Stable limits of log surfaces and Cohen-Macaulay singularities

Brendan Hassett *

March 5, 2001

Institute of Mathematical Sciences
Rm 501, Mong Man Wai Building
Chinese University of Hong Kong
Shatin, Hong Kong
bhassett@ims.cuhk.edu.hk

Math Department–MS 136
Rice University
6100 S. Main St.
Houston, TX 77005-1892
USA

Abstract

Given a family of surfaces of general type over a smooth curve, one can apply semistable reduction and the minimal model program to obtain a stable reduction. This is the basis for a geometric compactification for moduli spaces of surfaces of general type, due to Kollár, Shepherd-Barron, and Alexeev. However, this approach hinges on the fact that the resulting stable limit has relatively mild singularities; in particular, it should be Cohen-Macaulay. Unfortunately, the standard formalism does not guarantee that stable limits of families of *log* surfaces are Cohen-Macaulay. Here we prove that this is the case.

Subject classification: 14J17, 14J10
Keywords: Cohen-Macaulay, surfaces, moduli spaces

1 Introduction

Canonical models are fundamental in the study of surfaces of general type. However, they are generally singular and it is not always clear they are

*The author was partially supported by an NSF Postdoctoral Fellowship, NSF Grant 0070537, and the Institute of Mathematical Sciences of the Chinese University of Hong Kong.

smoothable. So in constructing compact moduli spaces for surfaces of general type, one considers stable limits arising as degenerations of surfaces with canonical singularities ([13] §5). Similarly, when compactifying moduli spaces for log surfaces of general type, one should consider degenerations of log surfaces with log canonical singularities. This is necessary if every log canonical model is to occur in some component of our compactification. In this paper, we prove theorems needed to construct compact moduli spaces with this property. Existing compactifications ([1] 3.18) do not have this property.

We fix some notation. Throughout this paper, we work over an algebraically closed field k of characteristic zero. We use Δ to denote the unit disc (or the algebraic analog $\operatorname{Spec} k[[t]]$) and write Δ^* for the punctured disc $\Delta - 0$. A family of log surfaces consists of a log variety with reduced boundary $(\mathcal{S}, \mathcal{C})$ and a flat morphism $\pi : \mathcal{S} \to B$ such that $\pi|\mathcal{C}$ is flat and the fibers are log surfaces. In particular, the fibers of π and $\pi|\mathcal{C}$ are reduced. When $B = \Delta$, the fiber over 0 (the *central fiber*) will be denoted (S_0, C_0). Our main result is:

Theorem 1.1 (Main Theorem) *Let $\pi^c : (\mathcal{S}^c, \mathcal{C}^c) \to \Delta$ be a family of log surfaces so that the pair $(\mathcal{S}^c, \mathcal{C}^c + S_0^c)$ is log canonical and the fibers over $\tilde{\Delta}^*$ are log canonical. Then $\mathcal{S}^c$ and $\mathcal{C}^c$ are Cohen-Macaulay.*

The inclusion of the central fiber in the boundary is a bookkeeping device; it keeps track of the singularities of S_0. Even a smooth surface fibered over the disc may have unpleasant singularities in the central fiber.

The main application is a description of the limiting surfaces arising in compactications of moduli spaces of log surfaces of general type. A family of log surfaces $\pi : (\mathcal{S}, \mathcal{C}) \to B$ is *allowable* if the fibers have semilog canonical singularities, each reflexive power of $\omega_\pi(\mathcal{C})$ commutes with base extension, and some power is locally free. The allowability assumption is a natural one: it excludes pathological cases where $(K_{S_t} + C_t)^2$ and the log plurigenera fail to be constant. The constancy of plurigenera in families of smooth surfaces is an important and useful property that we would like to preserve on compactifying the moduli space. Furthermore, a family of log surfaces $(\mathcal{S}, \mathcal{C}) \to \Delta$ with $K_{\mathcal{S}} + \mathcal{C}$ log canonical (and $\mathcal{S}$ Cohen-Macaulay) is automatically allowable. This is obvious when $K_{\mathcal{S}} + \mathcal{C}$ is locally free and is proven in general by passing to an index-one cover. See [10] for more on allowable families and compactifications of moduli spaces of surfaces.

