

Weak approximation for hypersurfaces of low degree

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ABSTRACT. We explore the arithmetic of rationally connected varieties over function fields of curves. The main technical issues revolve around the existence of free rational curves with prescribed geometric properties.

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Introduction

In this paper, we survey arithmetic questions over function fields of curves which have attracted considerable attention recently. The starting point is the theorem of Graber, Harris, and Starr [**GHS03**]: Every rationally-connected variety over the function field of a curve has a rational point. Once we have one rational point, it is natural to analyze the structure of the set of all rational points, e.g., density in the Zariski topology (or other topologies), weak approximation, etc.

The first result in this direction actually predates the work of Graber, Harris, and Starr. Let $\pi : \mathcal{X} \rightarrow B$ be a flat proper morphism to a smooth curve, with generic fiber smooth and rationally connected. Kollár, Miyaoka, and Mori proved that the existence of one section guarantees a Zariski-dense collection of sections. More precisely, fix points $b_1, \dots, b_m \in B$ of good reduction for π , i.e., with $\mathcal{X}_{b_i} = \pi^{-1}(b_i)$ smooth. Then for any points $x_i \in \mathcal{X}_{b_i}$ there exists a section $\sigma : B \rightarrow \mathcal{X}$ with

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$\sigma(b_i) = x_i$ for each i . In [HT06], we proved the existence of sections with prescribed jets at these points, i.e., weak approximation at places of good reduction.

Full weak approximation requires the analysis of *singular* fibers of π . Precisely, weak approximation holds if for any smooth model $\mathcal{X} \rightarrow B$ and points x_i in the smooth locus \mathcal{X}_b^{sm} , there exists a section $\sigma : B \rightarrow \mathcal{X}$ with $\sigma(b_i) = x_i$ for each i . Reaching smooth points of arbitrary fibers is much harder than reaching arbitrary points of smooth fibers.

Even the case of cubic surfaces is open at present. In [HT08], we analyzed fibrations $\pi : \mathcal{X} \rightarrow B$ with smooth total space such that the fibers satisfy strong rational-connectedness, i.e., for each $b \in B$, any two points in \mathcal{X}_b^{sm} can be joined by a *proper* rational curve contained in \mathcal{X}_b^{sm} . This suffices to prove weak approximation for generic cubic surfaces over function fields of curves, e.g., those with square-free discriminant.

We now summarize the principal new results of this paper. Theorem 1 establishes full weak approximation for smooth hypersurfaces of very low degree, e.g., for cubic fivefolds. Theorem 18 addresses fibrations $\pi : \mathcal{X} \rightarrow B$ with ordinary singularities and singular total space. Under suitable technical assumptions (cf. Hypothesis 14), we establish weak approximation. We give applications to cubic hypersurfaces in Theorem 25.

Our goal here is to highlight new techniques for ‘managing’ rational curves: Deformation theory, constructions of combs, specialization arguments, and producing curves with prescribed intersection properties. One overarching question is how to produce ‘good curves’ (e.g., free rational curves or sections of fibrations) in a given homology class. For instance, given an irreducible component $E \subset \mathcal{X}_b$ with multiplicity one, is there a section $\sigma : B \rightarrow \mathcal{X}$ with $\sigma(b) \in E$? The section given by the theorem of Graber-Harris-Starr might pass through a different component of \mathcal{X}_b . How can we induce sections to ‘jump’ from component to component?

Throughout, we work over an algebraically closed ground field K .

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1. Approximation for hypersurfaces of low degree

Assume K is of characteristic zero.

THEOREM 1. *There exists a function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ with the following property: Consider a smooth curve B with function field F and a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d over F . If $n \geq \phi(d)$ then X satisfies weak approximation over F .*

We give a recursive definition for $\phi(d)$. Let $\phi(1) = 1$ and define $\phi(d)$, $d > 1$ by the formula

$$\phi(d) = \binom{\phi(d-1) + d - 1}{\phi(d-1)}.$$

In particular, we have

d	1	2	3	4	5
$\phi(d)$	1	2	6	84	2331890

REMARK 2. The theorem is weaker than one might expect, in light of the strong results obtained over *number fields* using the circle method [Ski97], [Bir62]: Let F be a number field and $X \subset \mathbb{P}^n$ a smooth hypersurface over F of degree d . Assume $n \geq 2^d(d-1)$ and for each place ν of F we have $X(F_\nu) \neq \emptyset$. Then X satisfies weak approximation over F .

By way of comparison

$$\frac{d}{2^d(d-1)} \mid \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 4 & 16 & 48 & 128 \end{array},$$

which is much smaller than $\phi(d)$ for $d \gg 0$.

PROOF. We shall require some basic results on varieties of linear subspaces:

LEMMA 3 (cf. 23.3 of [PS92]). *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d over F . Consider the ℓ -dimensional projective subspaces contained in X*

$$\Pi_{\ell, X} = \{[\Lambda] \in \mathbb{G}(\ell, n) : \Lambda \subset X\},$$

realized as a subscheme of the Grassmannian. Suppose that its expected dimension

$$\text{expdim } \Pi_{\ell, X} := (n - \ell)(\ell + 1) - \binom{d + \ell}{d} > 0.$$

Then $\Pi_{\ell, X} \neq \emptyset$ and thus

$$\dim \Pi_{\ell, X} \geq \text{expdim } \Pi_{\ell, X}.$$

Consider the incidence correspondence

$$\mathcal{I} = \{(\Lambda, X) : \Lambda \subset X\} \subset \mathbb{G}(\ell, n) \times \mathcal{Hilb}$$

where $\mathcal{Hilb} \simeq \mathbb{P}^{\binom{n+d}{d}-1}$ is the Hilbert scheme of hypersurfaces of degree d in \mathbb{P}^n . Since \mathcal{I} is a projective bundle over the first factor, it is smooth and irreducible. By Bertini's Theorem, the generic fiber of the second projection is also smooth, i.e., $\Pi_{\ell, X}$ is smooth of the expected dimension for generic X . Moreover, the fibers of the projection onto \mathcal{Hilb} have the expected dimension over a subset with complement of codimension ≥ 2 .

LEMMA 4. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d over F , with $n \geq \phi(d)$. Then*

$$\Pi_{\ell, X}(F) \neq \emptyset$$

if $\ell < \phi(d-1)$.

We first consider the special case where $\Pi = \Pi_{\ell, X}$ is smooth of the expected dimension. If $S \rightarrow \mathbb{G}(\ell, n)$ is the universal subbundle of rank $\ell + 1$ then $\Pi_{\ell, X}$ is given by the vanishing of section of $\text{Sym}^d S^*$, which has rank $\binom{d+\ell}{d}$. The adjunction formula gives

$$K_\Pi = (K_{\mathbb{G}(\ell, n)} + c_1(\text{Sym}^d S^*))|_\Pi = \mathcal{O}_\Pi(-n - 1 + \binom{d+\ell}{d-1}),$$

which is negative provided $\ell < \phi(d-1)$ and $n \geq \phi(d)$. Thus Π is Fano, in the sense that its anticanonical divisor is ample. It is therefore rationally connected, and $\Pi(F) \neq \emptyset$ by [GHS03].

