Local stable reduction of plane curve singularities

Brendan Hassett *

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Abstract

Consider a family of curves over the disc, with smooth fibers except for the central fiber over the origin. By the local stable reduction theorem, after suitable blow-ups and base changes we obtain a family such that the central fiber has reduced normal crossings. This stable central fiber has two parts: the proper transform of the original central fiber and the 'tail'. Which tails arise when the original central fiber is a given plane curve singularity? We address this question using the technique of stable reduction for log surfaces. For certain singularities, we find that weighted plane curves naturally arise as tails.

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1 Introduction

One important consequence of the log minimal model program is the existence of compactifications for moduli spaces of log surfaces of general type. These compactifications are discussed in [16] and [1], and may be regarded as higher-dimensional analogs of moduli spaces of pointed stable curves. Implicit in these compactifications is a notion of *stable reduction*, i.e. a procedure for modifying a family of log surfaces acquiring arbitrarily complicated

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singularities to obtain a family of log surfaces with certain prescribed singularities. For instance, after stable reduction the boundary of a log surface has only nodal singularities. Unfortunately, this process is not well-understood in practice, even in the simplest cases.

In this paper, we consider stable reduction from a local standpoint. Given a germ of a family of log surfaces degenerating to a singular pair (S_0, C_0), what kinds of surfaces appear in the course of stable reduction? The simplest cases to consider are germs of isolated plane curve singularities. One attractive aspect of these special cases is that they give insight into the following elementary question: which stable curves are the limits of plane curves? Another, more 'modern' formulation of this question is: which stable maps to the plane are smoothable?

This paper is organized as follows. First, we review some basic properties of isolated plane curve singularities, their resolutions, and their classification. In particular, we introduce singularities of *toric type* (e.g. $x^p - y^q = 0$) and *quasitoric type* (e.g. $xy(x^p - y^q) = 0$); these include all the simple ADE singularities. In section three, we describe the notion of local stable reduction for curves and log surfaces, and prove some elementary properties of the stable limits. In section four, we give a partial description of these stable limits, the Main Component Theorem (Theorem 4.1). This is made explicit for singularities of toric and quasitoric type. In the fifth section, we describe some degenerations of surfaces that arise naturally from the log minimal model program. In the sixth section, we use these degenerations to describe the 'tails' of stable limits arising from smoothings of singularities of toric and quasitoric type. The key statements are Theorems 6.2 and 6.3, Proposition 6.4, and Conjectures 6.5 and 6.6. Certain examples, like the simple ADE singularities, are discussed in more detail. In the last section, we show how these ideas shed light on the geometry of the equisingular deformation space. We also describe certain boundary components of the compatification for the moduli space of pairs $(\mathbb{P}^2, C)$, where C is a smooth plane curve of degree $d \geq 4$. In a subsequent paper [13], we enumerate the stable limits of pairs $(\mathbb{P}^2, C)$, where C is a smooth plane quartic.

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Throughout this paper, we work over $\mathbb{C}$.

2 Singularities of toric and quasitoric type

Throughout this paper, $C_0 \subset S_0 = \operatorname{Spec} \mathbb{C}[[x, y]]$ denotes the germ of an isolated reduced plane curve singularity at the origin $x = y = 0$.

Definition 2.1 An imbedded resolution of C_0 is a projective birational morphism of smooth surfaces $r : P_1 \to S_0$, with exceptional locus E, such that:

1. C_1, the proper transform of C_0, is smooth;
2. $E \cup C_1$ is a normal crossings divisor.

An imbedded resolution (P_1, C_1) is minimal if every other imbedded resolution factors through it.

Since r is a birational projective morphism of smooth surfaces, it may be obtained by a succession of blow-ups of smooth points. The exceptional locus E is thus a tree of smooth rational curves $\{E_i\}$. The intersection form on P_1 restricts to a negative definite unimodular quadratic form on $\oplus_i \mathbb{Z} E_i$.

Proposition 2.2 *Let $C_0 \subset \operatorname{Spec} \mathbb{C}[[x, y]]$ be a plane curve singularity. Its minimal imbedded resolution (P_1, C_1) is characterized by the fact that each irreducible exceptional curve $E_i \subset P_1$ satisfies at least one of the following:*

1. $E_i^2 < -1$;
2. $(C_1 + E - E_i)E_i > 2$.

Given an arbitrary imbedded resolution, the minimal resolution is obtained by blowing down all exceptional curves satisfying neither of these properties. Evidently, these curves may be blown down without introducing any unwanted singularities.

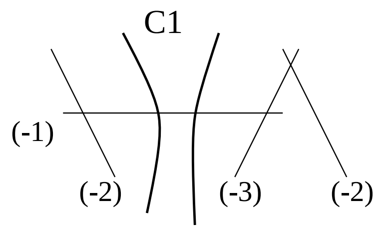

Figure 1: Topological type of $y^4 = x^{10}$

Let C_0 be an isolated plane curve singularity and let (P_1, C_1) be its minimal imbedded resolution. Let $E_1, E_2, \ldots, E_n$ denote the distinct irreducible components of the exceptional locus. Consider the graph $?_{C_0}$ with edges corresponding to the E_i and vertices corresponding to nonempty intersections $E_i \cap E_j$ with $i \neq j$. Clearly $?_{C_0}$ is a tree with two edges meeting at each vertex. The edges of $?_{C_0}$ have two natural labellings

$$\Sigma_{C_0} : E_i \to E_i^2 \qquad \Xi_{C_0} : E_i \to E_i C_1$$

by negative and non-negative integers respectively.

Definition 2.3 The topological type of C_0 is defined as the labelled tree $(?_{C_0}, \Sigma_{C_0}, \Xi_{C_0})$. Two curves C_0 and C_0' have the same topological type if there is a bijection $?_{C_0} \simeq ?_{C_0'}$ compatible with the labellings.

Of course, most triples $(?, \Sigma, \Xi)$ are not realized as the topological type of any singularity.

We shall often use diagrams to represent the topological type of the singularities we consider. On these diagrams, $?_{C_0}$ is represented as a tree of straight line segments, the values of Σ_{C_0} are indicated by the numbers in parentheses, and the values of Ξ_{C_0} by the number of intersections of the proper transform C_1 with the edges of the tree. For example, the topological type of $y^4 = x^{10}$ is given in figure 1.