Theorem 1.2 (Local Stable Reduction Theorem) *Let $\pi : (\mathcal{S},\mathcal{C}) \to \Delta$ be a family of log surfaces. Assume that $\pi|\Delta^*$ is allowable with normal fibers. Then there exists a base change $\tilde{\Delta} \to \Delta$, an allowable family of log surfaces $\pi^c : (\mathcal{S}^c,\mathcal{C}^c) \to \tilde{\Delta}$, and a birational projective morphism $\beta : (\mathcal{S}^c,\mathcal{C}^c) \to (\mathcal{S} \times_\Delta \tilde{\Delta}, \mathcal{C} \times_\Delta \tilde{\Delta})$ satisfying the following:*

1. *β induces an isomorphism over $\tilde{\Delta}^*$:*
$$(\mathcal{S}^c,\mathcal{C}^c)|\tilde{\Delta}^* \to (\mathcal{S} \times_\Delta \tilde{\Delta}^*, \mathcal{C} \times_\Delta \tilde{\Delta}^*);$$

2. *$(\mathcal{S}^c,\mathcal{C}^c+S^c_0)$ has log canonical singularities and $K_{\mathcal{S}^c}+\mathcal{C}^c$ is ample relative to β.*

We sketch the proof. By the semistable reduction theorem ([12] Theorem 7.17), we obtain a semistable resolution

$$\rho : (\tilde{\mathcal{S}}, \tilde{\mathcal{C}} + \tilde{S}_0 + \tilde{\mathcal{F}}) \to (\mathcal{S} \times_\Delta \tilde{\Delta}, \mathcal{C} \times_\Delta \tilde{\Delta} + S_0)$$

i.e., the boundary has reduced normal crossings. Here $\tilde{\mathcal{F}}$ denotes the exceptional divisors dominating the base. Let $(\mathcal{S}^c, \mathcal{C}^c + S^c_0)$ be the log canonical model relative to ρ; this evidently has log canonical singularities and relatively ample log canonical bundle. S^c_0 has normal crossings in codimension one and is S2 by the Main Theorem. The Main Theorem also implies that $\mathcal{C}^c$ is Cohen-Macaulay, so C^c_0 has no imbedded points and π^c is a family of log surfaces. The reflexive powers of $\omega_{\pi^c}(\mathcal{C}^c)$ commute with restriction to the central fiber because $(\mathcal{S}^c,\mathcal{C}^c)$ has log canonical singularities. Adjunction ([11] 17.2) yields that (S^c_0, C^c_0) and the other fibers are also semilog canonical. □

A similar argument gives a global version of this result. A (projective) log surface (S, C) is *stable* if it is semilog canonical and $K_S + C$ is ample.

Theorem 1.3 (Stable Reduction Theorem) *Let $\pi : (\mathcal{S},\mathcal{C}) \to \Delta$ be a family of log surfaces. Assume that $\pi|\Delta^*$ is allowable with normal, stable fibers. Then there exists a base change $\tilde{\Delta} \to \Delta$ and an allowable family of stable log surfaces $\pi^c : (\mathcal{S}^c,\mathcal{C}^c) \to \tilde{\Delta}$ such that $(\mathcal{S}^c,\mathcal{C}^c)$ is the pull-back of $(\mathcal{S},\mathcal{C})$ over $\tilde{\Delta}^*$.*

We should emphasize that $\mathcal{C}^c \to \tilde{\Delta}$ is a family of reduced nodal curves; if the adjunction formula $(K_{\mathcal{S}^c} + \mathcal{C}^c)|\mathcal{C}^c = K_{\mathcal{C}^c}$ holds, it is a family of stable curves. This is used to study plane curves in [6] and [7].

In section two, some ideas from commutative algebra are developed. The third section contains the proof of the Main Theorem. In the final section we sketch an application of our results to bielliptic curves.

The author has greatly benefitted from conversations with János Kollár, Sándor Kovács, Andrew Kresch, and Kenji Matsuki.

2 Algebraic Notions

We assume all schemes are separated, reduced, and of finite type over a field.

A morphism of reduced schemes $Y \to X$ is *birational* if it maps generic points of irreducible components bijectively to generic points of irreducible components and induces isomorphisms on the corresponding residue class fields.