To complete the lemma, we require a well-known specialization result (see §2 of [GHMS05]):

SUBLEMMA 5. *Let $T = \text{Spec } K[[t]]$ with closed point 0 and geometric generic point τ . Let $C \rightarrow T$ be a flat proper morphism with geometric generic fiber C_τ a smooth connected curve. Let $\mathcal{X} \rightarrow C$ be a flat proper morphism so that $\mathcal{X} \times_C C_\tau \rightarrow C_\tau$ admits a section. Then for each normal irreducible component $C'_0 \subset C_0$, the induced $\mathcal{X} \times_C C'_0 \rightarrow C'_0$ admits a section.*

Here is the idea: The section degenerates to a stable map over C_0 of degree one, which yields a section over each normal component.

Let X be an arbitrary hypersurface over $F = K(B)$ with classifying morphism $\mu : B \rightarrow \mathcal{H}ilb$. (For simplicity, assume now that B is projective.) Let $(\rho_s)_{s \in \mathbb{G}_m} \subset \text{GL}_{\binom{n+d}{d}}$ be a one-parameter subgroup acting without fixed points on B , so that for generic s :

- $\rho_s(B)$ is contained in the locus in $\mathcal{H}ilb$ parametrizing hypersurfaces Y with $\Pi_{\ell, Y}$ of the expected dimension;
- $\rho_s(B)$ meets the locus over which $\Pi_{\ell, Y}$ is smooth.

Let T be the completion of \mathbb{G}_m at the identity and consider the restriction

$$W := \mathcal{I} \times_{\mathcal{H}ilb} (B \times T) \rightarrow B \times T.$$

This might not be flat, but there is a flattening blow-up $\beta : C \rightarrow B \times T$ so that the dominating component of $Z \subset W \times_{B \times T} C$ is flat over C . We may take C to be nonsingular and β to be an isomorphism over the generic point of T ; in particular, C_0 has a component $C'_0 \simeq B$. Combining Sublemma 5 with our result in the case of smooth Fano varieties, we obtain a section of $Z \times_C B \rightarrow B$. However, the natural morphism over B

$$Z \times_C B \rightarrow \mathcal{I} \times_{\mathcal{H}ilb} B$$

induces a section of $\mathcal{I} \times_{\mathcal{H}ilb} B \rightarrow B$. (Recall we are constructing the fiber product using the projection to $\mathcal{H}ilb$ and the classifying morphism μ .) Restricting to the generic point of B , we obtain a rational point in $X(F)$.

The remainder of the proof is an induction on d . We now use the fact that X is smooth. Let $\Lambda \subset X$ be the linear subspace produced in the last lemma. Projecting from Λ gives a morphism

$$\pi_\Lambda : \text{Bl}_\Lambda X \rightarrow \mathbb{P}^{n-\ell-1},$$

with fibers hypersurfaces of degree $d-1$ in $\mathbb{P}^{\ell+1}$; the generic such fiber is smooth. The inductive hypothesis and our assumption that $\ell < \phi(d-1)$ guarantee these fibers satisfy weak approximation. Proposition 1.2 of [CTG04] implies weak approximation holds for X . Its hypothesis demands smoothness of the fibers, but weak approximation is a birational property, so we might as well restrict to the dense open set where π_Λ is smooth. \square

REMARK 6. It should be possible to weaken the smoothness hypothesis: Varieties having log terminal singularities and ample anticanonical class are rationally connected [Zha06].

REMARK 7. There are similar inductive proofs of the *unirationality* of small degree hypersurfaces over algebraically-closed fields [PS92] [HMP98]. The growth of n with respect to d is also an iterated exponential function.

2. Ordinary singularities reviewed

We assume the ground field has characteristic different from two.

DEFINITION 8. A variety V has *ordinary singularities* if it has a finite number of isolated singularities, each étale-locally isomorphic to a cone over a smooth quadric hypersurface.

PROPOSITION 9. *Let T be the spectrum of a complete DVR over the ground field with closed point 0 and geometric generic point τ . Consider $\mathcal{V} \rightarrow T$ with V_τ smooth and V_0 having ordinary singularities. In a formal (or étale) neighborhood of each singular point $v \in V_0$, $\mathcal{V} \rightarrow T$ takes the form*

$$x_1^2 + \dots + x_n^2 = t^r,$$

where $\dim V_0 = n - 1$ and r is a positive integer.

PROOF. Over a field of characteristic $\neq 2$, each ordinary singularity $v \in V_0$ is formally-locally isomorphic to

$$0 \in \{(x_1, \dots, x_n) : x_1^2 + \dots + x_n^2 = 0\} \subset \mathbb{A}^n.$$

We use standard properties of the versal deformation space $\text{Ver}(V_0, v)$: It is smooth of dimension one and the corresponding deformation may be written explicitly

$$s = y_1^2 + \dots + y_n^2.$$

The local structure of the deformation $\mathcal{V} \rightarrow T$ near v is determined by the classifying morphism

$$T \rightarrow \text{Ver}(V_0, v),$$

or the induced homomorphism of K -algebras

$$\begin{aligned} K[[s]] &\rightarrow K[[t]] \\ s &\mapsto c_r t^r + c_{r+1} t^{r+1} + \dots \quad c_r \neq 0. \end{aligned}$$

The characteristic is different from two, so we can solve

$$f(t)^2 = c_r + c_{r+1}t + \dots$$

for $f(t) \in K[[t]]$. Substituting $y_i = f(t)x_i$ for $i = 1, \dots, n$ gives the desired normal form. \square

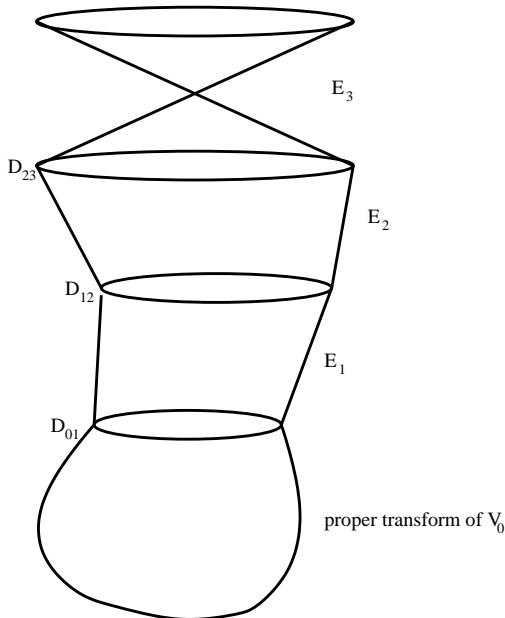
REMARK 10. For later use, we describe what happens when the classifying morphism

$$T \rightarrow \text{Ver}(V_0, v)$$

is constant. Here, the generic fiber V_τ remains singular, which is incompatible with the assumptions of Proposition 9. In this situation, $\mathcal{V} \rightarrow T$ admits an equisingular section $\sigma : T \rightarrow \mathcal{V}$ with $\sigma(0) = v$, i.e., a section along which the fibers all have ordinary double points. In local coordinates, \mathcal{V} can be written

$$x_1^2 + \dots + x_n^2 = 0$$

with the section σ equal to the origin. Blowing up along σ gives a simultaneous resolution of the fibers.