Intuitively, an *equisingular deformation* of an isolated plane curve singularity C_0 is a deformation of C_0 such that all the fibers have the same topological type as C_0. A rigorous definition was given by Wahl [22].

Our notion of topological type coincides with other concepts used to characterize the topology of the singularity, like Puiseux pairs and multiplicity sequences. See [4] chapter 8 for an introduction to these notions; [23] and [24] (Part I) show that all these definitions coincide.

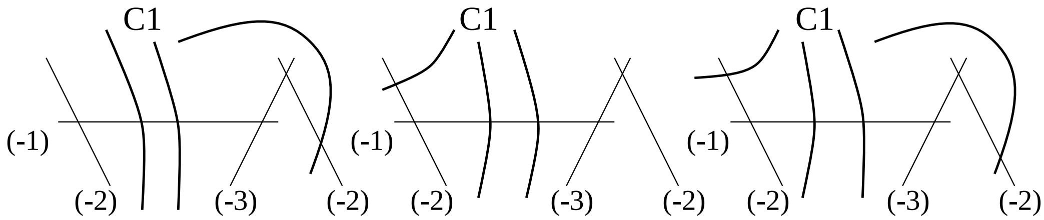

Figure 2: Topological types of $x(y^4-x^{10})=0, y(y^4-x^{10})=0, xy(y^4-x^{10})=0$

Definition 2.4 Let p and q be integers with $p \geq q > 1$. A plane curve singularity C_0 is said to be of toric type (p,q) if it has the same topological type as $x^p = y^q$. We say that C_0 is of quasitoric type if it has the same topological type as one of the following:

1. $x(x^p - y^q) = 0$ with $p > q$;
2. $y(x^p - y^q) = 0$ with $p > q$ and $q \not| p$;
3. $xy(x^p - y^q) = 0$ with $p > q$ and $q \not| p$;

The branches $x = 0$ and $y = 0$ are said to be distinguished.

The conditions on p and q guarantee that each of the equations above determines a distinct topological type. See figure 2 for diagrams representing certain singularities of quasitoric type.
Remark: Note that the simple singularities of types A_n, D_4, E_6, and E_8 are of toric type. Those of types $D_k(k > 4)$ and E_7 are of quasitoric type.

Proposition 2.5 *If C_0 is of type (p,q) then*

1. *$?_{C_0}$ is a chain;*
2. *$\Xi_{C_0} > 0$ for exactly one component E_n, the unique component with $\Sigma_{C_0} = -1$.*

Furthermore, E_n is an endpoint of the chain iff $q|p$.

Proposition 2.6 *If C_0 is of quasitoric type then*

1. *$?_{C_0}$ is a chain;*

2. *the proper transforms of the non-distinguished branches all intersect a single E_n, the unique component with $\Sigma_{C_0} = -1$;*

3. *the proper transforms of the distinguished branches meet components at the ends of the chain with $\Sigma_{C_0} < -1$ and $\Xi_{C_0} = 1$.*

To prove the propositions, we recast them in inductive form:

Lemma 2.7 *Assume p and q are integers with $p \geq q > 0$. The minimal resolution of $xy(x^p - y^q) = 0$ satisfies the following:*

1. *The union of the exceptional curves and the proper transforms of the distinguished branches $x = 0$ and $y = 0$ forms a chain, with the distinguished branches at opposite ends of the chain.*

2. *The non-distinguished branches all meet a single exceptional component E_n, which is the exceptional curve of the last blow-up in the resolution and the unique component with $\Sigma_{C_0} = -1$.*

3. *The proper transform of $x = 0$ (resp. $y = 0$) intersects E_n iff $q = p$ (resp. $q|p$).*

After blowing up once we obtain the local equations

$$xt(x^{p-q} - t^q) = 0 \quad sy(s^p y^{p-q} - 1) = 0$$

for the union of the exceptional divisors and the proper transform of C_0. Hence we proceed by induction on n, the number of blow-ups occurring in the minimal resolution. If $n = 1$ then $p = q$ and the result is evident. For the induction, we simply reinterpret the local equations above. Consider the exceptional curve E_1 and the proper transform of $y = 0$ as distinguished branches of $xt(x^{p-q} - t^q) = 0$. More precisely, set $x_1 = x$ and $y_1 = t$ if $p - q \geq q$ and $x_1 = t$ and $y_1 = x$ otherwise. The non-distinguished branches of this new singularity correspond with those of $xy(x^p - y^q) = 0$.

We extract the results on the minimal resolution. It is clear that only the last exceptional divisor E_n satisfies $E_n^2 = -1$, and all the non-distinguished branches meet E_n. This yields (2). The proper transform of $x = 0$ (i.e. $s = 0$) intersects E_1, but none of the other exceptional components, and thus is at one end of the chain. By induction, the union of the proper transform of $y = 0$ (i.e. $t = 0$) and the exceptional locus also forms a chain, with E_1 and the proper transform of $y = 0$ at opposite ends. Thus (1) follows. The proof of (3) is straightforward, and left to the reader. □

Proposition 2.8 *Let C_0 be a singularity with minimal resolution satisfying the conditions of Proposition 2.5 or 2.6. Then there exist integers p and q and a square-free weighted-homogeneous polynomial $g_0(x,y) = x^p + \ldots + y^q$ (where x and y have weights q and p respectively) so that C_0 admits an equisingular specialization to one of the following:*

1. *$g_0(x,y) = 0$ with $p \geq q > 1$;*
2. *$xg_0(x,y) = 0$ with $p > q > 1$;*
3. *$yg_0(x,y) = 0$ with $p > q > 1$ and $q \not| p$;*
4. *$xyg_0(x,y)$ with $p > q > 1$ and $q \not| p$.*

Again, the proposition is proved by induction on the number of exceptional curves in a minimal resolution. Number these curves $E_1, \ldots, E_n$ in the order they appear in the minimal resolution. If $n = 1$, then the singularity is evidently topologically equivalent to $x^p = y^p$ for some p. Precisely, for some coordinates x and y transverse to all the branches of $C_0 = \{f = 0\}$, the homogeneous part of f with minimal degree takes the form

$$g_0(x,y) = x^p + \ldots + y^p.$$

Here g_0 is square-free, and $f = 0$ has the same topological type as $g_0 = 0$.