Definition 2.1 *A reduced scheme X is weakly normal ([2]) (resp. seminormal ([18] [4])) if each morphism $Z \to X$ which is finite, bijective, and birational (resp. finite, bijective, birational, and induces trivial residue field extensions at each point) is an isomorphism.*

The going-up theorem ([3] 4.15) implies that any dominant finite morphism is surjective, so we could replace 'bijective' by 'injective' in the definition. Evidently, weakly normal and seminormal coincide in characteristic zero. We now resume our blanket assumption that the base field has characteristic zero. For simplicity, we shall only use the term 'seminormal'.

Remark 2.2 *If Y is a S2 surface with normal crossings in codimension one then Y is seminormal ([4] 2.7).*

Let $X^n = \operatorname{Spec} \mathcal{A}^n$ be the normalization of X, i.e. $\mathcal{A}^n$ is the algebra of elements of the total ring of fractions which are integral over X. Under our assumptions, the induced morphism $h : X^n \to X$ is finite ([14] pp. 261-264); it is also maximal, i.e. it factors through any other birational finite morphism $Z \to X$.

Definition 2.3 *([18] §1.2) The seminormalization X^{sn} of X is defined as $\operatorname{Spec} \mathcal{A}^{sn}$, where $\mathcal{A}^{sn}$ is the algebra of functions $f \in \mathcal{A}$ such that, for each $p \in X$ with residue field k_p and p_1, p_2 lying over p,*

1. $f(p_i) \in k_p \subset k_{p_i}$;

2. $f(p_1) = f(p_2)$.

We have a factorization

$$\begin{array}{ccccc} & & X^{sn} & & \\ & {}^{h_1}\nearrow & & \searrow^{h_2} & \\ X^n & & \xrightarrow{h} & & X \end{array}$$

so that h_2 is birational, finite, injective, induces isomorphisms of residue class fields, and is maximal with these properties.

It is well known that the normalization satisfies a universal property: If $m : Y \to X$ is a dominant morphism of integral schemes and Y is normal then m factors through X^n. The seminormalization satisfies an analogous universal property:

Proposition 2.4 (Universal Property of Seminormalization) *([9], ch. I, Proposition 7.2.3.3) Let $m : Y \to X$ be a morphism and assume that Y is seminormal. Then m factors uniquely through X^{sn}, the seminormalization of X.*

We now assume that all schemes are pure dimensional.

Let X^s be the S2-ification of X ([5] §5.10), i.e., $X^s = \operatorname{Spec} \mathcal{A}^s$ where $\mathcal{A}^s$ denotes the functions regular in the complement of a codimension-two subset of X. The induced map $g : X^s \to X$ is finite because the normalization map factors through it. In this context, we can emulate the definition of the seminormalization:

Definition 2.5 *The semiS2-ification X^{ss} of X is defined as $\operatorname{Spec} \mathcal{A}^{ss}$, where $\mathcal{A}^{ss}$ is the algebra of functions $f \in \mathcal{A}^s$ such that, for each $p \in X$ with residue field k_p and p_1, p_2 lying over p,*

1. $f(p_i) \in k_p \subset k_{p_i}$;
2. $f(p_1) = f(p_2)$.

We have a factorization

$$\begin{array}{ccccc} & & X^{ss} & & \\ & {}^{g_1}\nearrow & & \searrow^{g_2} & \\ X^s & & \xrightarrow{g} & & X \end{array}$$

so that g_2 is birational, finite, injective, induces isomorphisms of residue class fields, and is maximal with these properties.

We obtain a natural morphism $X^{sn} \to X^{ss}$ by the maximality of the seminormalization.

Definition 2.6 *X is topologically S2 at x if $g_1 : X^s \to X^{ss}$ is an isomorphism over x.*

This is equivalent to insisting that $g_1 : X^s \to X^{ss}$ is injective. For example, the union of two planes glued together at a single point is not topologically S2 at the point of intersection. Topologically S2 schemes have the following connectivity property:

Proposition 2.7 *Assume that X is topologically S2 at p. Then there exists no point x specializing to p such that the punctured formal neighborhood $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x} - x$ is disconnected.*

proof: Without changing the topology we may replace X by X^{ss}. Let $p \in X$ be a point where X is not S2; let $x \in X$ correspond to an irreducible component of the locus where X is not S2 so that x specializes to p. The codimension of x is at least two.