FIGURE 1. Degenerate fiber of the resolution when $r = 7$

Retain the notation of Proposition 9 and let (\mathcal{V}, v) denote the localization of \mathcal{V} at v . We describe a resolution of its singularities. If $r = 1$ then no resolution is necessary. When $r > 1$, consider the sequence of blow-ups

$$\beta : \mathcal{W}(\lfloor r/2 \rfloor) \rightarrow \mathcal{W}(\lfloor (r-2)/2 \rfloor) \rightarrow \dots \rightarrow \mathcal{W}(1) \rightarrow \mathcal{W}(0) := (\mathcal{V}, v),$$

where each $\mathcal{W}(j)$ has one singularity $w(j)$ locally isomorphic to

$$x_1^2 + \dots + x_n^2 = t^{r-2j}$$

and

$$\mathcal{W}(j+1) = \text{Bl}_{w(j)} \mathcal{W}(j).$$

The exceptional locus

$$\beta^{-1}(v) = E_1 \cup_{D_{12}} E_2 \cdots E_j \cup_{D_{j,j+1}} E_{j+1} \cdots E_{\lfloor r/2 \rfloor},$$

where each E_j is the proper transform of the exceptional divisor of $\mathcal{W}(j) \rightarrow \mathcal{W}(j-1)$ and

$$(2.1) \quad E_j \simeq \begin{cases} \mathbb{P}(\mathcal{O}_{Q_{n-1}}(-1) \oplus \mathcal{O}_{Q_{n-1}}) & \text{if } j < \lfloor r/2 \rfloor, \\ \text{Cone } Q_{n-1} & \text{if } j = (r-1)/2, \\ Q_n & \text{if } j = r/2, \end{cases}$$

where $Q_m \subset \mathbb{P}^m$ is a smooth quadric hypersurface of dimension $m-1$. The components E_j and E_{j+1} meet along $D_{j,j+1} \simeq Q_{n-1}$, embedded in E_j as the negative section and in E_{j+1} as a positive section (for $j+1 < \lfloor r/2 \rfloor$), or a generic hyperplane section (for $j+1 = \lfloor r/2 \rfloor$). The proper transform of V_0 meets E_1 along a variety $D_{01} \simeq Q_{n-1}$, where $D_{01} \subset \text{Bl}_v V_0$ is the exceptional divisor and $D_{01} \subset E_1$ is a generic hyperplane section.

3. Managing free curves

DEFINITION 11. A smooth separably rationally connected variety Y is *strongly rationally connected* if any of the following conditions hold:

- (1) for each point $y \in Y$, there exists a rational curve $f : \mathbb{P}^1 \rightarrow Y$ joining y and a generic point in Y ;
- (2) for each point $y \in Y$, there exists a *very* free rational curve containing y ;
- (3) for any finite collection of points $y_1, \dots, y_m \in Y$, there exists a very free rational curve containing the y_j as smooth points;
- (4) for any finite collection of jets

$$\text{Spec } K[\epsilon]/\langle \epsilon^{N+1} \rangle \subset Y, i = 1, \dots, m$$

supported at distinct points y_1, \dots, y_m , there exists a very free rational curve smooth at y_1, \dots, y_m and containing the prescribed jets.

For the equivalence of these formulations, see [HT08] (cf. [Kol96] 3.9.4). (In [HT08], the word ‘very’ was left out in the second condition; this omission does not affect the other results stated there.)

DEFINITION 12. A *comb with m reducible teeth* is a projective nodal curve C which is the union of $m + 1$ subcurves R, T_1, \dots, T_m such that

- R is smooth and irreducible;
- $T_\ell \cap T_{\ell'} = \emptyset$, for all $\ell \neq \ell'$;
- each T_ℓ meets R transversally in a single point; and
- each T_ℓ is a chain of \mathbb{P}^1 's.

Here R is called the *handle* and the T_ℓ the *teeth*.

We shall use the following technical result repeatedly:

PROPOSITION 13 (cf. Proposition 24 of [HT06]). *Let C be a nodal curve with dual graph a tree, Y a smooth algebraic space, $\iota : C \rightarrow Y$ an immersion with nodal image, and $\mathbf{p} = \{p_1, \dots, p_w\} \subset C$ a collection of smooth points. Suppose that for each component C_ℓ the normal bundle $\mathcal{N}_\iota(-\mathbf{p})|_{C_\ell}$ is globally generated and has no higher cohomology. Then ι deforms to an immersion of a smooth curve into Y containing $\iota(\mathbf{p})$.*

We will also use the following standard fact: If C is a treelike nodal curve and V is locally free over C , then V is globally generated (resp. has vanishing higher cohomology) on C iff $V|_{C_\ell}$ is globally generated (resp. has vanishing higher cohomology) over each irreducible component $C_\ell \subset C$. (See Proposition 21 of [HT06] for a proof.)

HYPOTHESIS 14 (Key Hypothesis). Let X be a projective variety with ordinary singularities, and $\rho : \tilde{X} \rightarrow X$ the minimal divisorial resolution, obtained by blowing up each singularity. Assume that:

- X^{sm} is strongly rationally connected;
- for each exceptional divisor $D \subset \tilde{X}$, there exists a rational curve $f : \mathbb{P}^1 \rightarrow \tilde{X}$ meeting D transversely in one point and avoiding the other exceptional divisors of ρ .

REMARK 15. There are Fano hypersurfaces with ordinary singularities that do not satisfy Hypothesis 14. We give an example at the end of Section 5.

PROPOSITION 16. *Suppose X satisfies Hypothesis 14 with minimal resolution $\rho : \tilde{X} \rightarrow X$ having exceptional divisors $\{D_i\}$. Then for each D_i and $p \in D_i$, there exists a very free curve in \tilde{X} meeting D_i transversely at p but disjoint from the $D_j, j \neq i$. Given nonnegative integers $\{m_i\}$, there exists a very free curve in \tilde{X} meeting D_i transversely at m_i points.*

PROOF. We claim there exists a very free curve meeting D_i transversely at one point. Let $R = f(\mathbb{P}^1)$ be the rational curve provided by Hypothesis 14. Construct a comb with handle R and teeth T_ℓ consisting of very free rational curves in X^{sm} , which exist since X^{sm} is strongly rationally connected. For a suitable choice of teeth, we can deform our comb to a very free rational curve $g : \mathbb{P}^1 \rightarrow \tilde{X}$ [Kol96] II.7.9-10. This still meets D_i transversally at one point.

For each i , choose m_i very free curves meeting D_i in distinct points, as described above. We use these as teeth for our comb. As for the handle, since X^{sm} is strongly rationally connected, there exists a very free curve $R \subset X^{sm}$ intersecting these curves transversely in distinct points. Adding additional teeth contained in X^{sm} , if necessary, we obtain a comb that deforms to a very free curve, meeting D_i transversely in m_i points. This proves the last assertion of the proposition.

It remains to show that there are very free curves meeting D_i at a prescribed point p . Let $g : \mathbb{P}^1 \rightarrow \tilde{X}$ be very free with image meeting D_i transversely at $q \in D_i$. Of course, we may deform g to a very free curve meeting D_i at a point nearby q in D_i . Recall $D_i \simeq Q_{n-1}$, a smooth quadric hypersurface of dimension $n-2$, where $n = \dim X + 1$. Any two points $p, q \in Q_{n-1}$ can be joined by a smooth rational curve $R \subset Q_{n-1}$ with $\deg R = 1, 2$. Indeed, if the line ℓ joining p and q lies in Q_{n-1} , we take $R = \ell$; otherwise, we take R to be a smooth conic. For simplicity, we choose g so that q is not contained in any line $\ell \subset Q_{n-1}$ through p , thus we are in the second case. In particular, $\mathcal{N}_{R/D_i} \simeq \mathcal{O}_{\mathbb{P}^1}(2)^{n-3}$.