Now we establish the inductive step. Let C_0 be a singularity with minimal resolution containing $n > 1$ exceptional divisors and satisfying the conditions of the Propositions 2.5 or 2.6. Blow up once, and consider $C' = C_0' \cup E_1$, the union of the proper transform and the exceptional curve. This has (at most) two distinct singularities r_1 and r_2; r_1 is where the non-distinguished branches meet E_1 and r_2 is where E_1 meets a distinguished branch D. By induction, we know that the singularity of C' at r_1 specializes to one of the equations above, with E_1 satisfying $x' = 0$ or $y' = 0$. Of course, C_0 is obtained by contracting one of these lines, and the image of the branches through r_1 satisfies equations 1 or 3 above. However, the image of the distinguished branch D (if it exists) satisfies some equation $x +$ higher order terms $= 0$, so the equation of C_0 takes the form

$$yg_0(x,y)(x + \text{h.o.t.}) = 0 \text{ or } g_0(x,y)(x + \text{h.o.t.}) = 0.$$

Since D meets each of the other branches generically, its equation can be specialized to $x = 0$ without changing the topological type of C_0. $\square$

Essentially the same proof yields the following corollary:

Corollary 2.9 *Let C_0 be of toric or quasitoric type. Then there exist local coordinates x and y and positive integers p and q such that C_0 has one of the following analytic equations*

1. $g_0(x,y) + h.o.t. = 0$, *with* $p \geq q > 1$;
2. $x(g_0(x,y) + h.o.t.) = 0$, *with* $p > q > 1$;
3. $y(g_0(x,y) + h.o.t.) = 0$, *with* $p > q > 1$ *and* $q \not| p$;
4. $xy(g_0(x,y) + h.o.t.) = 0$, *with* $p > q > 1$ *and* $q \not| p$;

where $g_0(x,y) = x^p + \ldots + y^q$ is weighted-homogeneous and square-free. (In particular, C_0 is semiquasihomogeneous.)

The weights on x and y are the same as in Proposition 2.8. Of course, 'higher order' means higher degree with respect to this grading.

The equations of Proposition 2.8 all define singularities of toric or quasitoric type. We therefore obtain the following corollary:

Corollary 2.10 *Each isolated plane curve singularity satisfying the conditions of Proposition 2.5 (resp. 2.6) is of toric (resp. quasitoric) type.*

Note also that Proposition 2.8 implies that each unibranch singularity of toric type admits an equisingular specialization to $x^p = y^q$. This is a well known result of Zariski (chapter III §2.3 of [25]), and serves as the foundation for the construction of moduli spaces for unibranch singularities of toric type. For a systematic study of parameter/moduli spaces for unibranch curve singularities, see the survey [11] and the lecture notes of Zariski (with appendix by Teissier) [25].

We will also use the following result of Wahl [22] on equisingular deformation spaces:

Theorem 2.11 *Let C_0 be an isolated plane curve singularity. Then the equisingular deformation space of C_0 is smooth. For the singularity $x^p = y^q$ it may be represented as*

$$y^q = x^p + \sum t_{ij} x^i y^j$$

where $0 \leq i \leq p-2$, $0 \leq j \leq q-2$, *and* $qi + pj \geq pq$.

By Proposition 2.8, we may regard the singularities appearing in this family as a parameter space for the singularities of type (p, q). See the papers [11] and [19] for an elaboration of this point of view. With this as motivation, we make the following definition:

Definition 2.12 The codimension $c(p, q)$ of the singularities of type (p, q) is the codimension of the family in Theorem 2.11 in the versal deformation space of $y^q = x^p$.

This will be computed in section 7.

3 Basic properties of local stable reduction

The stable reduction theorem for curves was proved by Deligne and Mumford [6]. Another proof may be found in Artin and Winters [3], and a good general introduction is the recent book of Harris and Morrison [12], chapter 3C.

Let C_0 be the germ of an isolated plane curve singularity. Set $\Delta = \operatorname{Spec}\mathbb{C}[[t]]$ or $\{t : |t| < 1\}$. Let $\pi : \mathcal{C} \to \Delta$ be a smoothing of C_0, i.e. π is flat, $C_0 = \pi^{-1}(0)$, and $C_t := \pi^{-1}(t)$ is smooth for $t \neq 0$. Local stable reduction is a procedure for obtaining a family of nodal curves $\pi^c : \mathcal{C}^c \to \tilde{\Delta}$. It involves the following steps:

1. Carry out semistable reduction, following [15]. We find a base change $\tilde{\Delta} \to \Delta$ and a resolution $\rho : \tilde{\mathcal{C}} \to \mathcal{C} \times_\Delta \tilde{\Delta}$ such that the central fiber $\tilde{C}_0$ is a reduced normal crossings divisor.

2. Obtain $\mathcal{C}^c$ by taking the log canonical model of $(\tilde{\mathcal{C}}, \tilde{C}_0)$ relative to the morphism ρ.

We use a relative version of the minimal model program, i.e., we take minimal models relative to a proper morphism of normal varieties. A good exposition may be found in [14].

Over the punctured disc $\tilde{\Delta}^*$, $\mathcal{C}^c$ coincides with the pull-back of the original family. The central fiber $C_0^c = C_1 \cup C_T$ where C_1 is the normalization of C_0 and $C_T = C_0^c - C_1$. (We are assuming C_0 is not a node.) Let $C_1 \cap C_T = \{p_1, \ldots, p_b\}$, where b is the number of branches of C_0. Note that the pointed curve $(C_1, p_1, \ldots, p_b)$ depends only on C_0, and not on the choice of smoothing.

Definition 3.1 The pointed curve $(C_T, p_1, \ldots, p_b)$ is called the tail of the local stable reduction.