Let $\hat{X} = \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ be the formal neighborhood of X at x. Note that X is S2 at x iff $\hat{X}$ is S2 at x ([14] pp. 136). Furthermore, $\hat{X}$ is S2 away from x. Consider the S2-ification $\hat{g} : \hat{X}^s \to \hat{X}$. Note that $\hat{X}^s - \hat{g}^{-1}(x) \to \hat{X} - x$ is an isomorphism, and regular functions on $\hat{X} - x$ correspond to regular functions on $\hat{X}^s$. Since $\hat{g}$ is finite, $\hat{X}^s$ is the disjoint union of the localizations at its maximal ideals, each of which is finite over $\hat{X}$ ([3] Corollary 7.6). By assumption $\hat{g}$ is not injective, so $\hat{X}^s$ and $\hat{X}^s - \hat{g}^{-1}(x)$ are disconnected. Therefore, $\hat{X} - x$ is also disconnected. □

This means that if X is not topologically S2, then X is not *locally analytically connected in codimension one*, i.e. it can be disconnected (analytically locally) by removing a subset of codimension > 1. By a result of Rim ([16] Prop. 3.3), X is not smoothable. Indeed, the proof yields the following criterion:

Proposition 2.8 *Let X be a reduced, pure-dimensional, separated scheme of finite type over a field k. Assume there is a flat family $\mathcal{X} \to \operatorname{Spec} k[[t]]$ such that $\mathcal{X}$ is normal and the central fiber is X. Then X is locally analytically connected in codimension one.*

3 Proof of Theorem 1.1

We first reduce to the case where $(\mathcal{S}^c, \mathcal{C}^c + S_0^c)$ admits a semistable resolution. The semistable reduction theorem ([12] Theorem 7.17) implies that such a resolution exists after pulling back via a finite base change $\tilde{\Delta} \to \Delta$. The base-changed $\mathcal{S}^c \times_\Delta \tilde{\Delta}$ is still normal (its fibers are reduced), so the base-changed pair is log canonical by the logarithmic ramification formula (see Prop. 20.3 of [11]). In particular, the base-changed pair still satisfies the hypotheses of the Theorem 1.1; the base-changed family is Cohen-Macaulay if and only if the original is. For simplicity, we shall still write $(\mathcal{S}^c, \mathcal{C}^c + S_0^c)$ for the base-changed family and

$$R : (\tilde{\mathcal{S}}, \tilde{\mathcal{C}} + \tilde{S}_0 + \tilde{\mathcal{F}}) \to (\mathcal{S}^c, \mathcal{C}^c + S_0^c)$$

for the semistable resolution. It suffices to prove Theorem 1.1 when such a resolution exists.

Consider the log minimal model of $(\tilde{\mathcal{S}}, \tilde{\mathcal{C}} + \tilde{S}_0 + \tilde{\mathcal{F}})$ relative to R, denoted $(\mathcal{S}^m, \mathcal{C}^m + S_0^m + \mathcal{F}^m)$, and the induced proper morphism $h : \mathcal{S}^m \to \mathcal{S}^c$. Both $(\mathcal{S}^m, \mathcal{C}^m + S_0^m + \mathcal{F}^m)$ and $(\mathcal{S}^m, \mathcal{C}^m + \mathcal{F}^m)$ have $\mathbb{Q}$-factorial weak Kawamata log terminal singularities, and the underlying space $\mathcal{S}^m$ has Kawamata log terminal singularities as well. At points of $\mathcal{S}^c$ not in the image $h(\mathcal{F}^m)$ the proof is straightforward. Over these points there are no exceptional divisors dominating the base with discrepancy zero, hence $(\mathcal{S}^c, \mathcal{C}^c)$ is divisorial log terminal and therefore weak Kawamata log terminal ([17]). It follows that $\mathcal{S}^c$ and $\mathcal{C}^c$ are Cohen-Macaulay ([11] 2.16 and 17.5). We therefore restrict to points in $h(\mathcal{F}^m)$.