We have an exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{N}_{R/D_i} & \rightarrow & \mathcal{N}_{R/\tilde{X}} & \rightarrow & \mathcal{N}_{D_i/\tilde{X}}|_R \rightarrow 0 \\ 0 & \rightarrow & \mathcal{O}_{\mathbb{P}^1}(2)^{n-3} & \rightarrow & \mathcal{O}_{\mathbb{P}^1}(2)^{n-3} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) & \rightarrow & \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow 0 \end{array}$$

Construct a comb C with handle R and teeth very free curves T_1, T_2, T_3 meeting D_i transversally at distinct points $q_1, q_2, q_3 \in R$, different from p . The normal bundle analysis from Proposition 23 of [HT06] yields

$$\mathcal{N}_{C/\tilde{X}}|_R \simeq \mathcal{O}_{\mathbb{P}^1}(2)^{n-3} \oplus \mathcal{O}_{\mathbb{P}^1}(+1);$$

and Proposition 13 guarantees that C smooths to a rational curve through p . Since $C \cdot D_i = 1$, the smoothing meets D_i transversally at p . \square

COROLLARY 17. *Let $\mathcal{X} \rightarrow B$ be a flat proper morphism. The locus*

$$\{b \in B : \mathcal{X}_b = \pi^{-1}(b) \text{ satisfies Hypothesis 14} \}$$

is open.

PROOF. Our analysis of versal deformations in Section 2 implies having ordinary singularities is an open condition. Having a very free curve in the smooth locus is also an open condition, as very free curves deform to very free curves (cf. Theorem 3.11 of [Kol96].) It remains to consider the condition that there exists a rational curve meeting each exceptional divisor once transversally. Let $x \in \mathcal{X}_b$ be the corresponding ordinary singularity. If x is smoothed in \mathcal{X} then there is nothing

to check. Otherwise, $\mathcal{X} \rightarrow B$ admits a simultaneous resolution locally over suitable étale neighborhoods of b and x (see Remark 10). By Proposition 16, the curve in the resolution $\widetilde{\mathcal{X}}_b$ meeting the exceptional divisor over x can be taken to be very free, and thus deforms to nearby fibers. \square

4. Weak approximation at places with ordinary double points

Let B be a smooth curve with function field F . If X is a proper variety over F , a *model* of X over B is a flat proper morphism $\pi : \mathcal{X} \rightarrow B$ with generic fiber X . The existence of such models is a general result of Nagata [Nag63]. For each $b \in B$, let $\mathcal{X}_b = \pi^{-1}(b)$ and $\mathcal{X}_b^{sm} \subset \mathcal{X}_b$ the locus of smooth points.

THEOREM 18. *Let X be a smooth proper variety over F . Assume that X admits a model $\pi : \mathcal{X} \rightarrow B$ such that, for each $b \in B$, \mathcal{X}_b satisfies Hypothesis 14. Then X satisfies weak approximation.*

In Section 2, we described the local structure of natural resolutions of one-parameter deformations of ordinary singularities. Let $\beta : \mathcal{Y} \rightarrow \mathcal{X}$ denote the corresponding global resolution and $\varphi : \mathcal{Y} \rightarrow B$ the induced fibration. Note that each section $B \rightarrow \mathcal{Y}$ factors through the smooth locus of φ , denoted \mathcal{Y}^{sm} . Each irreducible component of \mathcal{Y}_b has multiplicity one, so \mathcal{Y}_b^{sm} is dense in \mathcal{Y}_b .

What does proving weak approximation entail? Fix $N \in \mathbb{N}$ and sections over the completions of B at b_1, \dots, b_m ,

$$\hat{s}_i : \widehat{B}_{b_i} \rightarrow \mathcal{Y} \times_B \widehat{B}_{b_i}, \quad \widehat{B}_{b_i} = \text{Spec } \varprojlim \mathcal{O}_{B,b_i} / \mathfrak{m}_{B,b_i}^N.$$

We must produce a section $\sigma : B \rightarrow \mathcal{Y}$ approximating each \hat{s}_i to order N , i.e.,

$$\sigma \equiv \hat{s}_i \pmod{\mathfrak{m}_{B,b_i}^{N+1}}.$$

In other words, we need a section with prescribed jet data

$$J := \{\hat{s}_i \pmod{\mathfrak{m}_{B,b_i}^{N+1}}, i = 1, \dots, m\}.$$

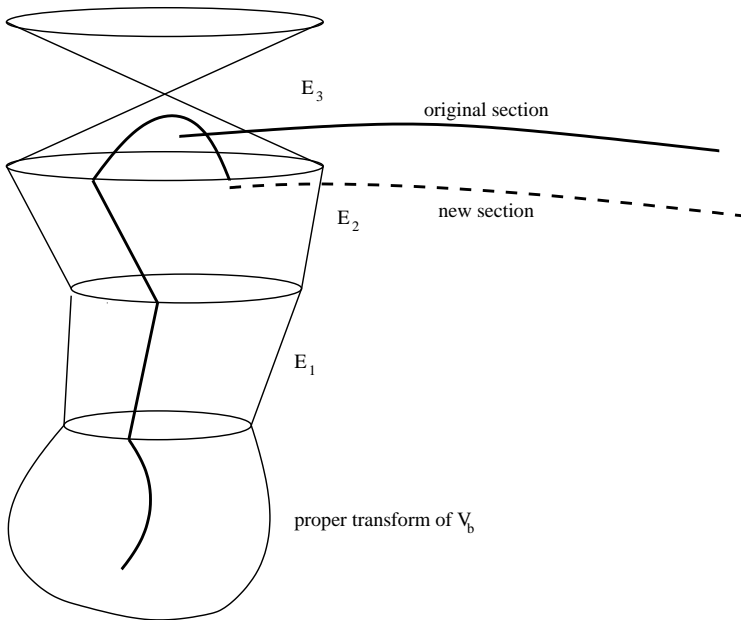
We use the iterated blow-up construction from Section 2.3 of [HT06]. Let

$$\gamma : \mathcal{Y}(J) \rightarrow \mathcal{Y}$$

be obtained by blowing up N times along each \hat{s}_i ; more precisely, at each stage we blow up the point of the proper transform of \hat{s}_i lying over b_i . Fix $y_i = \hat{s}'_i(b_i) \in \mathcal{Y}(J)_{b_i}$, where \hat{s}'_i is the proper transform of \hat{s}_i in $\mathcal{Y}(J)$. Any section $\sigma' : B \rightarrow \mathcal{Y}(J)$ with $\sigma'(b_i) = y_i, i = 1, \dots, m$ yields a section $\sigma := \gamma \circ \sigma' : B \rightarrow \mathcal{Y}$ with the desired jet data J .