Proposition 3.2 *Let C_0 be an isolated plane curve singularity and let $\mathcal{T}_{C_0}$ denote all the tails obtained from various smoothings of C_0. These tails are pointed stable curves and $\mathcal{T}_{C_0} \subset \overline{\mathcal{M}}_{\gamma,b}$ is a connected closed subvariety of the moduli space.*

Let $\mathcal{C} \to \Delta$ be a smoothing of C_0. After any base change $t = u^N$, the family $\mathcal{C} \times_\Delta \tilde{\Delta}$ remains normal. If $\mathcal{C}^c$ is the stable reduction, then we have a morphism of normal varieties $\phi : \mathcal{C}^c \to \mathcal{C} \times_\Delta \tilde{\Delta}$. The tail C_T is the fiber over the point $x = y = u = 0$ so it is connected by Zariski's Main Theorem. Since $(\mathcal{C}^c, C_0^c)$ is the log canonical model of $\mathcal{C} \times_\Delta \tilde{\Delta}$ relative to ρ, it follows that C_T is nodal and $\omega_{C_T}(p_1 + \dots p_b)$ is ample. In particular, $(C_T, p_1, \dots, p_b)$ is a pointed stable curve.

We now show that $\mathcal{T}_{C_0}$ is closed in the corresponding moduli space of stable pointed curves. Let Ver_{C_0} be the versal deformation space of C_0 and let Λ be a linear series globalizing Ver_{C_0}, i.e. some member $L_0 \in \Lambda$ has a single singularity p analytically equivalent to C_0, and the induced analytic map $\widehat{\Lambda}_{L_0} \to \mathrm{Ver}_{C_0}$ is an isomorphism. For instance, we may take Λ to be a linear system of plane curves with sufficiently large degree. If the generic member of Λ has genus g, we obtain a rational map $\Lambda \dashrightarrow \overline{\mathcal{M}}_g$. This map has indeterminacy at the point L_0 and the total transform of L_0 is a closed subvariety W of the moduli space. Since $(C_1, p_1, \dots, p_b)$ depends only on L_0, we find that $W \simeq \mathcal{T}_{C_0}$. $\square$

We want to describe $\mathcal{T}_{C_0}$ explicitly for certain singularities C_0. To do this, we consider pairs (S_0, C_0) where $S_0 = \mathrm{Spec}\, \mathbb{C}[[x,y]]$ and $C_0 \subset S_0$. We consider the smoothing $\mathcal{C} \to \Delta$ as a subscheme of the threefold $\mathcal{S} := S_0 \times \Delta$. Local stable reduction for the family of *pairs* $\pi : (\mathcal{S}, \mathcal{C}) \to \Delta$ involves the following steps:

1. Carry out semistable reduction, following [15] and [5] Theorem 7.17. We find a base change $\tilde{\Delta} \to \Delta$ and a resolution

$$\rho : (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}) \to (\mathcal{S} \times_\Delta \tilde{\Delta}, \mathcal{C} \times_\Delta \tilde{\Delta})$$

   such that the central fiber $\tilde{S}_0$ is reduced and $\tilde{\mathcal{C}} \cup \tilde{S}_0$ is a normal crossings divisor.

2. Obtain $(\mathcal{S}^c, \mathcal{C}^c)$ by taking the log canonical model of $(\tilde{\mathcal{S}}, \tilde{\mathcal{C}} + \tilde{S}_0)$ relative to the morphism ρ.

The central fiber (S_0^c, C_0^c) is called the *local stable limit.* Local stable reductions satisfy the following:

Proposition 3.3 *C_0^c is a Weil divisor, (S_0^c, C_0^c) is semilog canonical, and hence C_0^c is nodal. Furthermore, $K_{\mathcal{S}^c} + \mathcal{C}^c$ is Cartier along $\mathcal{C}^c$ and restricts to $K_{\mathcal{C}^c}$.*

For the convenience of the reader we sketch the proof (see also §2 of [1].) The pair $(\mathcal{S}^c, \mathcal{C}^c + S_0^c)$ is log canonical so C_0^c is generically reduced. Since the local stable reduction agrees with our original family away from the central fiber, the exceptional divisors with log discrepancy < 1 are supported in the central fiber. We obtain that $(\mathcal{S}^c, \mathcal{C}^c)$ is canonical hence $\mathcal{S}^c$ (and also S_0^c) is Cohen-Macaulay. Furthermore, $\mathcal{C}^c$ is also Cohen-Macaulay [17] 17.5 and thus C_0^c has no imbedded points. Hence we may apply adjunction ([17] 17.2) to obtain that (S_0^c, C_0^c) is semilog canonical. The fact that C_0^c is nodal follows from the classification of semilog canonical singularities [17] 12.2.4.

Now $K_{\mathcal{S}^c} + \mathcal{C}^c$ is Cartier along $\mathcal{C}^c$ provided that $K_{S_0^c} + C_0^c$ is Cartier along C_0^c. (The log canonical bundle necessarily restricts well to the central fiber.) Assume that $K_{S_0^c} + C_0^c$ is *not* Cartier at $p \in C_0^c$, and let $r > 1$ be the corresponding index. Restricting to a small analytic neighborhood about p, we can take an index-one cover $i : (T, D) \to (S_0^c, C_0^c)$. Note that i is an r-fold cover totally ramified at p and $K_T + D$ is Cartier. It follows that $i^* K_C = K_D(-(r-1)p)$, $(K_T + D)|D = K_D$, and hence $K_{S_0^c} + C_0^c$ (when restricted to C_0^c) has different $\frac{r-1}{r}$ at p ([17] 16.6). This contradicts the flatness of $\mathcal{C}_0^c \to \tilde{\Delta}$. The isomorphism $(K_{\mathcal{S}^c} + \mathcal{C}^c)|\mathcal{C}^c = K_{\mathcal{C}^c}$ follows by the adjunction formula ([17] 16.4.2).□

Corollary 3.4 *The boundary $\mathcal{C}^c$ coincides with the local stable reduction of $\mathcal{C} \to \Delta$ described above.*

We now set some notation. The central fiber $S_0^c = S_1 \cup S_T$ where S_1 is the proper transform of S_0 and S_T the closure of $S_0^c - S_1$. We write $S_T = S_2 \cup \ldots \cup S_n$ where the S_j are irreducible, and $C_j = S_j \cap C_0^c$ for the part of the tail C_T lying on S_j; it is possible that $C_j = \emptyset$ for some j. We also write $B_j = S_j \cap \overline{S_0^c - S_j}$ and $B_T = S_T \cap S_1$; note that $B_T = B_1$.

Definition 3.5 The log surface $(S_T, C_T + B_T)$ is called the surface tail of the local stable reduction.