In what follows, canonical bundles are all taken relative to h and its restrictions to subvarieties (which, for simplicity, are also denoted h). We shall use the following key exact sequence

$$\begin{aligned} 0 \;\to\; & \mathcal{O}_{\mathcal{S}^m}(NK_{\mathcal{S}^m} + (N-1)(\mathcal{C}^m + \mathcal{F}^m)) \to \mathcal{O}_{\mathcal{S}^m}(N(K_{\mathcal{S}^m} + \mathcal{C}^m + \mathcal{F}^m)) \\ \to\; & \mathcal{O}_{\mathcal{C}^m \cup \mathcal{F}^m}(N(K_{\mathcal{C}^m \cup \mathcal{F}^m} + \mathrm{Diff})) \to 0. \end{aligned}$$

Here Diff is the appropriate different ([11] chapter 16), an effective $\mathbb{Q}$-divisor with no irreducible components contained in the singular locus. We choose N so that the last two terms are locally free on $\mathcal{S}^m$ and $\mathcal{C}^m \cup \mathcal{F}^m$ respectively. The higher direct images of the first term are zero by a corollary to the Kawamata-Viehweg Vanishing Theorem ([8] 1.2.5 and 1.2.6).

We first show that $\mathcal{C}^c$ is Cohen-Macaulay. Observe that $\gamma : \mathcal{C}^m \cup \mathcal{F}^m \to \tilde{\Delta}$ is Cohen-Macaulay: $(\mathcal{S}^m, \mathcal{C}^m + \mathcal{F}^m)$ is weak Kawamata log terminal and thus divisorial log terminal ([11] 2.16), so $\mathcal{C}^m \cup \mathcal{F}^m$ is seminormal and S2 ([11] 17.5) and the fibers of γ have no imbedded points. The fibers of $\pi^m : (\mathcal{S}^m, \mathcal{C}^m + \mathcal{F}^m) \to \tilde{\Delta}$ have semilog canonical singularities by adjunction ([11] 17.2), hence γ is a family of reduced nodal curves. Furthermore, Diff is supported in the smooth locus of γ.

Our application of Kawamata-Viehweg vanishing shows that $h : \mathcal{C}^m \cup \mathcal{F}^m \to \mathcal{C}^c$ is induced by the sections of

$$\mathcal{O}_{\mathcal{C}^m \cup \mathcal{F}^m}(N(K_{\mathcal{C}^m \cup \mathcal{F}^m} + \mathrm{Diff}))$$

for N is suitably large and divisible. Evidently, the pluricanonical image

$$D = \mathrm{Proj} \oplus_{M \geq 0} h_* \mathcal{O}_{C_0^m \cup F_0^m}(NM(K_{C_0^m \cup F_0^m} + \mathrm{Diff}_0))$$

has no imbedded points, so it suffices to show that C_0^c coincides with D. This fails exactly when the map $D \to C_0^c$, induced by the restriction

$$h_* \mathcal{O}_{\mathcal{C}^m \cup \mathcal{F}^m}(MN(K_{\mathcal{C}^m \cup \mathcal{F}^m} + \mathrm{Diff})) \to h_* \mathcal{O}_{C_0^m \cup F_0^m}(MN(K_{C_0^m \cup F_0^m} + \mathrm{Diff}_0)),$$

has nontrivial cokernel. Such a cokernel yields torsion in

$$\mathbb{R}^1 h_* \mathcal{O}_{\mathcal{C}^m \cup \mathcal{F}^m}(N(K_{\mathcal{C}^m \cup \mathcal{F}^m} + \mathrm{Diff}))$$

as an $\mathcal{O}_{\tilde{\Delta}}$-module. We are reduced to showing there is no such torsion.

We employ a similar argument to prove that S_0^c is S2. Proposition 2.8 implies S_0^c is topologically S2, so it suffices to show S_0^c equals its seminormalization. The boundary divisor has normal crossings in codimension one, so S_0^m is seminormal by Remark 2.2. Let $k : S_0^m \to T$ denote the log canonical model of $(S_0^m, C_0^m + F_0^m)$ relative to h, i.e.,

$$T = \mathrm{Proj} \oplus_{M \geq 0} h_* \mathcal{O}_{S_0^m}(MN(K_{S_0^m} + C_0^m + F_0^m))$$

for suitably large and divisible N. Of course, T comes with an ample line bundle L such that $k^* L = \mathcal{O}_{S_0^m}(N(K_{S_0^m} + C_0^m + F_0^m))$.