To simplify notation, we will argue one place of B at a time: First resolve singularities and make iterated blow-ups in the fiber over b_1 , then in the fiber over b_2 , etc. until all m jet conditions are satisfied. Our main theorem then follows from the following local statement:

THEOREM 19. *Let X be a smooth proper variety of dimension $n - 1$ over F with model $\pi : \mathcal{V} \rightarrow B$. Let $b \in B$ be a place over which π satisfies Hypothesis 14 and $\{b_i\} \subset B$ a (possibly empty) set of additional places distinct from b . Fix a section $\sigma' : B \rightarrow \mathcal{V}$. Then for each local section \hat{s} at b and each N there exists a section $\sigma : B \rightarrow \mathcal{V}$ with $\sigma \equiv \hat{s} \pmod{\mathfrak{m}_{B,b}^{N+1}}$ and $\sigma(b_i) = \sigma'(b_i)$ for each i .*

FIGURE 2. Moving sections from E_3 to E_2

Let $v \in \mathcal{V}_b$ be an ordinary singularity. Let $\mathcal{W} \rightarrow \mathcal{V}$ be the minimal resolution of \mathcal{V} at v with exceptional divisors $E_1, \dots, E_{\lfloor r/2 \rfloor}$, as described in Section 2. We retain the notation introduced there and use E_0 to denote the proper transform of \mathcal{V}_b .

Suppose that $\mathcal{V} \rightarrow B$ admits a section $\sigma : B \rightarrow \mathcal{V}$. Its proper transform $\sigma' : B \rightarrow \mathcal{W}$ meets one of the divisors E_j .

Case I: Moving sections down

There exists a section $\tau : B \rightarrow \mathcal{W}$ with $\tau(b) \in E_{j-1}$ and $\tau(b_i) = \sigma'(b_i)$ for each i .

Suppose $\sigma' : B \rightarrow \mathcal{W}$ is a section with $\sigma'(b) \in E_j$, $j > 0$. We may assume that $\mathcal{N}_{\sigma'(B)/\mathcal{W}}(-b_1 - \dots - b_m)$ is globally generated and has no higher cohomology: After attaching very free curves in generic fibers, we obtain a comb with these properties, which smooths to a section with the same properties, cf. [GHS03] §2.

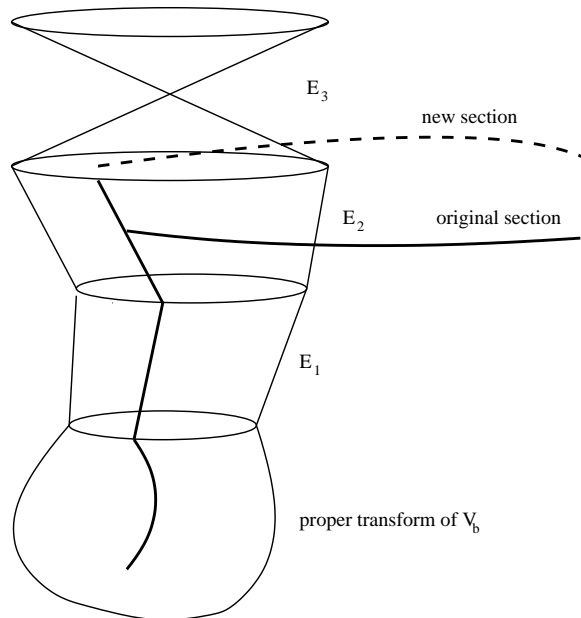
Claim 1: There exists a smooth rational curve $T_j \subset E_j$ with the following properties:

- (1) $\sigma'(b) \in T_j$;
- (2) T_j meets $D_{j-1,j}$ in two points transversely;
- (3) $\mathcal{N}_{T_j/E_j} \simeq \mathcal{O}_{\mathbb{P}^1}(+2)^{n-2}$;
- (4) T_j is disjoint from $D_{j,j+1}$.

By the description of fibers in (2.1) in Section 2, there exists a birational morphism

$$\psi : E_j \rightarrow Q' \subset \mathbb{P}^n$$

onto a quadric hypersurface. This contracts $D_{j,j+1}$ if $j < \lfloor r/2 \rfloor$ and is an isomorphism otherwise; the image of $D_{j-1,j}$ is a hyperplane section of Q' . Since $\sigma'(b)$ is a smooth point of \mathcal{W}_b , $\sigma'(b) \notin D_{j,j+1}$ and $q = \psi(\sigma'(b))$ is a smooth point of Q' . We

FIGURE 3. Moving sections from E_2 to E_3

take T_j to be the proper transform of a generic plane section of Q' containing q . It is straightforward to check this has the properties listed above.

Let $r_{j-1,j}$ be one of the points of $T_j \cap D_{j-1,j}$. The description of the components E_k at the end of Section 2 also gives:

Claim 2: There exist unique smooth rational curves T_{j-1}, \dots, T_1 with the following properties:

- (1) each T_k is a ruling of E_k , with respect to the \mathbb{P}^1 -bundle structure specified in (2.1);
- (2) T_{j-1} contains $r_{j-1,j}$ and $T_k \cap D_{k,k+1} = T_{k+1} \cap D_{k,k+1} := r_{k,k+1}$ for each $k = 1, \dots, j-1$.

Moreover we have $\mathcal{N}_{T_k/E_k} \simeq \mathcal{O}_{\mathbb{P}^1}^{n-2}$. Finally, Proposition 16 yields

Claim 3: There exists a very free rational curve $T_0 \subset E_0$ meeting D_{01} transversely at $r_{01} := T_1 \cap D_{01}$.

Consider the nodal curve

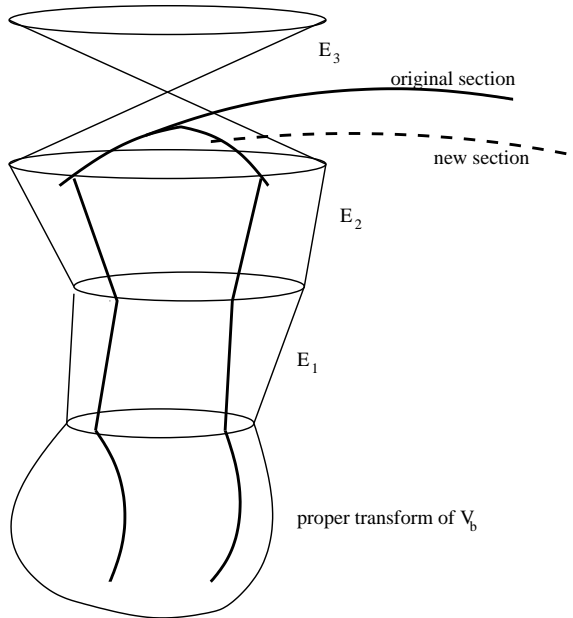
$$C = \sigma'(B) \cup_q T_j \cup_{r_{j-1,j}} T_j \cup \dots \cup_{r_{01}} T_0.$$

The restriction of $\mathcal{N}_{C/\mathcal{W}}(-b_1 - \dots - b_m)$ to each component of C is globally generated and has no higher cohomology. By Proposition 13, C deforms to a section $\tau : B \rightarrow \mathcal{W}$ containing $\{\sigma'(b_i)\}$. Computing intersection numbers, we deduce that $\tau(B) \cap E_{j-1} \neq \emptyset$.

Case II: Moving sections up

There exists a section $\tau : B \rightarrow \mathcal{W}$ with $\tau(b) \in E_{j+1}$ and $\tau(b_i) = \sigma'(b_i)$ for each i .