Proposition 3.6 *Let C_0 be an isolated plane curve singularity and let $\mathcal{T}'_{C_0}$ denote all the surface tails obtained from various smoothings of C_0. These tails are stable log surfaces and form a connected closed subvariety of the appropriate moduli space. There is a natural surjective forgetting morphism $\mathcal{T}'_{C_0} \to \mathcal{T}_{C_0}$.*

This proof follows the one for Proposition 3.2, using basic properties of stable reduction for log surfaces. We shall prove that $(S_1, B_1 + C_1)$ depends only on C_0 and not on the choice of smoothing in Theorem 4.1. Since $\phi : \mathcal{S}^c \to \mathcal{S} \times_\Delta \tilde{\Delta}$ is projective and birational, it is a blow-up of a suitable ideal sheaf. The support of the exceptional divisor is S_T. Because $\mathcal{S}^c$ is Cohen-Macaulay (Proposition 3.3), the exceptional divisor (a Cartier divisor) is also Cohen-Macaulay, as is S_T. The pair $(S_T, C_T + B_T)$ is therefore semilog canonical. The log canonical bundle $K_{\mathcal{S}^c} + \mathcal{C}^c$ is ample relative to ϕ. By the adjunction formula ([17] 16.4.3) $(K_{\mathcal{S}^c} + \mathcal{C}^c)|S_T = K_{S_T} + C_T + B_T$, so $(S_T, C_T + B_T)$ is stable. This establishes the first claim. The second claim is proved precisely as in Proposition 3.2. The existence of the forgetting morphism follows from Corollary 3.4. □

Remark: Can the forgetting map $\mathcal{T}'_{C_0} \to \mathcal{T}_{C_0}$ have positive dimensional fibers?

4 Computing local stable reductions

4.1 The Main Component Theorem

We give a simple recipe for computing the component S_1 of the local stable reduction:

Theorem 4.1 (Main Component Theorem) *Let $C_0 \subset S_0$ be an isolated plane curve singularity, (S_0^c, C_0^c) the local stable limit of some smoothing of C_0, $S_1 \subset S_0^c$ the proper transform of S_0, $B_1 = S_1 \cap \overline{S_0^c - S_1}$, and $C_1 = C_0^c \cap S_1$. There is a birational morphism $\sigma : S_1 \to S_0$ such that the divisor $K_{S_1} + B_1 + C_1$ is ample relative to σ. Indeed, $(S_1, B_1 + C_1)$ is obtained from an imbedded resolution $r : P_1 \to S_0$ by taking the log canonical model of $(P_1, E + C_1)$ relative to r.*

The minimal model program gives us an induced map $\phi : \mathcal{S}^c \to \mathcal{S} \times_\Delta \tilde{\Delta}$ which is an isomorphism except over the origin $0 = (x = y = t = 0)$. The exceptional divisors of ϕ correspond to $S_0^c \backslash S_1$, and ϕ restricts to a birational morphism $\sigma : S_1 \to S_0$. By the adjunction formula ([17] ch. 16.4.3), we have

$$K_{S_1} + B_1 + C_1 = (K_{\mathcal{S}^c} + S_0^c + \mathcal{C}^c)|S_1$$

which is σ-ample because $(\mathcal{S}^c, \mathcal{C}^c)$ is the log canonical model.

To prove the second part of the theorem, we must verify that B_1 equals the image of the divisor E defined above. In other words, we must show that every σ-exceptional divisor is contained in some ϕ-exceptional divisor. Assume this is not the case, so the image of some component $E' \subset E$ in S_1 is not contained in any of the ϕ-exceptional divisors. Then at the generic point of image of E', which is a smooth point of $\mathcal{S}^c$, ϕ has one-dimensional exceptional locus. However $\mathcal{S} \times_\Delta \tilde{\Delta}$ is also smooth, and the exceptional locus of any birational morphism of smooth varieties has pure codimension one. □

Generally, the map $\sigma : S_1 \to S_0$ can be expressed as the blow-up of a natural ideal sheaf. Consider the log canonical divisor

$$D :\equiv h^*(K_{S_1} + C_1 + B_1) = K_{P_1} + C_1 + \sum_{i=1}^{n} a_i E_i$$

on the resolution $h : P_1 \to S_1$. Let N be the smallest positive number such that ND is integral, globally generated relative to r, and generates $\oplus_{m \geq 0} r_* \mathcal{O}_{P_1}(mND)$ as an $\mathcal{O}_{S_0}$ algebra. (In practice, it often suffices to assume that ND is integral.) The *log adjoint ideal* $\mathcal{A}_{C_0}$ is defined as the push-forward $r_* \mathcal{O}_{P_1}(ND) \subset \mathcal{O}_{S_0}$. This can interpretted geometrically: the images in S_0 of the members of $|ND|$ yield a linear series on S_0, the *log adjoint series.* The base locus of the log adjoint series is the log adjoint ideal. Using basic properties of blowing up, we obtain the following:

Proposition 4.2 *Retain the notation of Theorem 4.1. Then $\sigma : S_1 \to S_0$ coincides with the blow-up of S_0 along the log adjoint ideal $\mathcal{A}_{C_0}$.*

Finally, we should point out when local stable reductions coincide with global stable reductions. Assume we are given a family of log surfaces $(\mathcal{S}, \mathcal{C}) \to \Delta$, such that the fibers over the punctured disc Δ^* are smooth and stable. In particular, $K_{S_t} + C_t$ is ample for $t \neq 0$. Furthermore, assume that S_0 is smooth but that C_0 has an isolated singularity. We can apply *local* stable reduction to obtain a family $(\mathcal{S}^c, \mathcal{C}^c) \to \tilde{\Delta}$ dominating $(\mathcal{S} \times_\Delta \tilde{\Delta}, \mathcal{C} \times_\Delta \tilde{\Delta})$, so that the central fiber (S_0^c, C_0^c) has semilog canonical singularities. When is this also the *global* stable reduction? Of course, (S_0^c, C_0^c) is semilog canonical and global stable reductions are unique, so it suffices to check whether $K_{S_0^c} + C_0^c$ is ample. This is clearly valid for each irreducible component of S_0^c that is exceptional for the map $\phi : \mathcal{S}^c \to \mathcal{S} \times_\Delta \tilde{\Delta}$. It follows that the local stable reduction is the global stable reduction whenever $K_{S_1} + B_1 + C_1$ is ample (using the notation of Theorem 4.1).

