathcal{N}|_{U_T},$$

which in turn is equivalent to $(\mathcal{F}_T)^{**} \simeq (\mathcal{L}_T)^{**} \otimes f_T^* \mathcal{N}$. Applying 3.8 along with Assumptions 3.12.1 and 3.12.2 yields that $\mathcal{L}_T$ is reflexive for each T, so the last isomorphism exists iff $(\mathcal{F}_T)^{**} \simeq \mathcal{L}_T \otimes f_T^* \mathcal{N}$. $\square$

References

1. V. A. Alexeev, *Boundedness and K^2 for log surfaces*, Internat. J. Math. **5** (1994), no. 6, 779–810.
2. W. Bruno and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Univ. Press, 1993.
3. B. Conrad, *Grothendieck duality and base change*, Lecture Notes in Mathematics, 1750, Springer-Verlag, 2000.
4. D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, New York, 1995.
5. A. Grothendieck and J. Dieudonné, *Eléments de Géométrie Algébrique*, Inst. Hautes Études Sci. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32**, 1960–67.
6. R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York-Heidelberg, 1977.
7. B. Hassett , *Stable limits of log surfaces and Cohen-Macaulay singularities,* J. Algebra **242** (2001) no. 1, 225–235.
8. K. Karu, *Minimal models and boundedness of stable varieties*, J. Algebraic Geom. **9** (2000), no. 1, 93–109.
9. S. Kleiman, *Relative duality for quasicoherent sheaves*, Compositio Math. **41** (1980), no. 1, 39–60.
10. J. Kollár, *Toward moduli of singular varieties*, Compositio Math. **56** (1985), no.3, 369–398.
11. J. Kollár, *Projectivity of complete moduli*, J. Differential Geom. **32** (1990), no. 1, 235–268.
12. J. Kollár, *Push forward and base change for open immersions*, unpublished manuscript, 1994.
13. J. Kollár and N. Shepherd-Barron , *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338.
14. E. Viehweg, *Quasi-Projective Moduli of Polarized Manifolds*, Springer-Verlag, Berlin, 1995.

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