Suppose $\sigma' : B \rightarrow \mathcal{W}$ is a section with $\sigma'(b) \in E_j$, $j < \lfloor r/2 \rfloor$, chosen so that $\mathcal{N}_{\sigma'(B)/\mathcal{W}}(-b_1 - \dots - b_m)$ is globally generated and has no higher cohomology.

FIGURE 4. Moving sections within E_3

As in case I, we construct a nodal curve

$$C = \sigma'(B) \cup_q T_j \cup \dots \cup T_0$$

where $T_k \subset E_k$ is a ruling for $k = 1, \dots, j$. This differs from the previous case in that T_j is also a ruling of E_j , and thus intersects $D_{j,j+1}$ transversely in one point. Repeating the deformation argument as above gives the desired section τ .

REMARK 20. Since the sections we construct have globally generated normal bundles, they deform to sections passing through the generic point of each E_j . We will tacitly assume this below.

Case III: Moving sections within a component

Let $\mathcal{W} \rightarrow \mathcal{V}$ denote the minimal resolution of *all* the singularities of \mathcal{V} in the fiber \mathcal{V}_b . Before giving the details of our argument, we put our present problem in a more general context:

QUESTION 21. For $b \in B$ let $\mathcal{W} \rightarrow B$ be a model with nonsingular total space over b . Choose an irreducible component $E \subset \mathcal{W}_b$ with multiplicity 1, and suppose there exists a section $\tau : B \rightarrow \mathcal{W}$ with $\tau(b) \in E$. Does weak approximation hold for jets of sections supported in E ?

We solve this in our specific situation; however, our analysis applies more generally.

PROPOSITION 22. Fix $b \in B$ and let $\mathcal{W} \rightarrow B$ be flat proper morphism with rationally connected generic fiber so that \mathcal{W} is nonsingular over b . Suppose \mathcal{W}_b satisfies the following: For each $p \in \mathcal{W}_b^{\text{sm}}$ and q a generic point of the irreducible component $E \subset \mathcal{W}_b$ containing p , there exists a connected nodal curve C of genus

zero, distinguished smooth points $0, \infty \in C$ in the same irreducible component, and a differential-geometric immersion $\iota : C \rightarrow \mathcal{W}_b$ with the following properties

- (i) $\iota(0) = p$ and $\iota(\infty) = q$;
- (ii) each irreducible component of \mathcal{W}_b intersects C with degree zero;
- (iii) ι takes C to the open subset of \mathcal{W}_b with normal crossings singularities of multiplicity at most two; $\iota^{-1}(\mathcal{W}_b^{\text{sing}}) \subset C^{\text{sing}}$ and at points of $\mathcal{W}_b^{\text{sing}} \cap \iota(C)$ there is one branch of C through each component of \mathcal{W}_b ;
- (iv) $\mathcal{N}_\iota(-p)$ is globally generated.

Then weak approximation holds for jets of sections supported in E .

As usual, if we start with a section $\sigma' : B \rightarrow \mathcal{W}$ with prescribed values $\{x_i\}$ at $\{b_i\}$, the sections produced by Proposition 22 can be chosen with the same values.

REMARK 23. Assumption (iii) implies that ι is a local complete intersection morphism over points of $\mathcal{W}_b^{\text{sing}} \cap \iota(C)$. In particular, the normal sheaf \mathcal{N}_ι is locally free on C .

In suitable local coordinates centered at such a point

$$\begin{array}{ccc} C & \rightarrow & \mathcal{W}_b \\ \{x_3 = x_4 = \dots = x_n = 0\} & \subset & \{x_1 x_2 = 0\} \end{array} .$$

If C_1 and C_2 are branches of C and W_1 and W_2 are the corresponding branches of \mathcal{W}_b then $\mathcal{N}_\iota|_{C_k} = \mathcal{N}_{C_k/W_k}$.

The assumptions of Proposition 22 guarantee the analogous conditions when the two points are ‘infinitely near’:

LEMMA 24. For each $p \in \mathcal{W}_b^{\text{sm}}$ and tangent direction $v \in \mathbb{P}(\mathcal{T}_p \mathcal{W}_b)$, there exists a connected nodal curve C of genus zero with one distinguished smooth point $0 \in C$ lying on C_1 and an immersion ι with the properties enumerated above, except that condition (i) is replaced by:

- i'. $d\iota(\mathcal{T}_0 C_1) = v$, i.e., C_1 has the prescribed tangency at p .

We sketch the argument, following the analysis of strongly rationally connected varieties in [HT08]. The hypotheses of the proposition guarantee the existence of two curves C and C' smooth at p with distinct tangent directions w and w' , so that $\{w, w', v\}$ lie on a line $\ell \subset \mathbb{P}(\mathcal{T}_p \mathcal{W}_b)$. Let $\widetilde{\mathcal{W}}_b = \text{Bl}_p \mathcal{W}_b$ and consider the proper transforms $C, C' \rightarrow \widetilde{\mathcal{W}}_b$. Concatenate to get the genus-zero curve

$$C'' := C \cup_w \ell \cup_{w'} C'$$

with an immersion $\iota'' : C'' \rightarrow \widetilde{\mathcal{W}}_b$. By Proposition 23 of [HT06] we have

$$\mathcal{N}_{\iota''}(-v)|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}^{n-2};$$

the assumptions of Proposition 22 guarantee $\mathcal{N}_{\iota''}(-v)|_C$ and $\mathcal{N}_{\iota''}(-v)|_{C'}$ are also globally generated.

We see that $\mathcal{N}_{\iota''}(-v)$ is globally generated—in particular, there are sections which smooth w and w' to first order. Since C'' is a nodal connected curve of genus zero, it follows that $\mathcal{N}_{\iota''}(-v)$ has vanishing higher cohomology. By a variant of Proposition 13 applicable to normal crossings varieties, we can deform ι'' to an immersion $\bar{\iota} : \bar{C} \rightarrow \widetilde{\mathcal{W}}_b$ which is smooth near the exceptional locus of $\widetilde{\mathcal{W}}_b \rightarrow \mathcal{W}_b$ and has $v \in \bar{\iota}(\bar{C})$. An intersection computation shows this curve meets the exceptional locus of $\widetilde{\mathcal{W}}_b \rightarrow \mathcal{W}_b$ with multiplicity one. The resulting curve $\bar{C} \rightarrow$

\mathcal{W}_b has the desired tangency; since properties (ii)-(iv) are open conditions (in the Hilbert scheme parametrizing immersed genus-zero curves in \mathcal{W}_b), they hold as well.

We now prove Proposition 22 by induction on N , the order of the jet we seek to approximate. The $N = 0$ case entails producing a section through an arbitrary point $p \in E \cap \mathcal{W}_b$. Let $\sigma' : B \rightarrow \mathcal{W}_b$ be a section through q , the generic point of E and C the genus-zero curve guaranteed by the hypothesis of Proposition 22. We claim the nodal curve

$$C' := \sigma'(B) \cup_q C$$

has the following properties:

- (1) the restriction of $\mathcal{N}_{C'/\mathcal{W}}(-p)$ to each irreducible component of C' is globally generated and has no higher cohomology;
- (2) C' intersects fibers of $\mathcal{W} \rightarrow B$ with multiplicity one.