4.2 Computing log canonical models

We recall some results on surface singularities. Consider the cyclic quotient surface singularity arising from the group action

$$(x, z) \to (\epsilon^a x, \epsilon z)$$

where ϵ is a primitive rth root of unity, $1 \leq a < r$, and $(a, r) = 1$. The minimal resolution of this singularity can be described quite explicitly. It consists of a chain of rational curves $E_1, E_2, \ldots, E_n$ with self-intersections $E_i^2 = -b_i$

$$\overset{-b_1}{\circ} - \cdots - \overset{-b_n}{\circ}.$$

The b_i are computed from the continued fraction representation

$$\frac{r}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ldots}}.$$

The proper transforms of $x = 0$ and $z = 0$ meet the first and last exceptional curves of this chain. (See [10] §2.6 for a good exposition of this subject.) We shall use the fraction $\frac{r}{a}$ to label the cyclic quotient. For instance, the singularity A_g corresponds to the fraction $\frac{g+1}{g}$. This notation depends on the choice of one of two possible orderings for the exceptional divisors of the minimal resolution.

We now describe the log minimal and log canonical models associated to singularities of toric type.

Proposition 4.3 *Retain the notation of Theorem 4.1, and assume further that C_0 is of type (p, q). Then S_1 is obtained from P_1 by contracting all the exceptional curves disjoint from C_1, and B_1 is the image of the unique exceptional curve meeting C_1. In particular, we have*

1. *If $p = q > 2$ then S_1 is smooth and equal to the minimal imbedded resolution of C_0.*

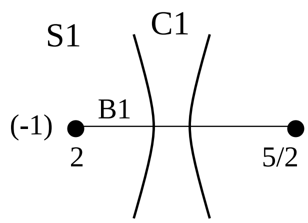

Figure 3: Log canonical model of $y^4 = x^{10}$

2. *If $q|p$ but $q \neq p$, then S_1 has one cyclic quotient singularity along B_1 (but disjoint from C_1).*

3. *If $q \not| p$, then S_1 has two cyclic quotient singularities along B_1 (but disjoint from C_1).*

Finally, $(S_1, C_1 + B_1)$ has log terminal singularities and thus coincides with the log minimal model (except in the case where $p = q = 2$).

See figure 3 for a schematic representation of such a log canonical model. The number in parentheses is the self-intersection of B_1 in the minimal resolution of S_1. The fraction denotes the corresponding quotient singularity.

We assume that C_0 is not an ordinary node; if C_0 is a node then the proposition is evident. Here we use Proposition 2.5 and Lemma 2.7. Let E_n be the unique exceptional curve meeting C_1. We claim that every exceptional curve besides E_n is contracted when we take the log minimal model; B_1 corresponds to the image of E_n. Let F be the union of the exceptional curves besides E_n; it has zero (resp. one, resp. two) connected components if $q = p$ (resp. $q|p$, resp. $q \not| p$). Each such component is a chain of exceptional curves, with one curve at the end of the chain intersecting E_n once. Such chains are log terminal [17] ch.3, so they are necessarily contracted when we take the log minimal model. As we have seen, contracting such a chain of rational curves yields a cyclic quotient singularity.

On the other hand, E_n may not be contracted even after taking the log canonical model. If we contracted E_n, then the boundary would have singularities worse than normal crossings, which contradicts the fact that the singularities are log canonical([17] chapter 3). □

Similar results hold for singularities of quasitoric type:

Proposition 4.4 *Retain the notation of Theorem 4.1, and assume further that C_0 is of quasitoric type. Then S_1 is obtained from P_1 by contracting*

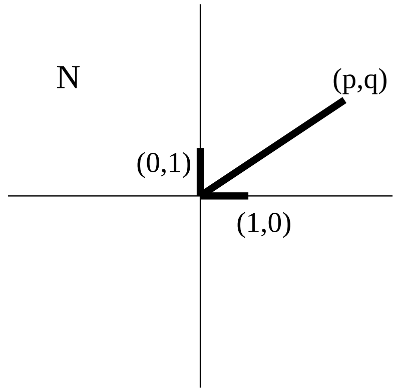

Figure 4: Toroidal representation of the log canonical model S_1

all the exceptional curves disjoint from the nondistinguished branches of C_1; B_1 is the image of the unique exceptional curve meeting those branches. S_1 has (at most) two cyclic quotient singularities along B_1; each distinguished branch passes through one of these quotient singularties.

The proof uses Proposition 2.6 and the classification of log canonical singularities.

Remark: For singularities of toric and quasitoric type $\sigma : S_1 \to S_0$ has a simple toroidal interpretation. Choose coordinates x and y and integers p and q satisfying the conclusions of Corollary 2.9. Let $b = \gcd(p,q)$ and consider the ideal $\mathcal{A}$ generated by the weighted monomials $\{x^i y^j : pj + qi \geq \frac{pq}{b}\}$. Blowing up $\mathcal{A}$ yields a resolution of C_0, imbedded in a normal surface with cyclic quotient singularities. This corresponds to the toric blow-up represented by the fan depicted in figure 4. The horizontal and vertical vectors in the fan correspond to the proper transforms of $y = 0$ and $x = 0$ respectively. The propositions above imply that this toroidal blow-up equals S_1, which gives a combinatorial method for describing the singularities of S_1 (see [10] §2.6). Finally, we should point out that $\mathcal{A}$ is precisely the log adjoint ideal $\mathcal{A}_{C_0}$ defined in the previous section.

5 Constructions of limiting surfaces

5.1 Geometry of weighted projective spaces

We recall some basic facts about the weighted projective plane $\mathbb{P}(r,s,1)$ (see [7] for a comprehensive survey). It may be represented as the toric surface obtained from the complete fan with one-dimensional faces $(1,0),(0,1)$, and $(-r,-s)$ [10] §2.2. In particular, $\mathbb{P}(r,s,1)$ is isomorphic to $\mathbb{P}(nr,ns,1)$, so we assume that r and s are relatively prime and $r \geq s$. Let B_2 denote the distinguished divisor corresponding to the face $(-r,-s)$, and let D_1 and D_2 be the divisors corresponding to $(1,0)$ and $(0,1)$. These satisfy the rational equivalences $rB_2 \equiv D_1$ and $sB_2 \equiv D_2$; the intersection form is

	D_1	D_2	B_2
D_1	r/s	1	$1/s$
D_2	1	s/r	$1/r$
B_2	$1/s$	$1/r$	$1/rs$

.