We assert the seminormalization $g_2 : T^{sn} \to T$ is an isomorphism. Indeed, since S_0^m is seminormal Proposition 2.4 yields

$$\begin{array}{ccccc} & & T^{sn} & & \\ & \nearrow & & \overset{g_2}{\searrow} & \\ S_0^m & & \xrightarrow{k} & & T \end{array} .$$

Since g_2 is finite, g_2^*L is ample on T^{sn} and pulls back to $\mathcal{O}_{S_0^m}(N(K_{S_0^m}+C_0^m+F_0^m))$. A suitably high power of g_2^*L is very ample and its sections pull back to sections of some $\mathcal{O}_{S_0^m}(nN(K_{S_0^m}+C_0^m+F_0^m))$.

We next construct the desired morphism $r : T \to S_0^c$. Since

$$\mathcal{S}^c = \mathrm{Proj} \oplus_{M \geq 0} h_* \mathcal{O}_{\mathcal{S}^m}(MN(K_{\mathcal{S}^m}+\mathcal{C}^m+\mathcal{F}^m))$$

for suitably large N, r is induced by the restriction

$$\oplus_{M\geq 0} h_*\mathcal{O}_{\mathcal{S}^m}(MN(K_{\mathcal{S}^m}+\mathcal{C}^m+\mathcal{F}^m)) \to \oplus_{M\geq 0} h_*\mathcal{O}_{S_0^m}(MN(K_{S_0^m}+C_0^m+F_0^m)).$$

We assert the higher direct image sheaves

$$\mathbb{R}^i h_* \mathcal{O}_{\mathcal{S}^m}(N(K_{\mathcal{S}^m}+\mathcal{C}^m+\mathcal{F}^m))$$

are torsion-free $\mathcal{O}_{\tilde{\Delta}}$-modules for $i > 0$ and N suitably divisible. This implies that

$$h_*\mathcal{O}_{\mathcal{S}^m}(N(K_{\mathcal{S}^m}+\mathcal{C}^m+\mathcal{F}^m)) \to h_*\mathcal{O}_{S_0^m}(N(K_{S_0^m}+C_0^m+F_0^m))$$

is surjective; its cokernel consists of elements of $\mathbb{R}^1 h_*\mathcal{O}_{\mathcal{S}^m}(N(K_{\mathcal{S}^m}+\mathcal{C}^m+\mathcal{F}^m))$ supported in the central fiber. Again, Kawamata-Viehweg vanishing and the key exact sequence reduces our assertion to the claim that $\mathbb{R}^1 h_* \mathcal{O}_{\mathcal{C}^m\cup\mathcal{F}^m}(N(K_{\mathcal{C}^m\cup\mathcal{F}^m}+\mathrm{Diff}))$ has no torsion over $\tilde{\Delta}$.

Now we establish our claim on the vanishing of torsion. Some care is required because the higher direct image *may* have torsion as an $\mathcal{O}_{\mathcal{S}^c}$-module, (e.g. if we are contracting a family of simple elliptic singularities). We apply the following lemma with $W = \mathcal{C}_m \cup \mathcal{F}_m$, $X = \mathcal{S}^c$, and L the log pluricanonical bundle:

Lemma 3.1 *Let $\gamma : W \to \tilde{\Delta}$ be a flat family of reduced nodal curves, $X \to \tilde{\Delta}$ another flat morphism, and $h : W \to X$ a proper morphism over $\tilde{\Delta}$. Let L be an invertible sheaf on X such that $h^*L = \mathcal{O}_W(N(K_W + A))$, where $N > 0$ and A is an effective $\mathbb{Q}$-divisor flat over $\tilde{\Delta}$ and supported in the smooth locus of γ. Then $\mathbb{R}^1 h_* h^* L$ is torsion-free as an $\mathcal{O}_{\tilde{\Delta}}$-module.*

proof: Pick a point p of the central fiber in the support of $\mathbb{R}^1 h_* h^* L$. We restrict to a neighborhood of p containing no other such points (the intersection of the support with each fiber is a finite set).