Proposition 13 then implies C' deforms to a smooth curve containing p , which is a section of $\mathcal{W} \rightarrow B$.

The second property is an immediate consequence of assumption (ii); C meets each component of \mathcal{W}_b trivially, so C' only meets the component E . For the first property, we have the exact sequence

$$0 \rightarrow \mathcal{N}_{C/\mathcal{W}_b} \rightarrow \mathcal{N}_{C/\mathcal{W}} \rightarrow \mathcal{N}_{\mathcal{W}_b/\mathcal{W}}|_C \rightarrow 0$$

which yields

$$0 \rightarrow \mathcal{N}_{C/\mathcal{W}_b}(-p) \rightarrow \mathcal{N}_{C/\mathcal{W}}(-p) \rightarrow \mathcal{O}_C(-p) \rightarrow 0.$$

(For consistency, we write $\mathcal{N}_{C/\mathcal{W}_b}$ for \mathcal{N}_L .) We also have the extension

$$0 \rightarrow \mathcal{N}_{C/\mathcal{W}} \rightarrow \mathcal{N}_{C'/\mathcal{W}}|_C \rightarrow Q \rightarrow 0$$

where Q is a length-one torsion sheaf supported at q . By Proposition 23 of [HT06], $\mathcal{N}_{C/\mathcal{W}_b}$ is saturated in this extension and thus

$$\mathcal{N}_{C'/\mathcal{W}}(-p)|_C / \mathcal{N}_{C/\mathcal{W}_b}(-p) \simeq \mathcal{O}_C(-p+q) \simeq \mathcal{O}_C.$$

The last isomorphism uses the fact that p and q lie on the same irreducible component of a nodal curve of genus zero. We deduce $\mathcal{N}_{C'/\mathcal{W}}(-p)|_C$ is an extension of locally free sheaves that are globally generated with no higher cohomology; hence it also is globally generated with no higher cohomology.

For the inductive step, let $j_b := \hat{s} \pmod{\mathfrak{m}_{B,b}^{N+1}}$ denote the N -jet we seek to approximate, and $j'_b := \hat{s} \pmod{\mathfrak{m}_{B,b}^N}$ its truncation to order $N-1$. Let $\sigma : B \rightarrow \mathcal{W}$ be a section with

$$\sigma \equiv j'_b \pmod{\mathfrak{m}_{B,b}^N}$$

and let $\gamma : \mathcal{W}' \rightarrow \mathcal{W}$ be the *iterated blow-up*: We blow-up successively N times along the proper transform of $\sigma(B)$ over the point b . Let $\sigma' : B \rightarrow \mathcal{W}'$ denote the proper transform of σ and G_1, \dots, G_N the exceptional components in \mathcal{W}' , in the order they appear. We have the following:

- (1) $G_N \simeq \mathbb{P}^{n-1}$ and $G_k \simeq \text{Bl}_{u_k} \mathbb{P}^{n-1}$ for $1 \leq k < N$; here u_k is the evaluation of the k -th proper transform of σ' at b ;
- (2) $\sigma'(b) \in G_N$;
- (3) there exists a point $x \in G_N$ so that if $\tau : B \rightarrow \mathcal{W}'$ is a section with $\tau(b) = x$ then $\gamma \circ \tau : B \rightarrow \mathcal{W}$ has the desired jet j_b .

We construct rational curves $T_k \subset G_k, k = 1, \dots, N$ recursively as follows. Let $T_N \subset G_N \simeq \mathbb{P}^{n-1}$ denote the line joining x and $\sigma'(b)$ (if these happen to coincide we have nothing to prove). Let $T_{N-1} \subset G_{N-1} \simeq \text{Bl}_{u_{N-1}} \mathbb{P}^{n-1}$ denote the proper transform of the line in \mathbb{P}^{n-1} joining u_{N-1} and $T_N \cap G_{N-1}$. For each $k > 1$, let $T_{k-1} \subset G_{k-1}$ denote the proper transform of the line in \mathbb{P}^{n-1} joining u_{k-1} and $T_k \cap G_{k-1}$.

Let E' denote the proper transform of E in \mathcal{W}' ; we have $E' = \text{Bl}_p E$. Let v denote the intersection of T_1 with E' , which corresponds to an element of $\mathbb{P}(\mathcal{T}_p E)$. Let C be the curve guaranteed by Lemma 24 and T_0 its proper transform in \mathcal{W}' . The nodal chain of curves

$$C' = \sigma'(B) \cup T_N \cup T_{N-1} \dots \cup T_0$$

satisfies

- (1) the restriction of $\mathcal{N}_{C'/\mathcal{W}'}(-x)$ to each irreducible component of C' is globally generated and has no higher cohomology;
- (2) C' intersects fibers of $\mathcal{W}' \rightarrow B$ with multiplicity one.

These properties are verified by computations analogous to those in the base case $N = 0$. By Proposition 13, C' deforms to a smooth curve containing x , which is a section of $\mathcal{W}' \rightarrow B$. The image $\gamma(C')$ has the specified N -jet at b .

We complete the proof of Theorem 19. We just have to verify that there exist genus-zero curves satisfying the assumptions of Proposition 22. Again $E_j \subset \mathcal{W}_b, j = 0, \dots, \lfloor r/2 \rfloor$ denote the components of our degenerate fiber. As in Case I, we have a morphism $\psi : E_j \rightarrow Q'$ to a quadric surface. We also make reference to the description of components in Equation (2.1).

When $j = 0$ we take C to be a very free curve in $E_0 \cap \mathcal{W}_b^{sm} = \mathcal{V}_b^{sm}$ joining p and q , which exists since \mathcal{V}_b^{sm} is strongly rationally connected. For $j > 0$, C takes the form

$$T'_0 \cup T'_1 \cup \dots \cup T'_{j-1} \cup T_j \cup T''_{j-1} \cup \dots \cup T''_0,$$

where the T'_k and T''_k are recursively defined as follows:

- (1) T_j is a plane section of E_j joining p and q . Since $q \in E_j$ is generic, we may assume $T_j \cap D_{j-1,j} = \{r'_{j-1,j}, r''_{j-1,j}\}$. We have $\mathcal{N}_{T_j/E_j} \simeq \mathcal{O}_{\mathbb{P}^1}(+2)^{n-2}$.
- (2) For each $k \geq 1$, T'_k (resp. T''_k) is the ruling of E_k passing through $r'_{k,k+1} := T'_{k+1} \cap D_{k,k+1}$ (resp. $r''_{k,k+1} := T''_{k+1} \cap D_{k,k+1}$). Its normal bundle in E_k is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{n-2}$.
- (3) $T'_0 \simeq \mathbb{P}^1$ (resp. T''_0) is a very free rational curve in E_0 meeting D_{01} at $r'_{01} := T'_1 \cap D_{01}$ (resp. $r''_{01} := T''_1 \cap D_{01}$). This exists by Proposition 16; its normal bundle is positive.

The T'_k and T''_k have normal bundles which are globally generated and have vanishing higher cohomology. Furthermore, $\mathcal{N}_{T_j/E_j}(-p)$ has the same properties. An intersection computation shows that each component of \mathcal{W}_b meets C with degree zero. Each component of C is immersed in the corresponding component of \mathcal{W}_b . Finally, the nodes of C are embedded in \mathcal{W}_b at points where two irreducible components meet. Proposition 22 applies and weak approximation for jets supported along E_j follows. This completes the proof of Theorem 19.