The Picard group is generated by $L = rsB_2$ ([10] 3.3) and the canonical divisor $K_{\mathbb{P}(r,s,1)} = -(B_2 + D_1 + D_2)$ ([10] 4.4).

If $r > s = 1$ (resp. $s > 1$) then $\mathbb{P}(r,s,1)$ has one (resp. two) cyclic quotient singularities, occuring at the intersection $D_2 \cap B_2$ (resp. at the intersections $D_2 \cap B_2$ and $D_1 \cap B_2$). We should point out that $(\mathbb{P}(r,s,1), B_2 + D_1 + D_2)$ is always log canonical.

We describe some natural 'linear series' on $\mathbb{P}(r,s,1)$, i.e. spaces of sections of $\mathcal{O}_{\mathbb{P}(r,s,1)}(nB_2)$ for $n \geq 0$. Of course, these are not generally invertible sheaves, but rather rank-one reflexive sheaves. The dimensions of their global sections are determined by the isomorphism of graded rings [7] 1.4

$$\operatorname{Proj} \mathbb{C}[x^r, y^s, z] \cong \oplus_{n \geq 0} H^0(\mathcal{O}_{\mathbb{P}(r,s,1)}(nB_2)). \qquad (1)$$

If $r > s$ then $h^0(\mathcal{O}_{\mathbb{P}(r,s,1)}(D_2)) = 2$ and $h^0(\mathcal{O}_{\mathbb{P}(r,s,1)}(D_1)) = \lfloor \frac{r}{s} \rfloor + 2$. If $s = 1$ then $\mathbb{P}(r,1,1)$ is the cone over a rational normal curve of degree r in $\mathbb{P}^r$; $B_2 \equiv D_2$ is the class of a ruling and D_1 is the class of a hyperplane section. If $s > 1$ then $\mathbb{P}(r,s,1)$ is still ruled by the divisors linearly equivalent to D_2.

For each positive integer b, the Cartier divisor bL is very ample and its higher cohomologies vanish ([10] pp. 70-74). It follows that the general member of $|bL|$ is smooth of genus $\frac{b}{2}(brs - r - s - 1) + 1$, contained in the smooth locus of $\mathbb{P}(r,s,1)$, and $h^0(bL) = \frac{b}{2}(brs + r + s + 1) + 1$. Finally,

$|bL|$ maps B_2 to a rational normal curve of degree b. Thus there is a smooth member of $|bL|$ passing through any collection of b smooth points of $\mathbb{P}(r, s, 1)$ lying on B_2.

If $r > s$ then a generic Weil divisor $C_2 \in |bL + D_2|$ is smooth of genus $\frac{b}{2}(brs+s-r-1)$, but it contains the singularity at $B_2 \cap D_2$. In the local divisor class group at $B_2 \cap D_2$ C_2 is linearly equivalent to D_2, so $(\mathbb{P}(r, s, 1), C_2 + B_2)$ is log canonical. If $s > 1$ then the generic member $C_2 \in |bL + D_1|$ is smooth of genus $\frac{b}{2}(brs - s + r - 1)$, but it contains the singularity at $B_2 \cap D_1$; again $(\mathbb{P}(r, s, 1), C_2 + B_2)$ is log canonical. Similarly, the generic member $C_2 \in |bL + D_1 + D_2|$ is smooth of genus $\frac{b}{2}(brs + r + s - 1)$, contains both singularities $B_2 \cap D_1$ and $B_2 \cap D_2$, and $(\mathbb{P}(r, s, 1), C_2 + B_2)$ is log canonical.

We will use the following result for dimension counts later in this paper. It is a consequence of the cohomology computations of [7] §2, or of the description of the automorphisms of $\mathbb{P}(r, s, 1)$ in terms of the automorphisms of the corresponding graded algebras.

Proposition 5.1 *Let r and s be relatively prime integers with $r \geq s > 0$. Then the automorphism group of $\mathbb{P}(r, s, 1)$ has dimension $\lfloor \frac{r}{s} \rfloor + 4$ if $s \neq 1$, dimension $r + 5$ if $s = 1$ and $r \neq 1$, and dimension 8 if $r = s = 1$.*

5.2 Construction of degenerations

In this section, we construct certain surfaces S_0^c containing tails arising from plane curve singularities C_0 of toric and quasitoric type. We describe how these surfaces are obtained as central fibers of birational modifications of the trivial families $\mathcal{S} = S_0 \times \Delta$.

Theorem 5.2 *Let $C_0 \subset S_0$ be a singularity toric or quasitoric type which is not an ordinary node. There exists a surface S_0^c with the following properties*

1. *S_0^c has two irreducible components S_1 and S_2 and has semilog terminal singularities.*

2. *S_1 is the log canonical model of the imbedded resolution of C_0 (cf. Theorem 4.1).*

3. *S_2 is isomorphic to a weighted projective space $\mathbb{P}(p, q, 1)$.*

4. *S_0^c can be smoothed to S_0 so that the total space $\mathcal{S}^c$ has terminal singularities.*