Let $E'_0 \subset h^{-1}(p)$ denote the union of the one-dimensional (i.e., non-imbedded) points of the fiber; it is a closed subscheme of $W_0 := \gamma^{-1}(0)$ and thus is a nodal reduced proper curve. By the standard classification results ([11] §3), each connected component $E_0 \subset E'_0$ has arithmetic genus zero or one. In the genus-zero case, E_0 is a tree of smooth rational curves intersecting A or $W_0 - E_0$ nontrivially. (Here $W_0 - E_0$ denotes the closure of the components other than E_0.) The pull-back of L to E_0 has no higher cohomology, so it does not contribute to $\mathbb{R}^1 h_* h^* L$. If the genus of E_0 is one, we have either a smooth genus one curve, a rational curve with one node, or a cycle of smooth rational curves, in each case intersecting A and $W_0 - E_0$ trivially. The pull-back of L to E_0 does then have higher cohomology, i.e., $h^1(E_0, L|E_0) = 1$. Let m denote the number of genus-one components in E'_0. The computation above shows that the dimension of the fiber of $\mathbb{R}^1 h_* h^* L$ at p is equal to m, and thus the dimension over $0 \in \tilde{\Delta}$ is also m. On the other hand, for each genus-one component E_0, there exists a connected component $E \subset W$ so that $E \cap W_0 = E_0$ (because E_0 is also a connected component of W_0). Hence the dimension of the fiber of $\mathbb{R}^1 h_* h^* L$ over the generic point of $\tilde{\Delta}$ is also m. Thus $\mathbb{R}^1 h_* h^* L$ is torsion-free as an $\mathcal{O}_{\tilde{\Delta}}$-module.□

4 An example: compact moduli for bielliptic curves

Here we work over an algebraically closed field of characteristic zero.

There are simple examples of log surfaces of general type such that the log canonical model has elliptic singularities which are not smoothable. Such log surfaces arise quite naturally in the study of bielliptic curves.

A smooth projective curve C (connected of genus $g \geq 2$) is *bielliptic* if it admits a degree-two finite map onto a smooth curve of genus one. We write $r : C \to E$ for the double cover, $B \subset E$ for the branch locus, and $V := r_* \mathcal{O}_C \cong \mathcal{O}_E \oplus \mathcal{L}^{-1}$ where $\mathcal{L}^2 = \mathcal{O}_E(B)$. The natural map $r^* V \to \mathcal{O}_C$ yields an imbedding $C \hookrightarrow \tilde{S} := \mathbb{P}(V^*)$. Let $\tau : \tilde{S} \to E$ be the projection map, $\mathcal{O}_{\tilde{S}}(+1)$ the relative hyperplane, and Σ the distinguished section of τ arising from the trace map. Note that $\mathcal{O}_{\tilde{S}}(C) = \tau^* \mathcal{O}_E(B)(+2), \mathcal{O}_{\tilde{S}}(\Sigma) = \mathcal{O}_{\tilde{S}}(+1), \Sigma \cap C = \emptyset$, and $\mathcal{O}_\Sigma(\Sigma) = \mathcal{L}^{-1}$.

Consider the log surface $(\tilde{S}, C + \Sigma)$ with logarithmic canonical bundle

$$K_{\tilde{S}} + C + \Sigma = (\tau^* \mathcal{L})(+1).$$

We have natural identifications

$$H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + C + \Sigma)) = H^0(E, V \otimes \mathcal{L}) = H^0(E, \mathcal{O}_E \oplus \mathcal{L}).$$

If $g > 2$, $\mathcal{O}_E \oplus \mathcal{L}$ is globally generated by g sections and the log canonical series yields a morphism

$$\mu : \tilde{S} \to \mathbb{P}^{g-1}.$$

If $g > 3$, μ is birational onto its image S, a cone over an elliptic curve of degree $g-1$, and contracts Σ to the vertex. For $g = 2$ or 3 a suitable multiple of the log canonical series has the same effect. To summarize:

Proposition 4.1 *Let C be a smooth projective curve of genus $g \geq 2$, which may be represented as a double cover of a genus-one curve E. Then C admits a natural imbedding into a projective surface S, which is birationally ruled over E and has a simple elliptic singularity of multiplicity $g-1$. The resulting log surface (S, C) is stable. For $g > 3$, S may be realized as the cone over E, where $E \subset \mathbb{P}^{g-2}$ is projectively normal of degree $g - 1$.*

The cone over a projectively normal elliptic curve of degree > 9 does not possess a smoothing ([15] ch. 9). Thus if $g > 10$, we obtain natural examples of stable log surfaces (S, C) with elliptic singularities that are not smoothable. The Main Theorem allows us to construct compact moduli spaces for such pairs, and we obtain a new compactification for the locus of bielliptic curves.

The cases where $g \leq 10$ are also quite interesting geometrically, although they do not require the full force of the Main Theorem. Here S deforms to a Del Pezzo surface T with $K_T^2 = g - 1$ and C deforms to some $D \in |-2K_T|$. The classification of Del Pezzo surfaces implies that T is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow-up of $\mathbb{P}^2$ at $10 - g$ (sufficiently general) points. In the second case, the blow-down $T \to \mathbb{P}^2$ maps D to a singular sextic plane curve, and we may regard the bielliptic curves as examples of curves admitting a g^2_6.

References

[1] V. Alexeev, Moduli spaces $M_{g,n}(W)$ for surfaces, *in* "Higher Dimensional Complex Varieties, (Trento, Italy 1994)," ed. M. Andreatta and T. Peternell, Walter de Guyter, Berlin, 1996, 1-22.

[2] A. Andreotti and E. Bombieri, Sugli omeomorfismi della varietá algebriche, *Annali della Scuola Norm. Sup., Serie III* **23** (1969), 430-450.

[3] D. Eisenbud, "Commutative Algebra with a View Toward Algebraic Geometry," Springer Verlag, New York, Berlin, Heidelberg, 1996.

[4] S. Greco and C. Traverso, On seminormal schemes, *Compositio Math.* **40** (1980), no. 3, 325-365

[5] A. Grothendieck and J. Dieudonné, Éléments de géométrie algébrique IV, Étude locale des schémas et des morphismes de schémas (Seconde Partie) *Institut des Hautes Études Scientifiques publications mathématiques* **24** (1965).

[6] B. Hassett, Local stable reduction of plane curve singularities, *Journal für die Reine und Angewandte Mathematik* **520** (2000), 169-194.

[7] B. Hassett, Stable log surfaces and limits of quartic plane curves, *manuscripta mathematica* **100** (1999), 469-497.

[8] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model program, *in* "Advanced Studies in Pure Mathematics," Vol. 10, ed. H. Morikawa, North-Holland Press, Amsterdam (1987), 283-360.

[9] J. Kollár, "Rational Curves on Algebraic Varieties," Springer-Verlag, Berlin, 1996.

[10] J. Kollár, Projectivity of complete moduli, *J. Differential Geom.* **32** (1990), no. 1, 235-268

[11] J. Kollár et al., Flips and Abundance for Algebraic Threefolds, *Astérisque* **211**, Société Mathématique de France, Montrouge 1992.

[12] J. Kollár and S. Mori with the collaboration of H. Clemens and A. Corti, Semistable minimal models, *in* "Birational Geometry of Algebraic Varieties," Cambridge University Press, Cambridge, 1998.

[13] J. Kollár and N. Shepherd-Barron, Threefolds and deformations of surface singularities, *Invent. math.* **91** (1988), 299-338.

[14] H. Matsumura, "Commutative ring theory," Cambridge University Press, Cambridge, 1986

[15] H. Pinkham, Deformations of algebraic varieties with G_m action, *Astérisque* **20**, Société Mathématique de France, 1974.

[16] D. S. Rim, Torsion differentials and deformation, *Trans. Am. Math. Soc.* **169** (1972), 257-278.

[17] E. Szabó, Divisorial log terminal singularities, *J. Math. Sci. Univ. Tokyo* **1** (1994), 631-639.

[18] C. Traverso, Seminormality and Picard group, *Annali della Scuola Norm. Sup., Serie III* **24** (1970), 585-595.