5. Applications to cubic hypersurfaces

Weak approximation is known for cubic surfaces over the function field of a curve, provided there exists a nonsingular model $\mathcal{X} \rightarrow B$ such that the singular fibers are cubic surfaces with rational double points [HT08]. Of course, not all rational double points can arise—see [Cor96] for a comprehensive analysis of possible models of cubic surfaces. Here we focus on the case where the fibers have only ordinary singularities:

THEOREM 25. *Let X be a smooth cubic hypersurface of dimension at least two over the function field of a curve B . Suppose that X admits a (possibly singular) model $\mathcal{X} \rightarrow B$ with fibers cubic hypersurfaces with only ordinary singularities. When the dimension is two we also assume that no fiber is isomorphic to the Cayley cubic surface*

$$wxy + xyz + yzw + zwx = 0.$$

Then X satisfies weak approximation over B .

We emphasize here that B need not be compact; we can omit places where the singularities are worse than ordinary double points.

PROOF. The proof in [HT08] shows that each fiber \mathcal{X}_b^{sm} is strongly rationally connected in relative dimension two. In dimensions > 2 , \mathcal{X}_b^{sm} is strongly rationally connected if, for each $x \in \mathcal{X}_b^{sm}$, there exists a hyperplane section of \mathcal{X}_b^{sm} through x which is strongly rationally connected. However, the generic such hyperplane section is nonsingular and thus (strongly) rationally connected.

It remains to verify the second part of Hypothesis 14. Cutting by generic hyperplane sections as above, we are reduced to the case of cubic surfaces with a *single* double point. (A generic hyperplane section of an ordinary singularity has an ordinary singularity.) It is a classical fact that through each ordinary singularity of a cubic surface, there passes a line. We are done if at least one of these lines does not pass through a second singularity of the surface.

Unfortunately this is not always true, but we can classify the problematic cases:

LEMMA 26. *Let S be a cubic surface with ordinary singularities. Assume that each line containing one singularity of S also contains a second singularity. Then S is isomorphic to the Cayley cubic surface.*

We prove the lemma. The multiplicities of the lines on a singular cubic surface are well known (e.g., [GH78], pp. 640). Let $s_1 \in S$ be an ordinary double point. There are six lines $\ell \subset S$ containing s_1 , counted with multiplicity. The lines containing a second double point have multiplicity two, and any line contains at most two ordinary double points. If every line through s_1 also passes through a second ordinary double point, then there are three such lines ℓ_2, ℓ_3, ℓ_4 and three double points $s_2, s_3, s_4 \in S$ besides s_1 . In particular, S contains four ordinary double points and the six lines joining pairs of these. But the Cayley cubic is the unique cubic surface with four ordinary singularities [BW79]. This completes the proof of the lemma and the theorem. \square

Hypothesis 14 *fails* for the Cayley surface. Let $\tilde{S} \rightarrow S$ denote the minimal resolution of the Cayley cubic, with exceptional curves D_1, D_2, D_3, D_4 . Let R be a class in the Néron-Severi group of \tilde{S} with intersection numbers $m_i = (R \cdot D_i)$. The

divisor class $D_1 + D_2 + D_3 + D_4$ is two-divisible in the Néron-Severi group, so we have

$$m_1 + m_2 + m_3 + m_4 \equiv 0 \pmod{2}.$$

In particular, R cannot meet D_1 once without meeting one of the other D_i .

Suppose $\mathcal{X} \rightarrow B$ is a cubic surface fibration with \mathcal{X}_b a Cayley cubic and generic fiber a smooth cubic. Suppose that for each of the four singularities of \mathcal{X}_b , the local equation of \mathcal{X} is of the form

$$t^3 = x_0^2 + x_1^2 + x_2^2.$$

The resolution $\mathcal{W} \rightarrow \mathcal{X}$ of the singularities over b has four exceptional divisors E^1, E^2, E^3, E^4 , each of which is the cone over a smooth plane conic. Let E_0 denote the proper transform of \mathcal{X}_b .

Case III of the proof of Theorem 19 shows that, given a section through E^i or E_0 , weak approximation holds for jets supported in that component. However, the methods of Cases I and II do not apply in this example. In particular, we cannot deduce the existence of a section through *each* component of \mathcal{W}_b .

Appendix: Recent developments for hypersurfaces of low degree

We summarize progress since this paper was first written in March 2006. Work of de Jong and Starr yields weak approximation results for hypersurfaces substantially stronger than those presented in Section 1.

The key ingredient is the notion of rational *simple* connectedness. Rational connectedness in algebraic geometry plays a rôle analogous to path connectedness in topology. This suggests formulating algebro-geometric conditions analogous to *simple* connectedness. While there are a number of competing definitions, several have useful applications to weak approximation.

Let X be a smooth projective complex variety with $H_2(X, \mathbb{Z}) \simeq \mathbb{Z}$. For each $e > 0$, let $\overline{\mathcal{M}}_{0,n}(X, e)$ denote the *Kontsevich space of stable maps* (C, p_1, \dots, p_n, h) , where C is a connected nodal proper curve of arithmetic genus zero, $p_1, \dots, p_n \in C$ are smooth points, and $h : C \rightarrow X$ is a morphism of degree e with finitely-many automorphisms. There is an evaluation morphism

$$\begin{aligned} \text{ev} : \overline{\mathcal{M}}_{0,n}(X, e) &\rightarrow \underbrace{X \times \dots \times X}_{n \text{ times}} \\ (C, p_1, \dots, p_n, h) &\mapsto (h(p_1), \dots, h(p_n)). \end{aligned}$$

DEFINITION 27. X is *strongly rationally simply connected* if, given $n \geq 2$, for each e sufficiently large there is a canonically-defined irreducible component $M_{e,n} \subset \overline{\mathcal{M}}_{0,n}(X, e)$ such that

$$\text{ev} : M_{e,n} \rightarrow \underbrace{X \times \dots \times X}_{n \text{ times}}$$

is dominant with rationally-connected generic fiber. X is *rationally simply connected* if this holds for $n = 2$.

Here ‘canonically defined’ means defined over any field of definition of X .

We explain the relevance of these notions to weak approximation: Let X be a smooth projective variety defined over $F = \mathbb{C}(B)$, where B is a complex curve. Assume that $H_2(X_{\mathbb{C}}, \mathbb{Z}) \simeq \mathbb{Z}$. The first author has shown that if X is strongly rationally simply connected then X satisfies weak approximation over F ; Jason Starr

has shown that it suffices to assume X is rationally simply connected. (These theorems will be proven elsewhere.) De Jong and Starr have established the following general criteria for rational simple connectedness:

THEOREM 28. [**dJS06**, Theorems 1.1 and 1.2] *Let X be a smooth complete intersection of hypersurfaces of degrees (d_1, \dots, d_c) in \mathbb{P}^n , with each $d_j \geq 2$. Then X is rationally simply connected if*

$$n + 1 \geq \sum_{j=1}^c d_j^2,$$

with the exception of a quadric surface in \mathbb{P}^3 . It is strongly rationally simply connected if

$$n + 1 \geq \sum_{j=1}^c (2d_j^2 - d_j).$$

As a corollary, we deduce weak approximation for complete intersections in these degrees.

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