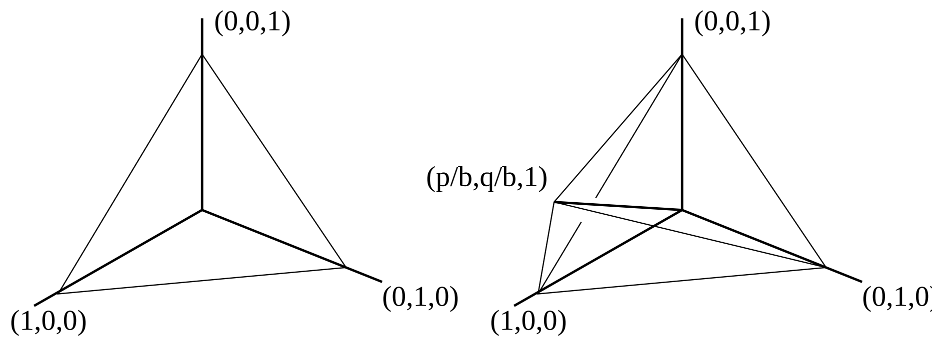

Figure 5: The toroidal blow-up $\phi : \mathcal{S}^c \to \mathcal{S}$

Choose analytic coordinates x and y and integers p and q satisfying the conclusions of Corollary 2.9. We assume that $p \geq q$ and set $b = \gcd(p, q)$. Consider the ideal $\mathcal{I}^c \subset \mathcal{O}_{\mathcal{S}}$ generated by t and generators of the log adjoint ideal $\mathcal{A}_{C_0}$. By the remark at the end of section 4.2, this may be represented torically as $\{t, x^i y^j : pj + qi \geq \frac{pq}{b}\}$. Let $\phi : \mathcal{S}^c \to \mathcal{S}$ be the blow-up of $\mathcal{S}$ along $\mathcal{I}^c$. We express this torically as follows: represent $\mathcal{S}$ by a fan with one cone generated by the vectors $v_1 = (1, 0, 0), v_2 = (0, 1, 0)$, and $v_3 = (0, 0, 1)$, corresponding to the divisors $y = 0, x = 0$, and $t = 0$. Then ϕ corresponds to the toric blow-up represented by a fan with three cones, each spanned by two of the v_i along with the vector $(\frac{p}{b}, \frac{q}{b}, 1)$. (See figure 5.)

Basic properties of divisors in toric varieties imply that the central fiber $S_0^c \subset \mathcal{S}^c$ (defined by $t = 0$) is reduced. By the description of orbit closures in [10] 3.1, it consists of two components isomorphic to the toric blow-up $S_1 \to S_0$ and the weighted projective space $S_2 = \mathbb{P}(p/b, q/b, 1)$ respectively. Each of these surfaces has (up to) two quotient singularities, lying along their common intersection $B_1 = B_2$. Away from these quotient singularities, S_1 and S_2 intersect in normal crossings.

We claim the threefold $\mathcal{S}^c$ has terminal singularities; this implies (by adjunction) that S_0^c is semilog terminal. It suffices to prove that an affine toric threefold U with fan generated by $(1, 0, 0), (0, 1, 0), (1, s, r)$ (with r, s relatively prime) has terminal singularities. Each affine neighborhood of $\mathcal{S}^c$ has this form after an appropriate permutation. We represent this as a quotient singularity of index r. Set $w_1 = (1, 0, 0), w_2 = (0, 1, 0), w_3 = (1, s, r)$ and let N_1 be the corresponding lattice spanned by these elements. Now N_1

is an index r subgroup of the full torus N and

$$N = N_1 + \frac{\mathbb{Z}}{r}(w_3 - w_1 - rw_2) = N_1 + \frac{\mathbb{Z}}{r}((r-1)w_1 + (mr-s))w_2 + w_3)$$

where $m \in \mathbb{N}$ and $0 < mr - s < r$. The toric variety with torus corresponding to N_1 and fan consisting of the cone generated by w_1, w_2, w_3 is just $\mathbb{A}^3$. The toric variety $U \cong \mathbb{A}_3/(\mathbb{Z}/r\mathbb{Z})$, where $\mathbb{Z}/r\mathbb{Z}$ acts with weights $(r-1, mr-s, 1)$. It follows that U has terminal singularities [20]. $\square$

6 Description of the tails

6.1 General theorems

Let $C_0 \subset S_0$ be a singularity of toric or quasitoric type which is not an ordinary node, S_0^c the surface constructed in Theorem 5.2, and $C_1 \subset S_1$ the curve constructed in Theorem 4.1. Let $C_2 \subset S_2$ be a Weil divisor in the linear series

1. $|\mathcal{O}_{\mathbb{P}(p/b,q/b,1)}(bL)|$ if C_0 is of type (p,q);
2. $|\mathcal{O}_{\mathbb{P}(p/b,q/b,1)}(bL + D_2)|$ if C_0 is quasitoric of type $x(x^p - y^q) = 0$ (with $p > q > 1$);
3. $|\mathcal{O}_{\mathbb{P}(p/b,q/b,1)}(bL + D_1)|$ (resp. $|\mathcal{O}_{\mathbb{P}(p/b,q/b,1)}(bL + D_1 + D_2)|$) if C_0 is quasitoric of type $y(x^p - y^q) = 0$ (resp. $xy(x^p - y^q) = 0$) (with $p > q > 1$ and $q \not| p$).

See section 5.1 for notation; we use $b = \gcd(p,q), r = \frac{p}{b}$, and $s = \frac{q}{b}$ throughout. Furthermore, assume that C_2 is nodal and $C_1 \cap B_1 = C_2 \cap B_2$. Let $C_0^c \subset S_0^c$ be the Weil divisor consisting of the union of C_1 and C_2. We should emphasize that C_0^c is Cartier iff C_0 is of toric type.

Theorem 6.1 *Retain the notation of the paragraph above and let $\mathcal{S}^c \to \Delta$ be the smoothing of S_0^c constructed in Theorem 5.2. Then there exists a Weil divisor $\mathcal{C}^c \subset \mathcal{S}^c$ such that $\mathcal{C}^c|S_0^c = C_0^c$ and C_t^c is smooth for $t \neq 0$. Furthermore, $(\mathcal{S}^c, \mathcal{C}^c)$ is the local stable reduction of a smoothing of C_0.*

We first claim each rank-one reflexive sheaf $\mathcal{F}$ on $\mathcal{S}^c$ restricts to a depth-two sheaf on the central fiber. In the proof of Theorem 5.2, we saw that $\mathcal{S}^c$ is covered by affines U which are quotients of $\mathbb{A}^3$ by $\mathbb{Z}/r\mathbb{Z}$ acting with weights $(r-1, mr-s, 1)$. In other words, U is the cone over some weighted projective space $\mathbb{P}(r-1, mr-s, 1)$. Each divisor class on U may be represented as the cone over a divisor of the weighted projective space, and the corresponding reflexive sheaves take the form $\mathcal{O}_U(n\mathrm{C}one(B_2))$ for some n. Using standard results relating the local cohomology of a graded module to the ordinary cohomology of its projectivization [8] Theorem A.4.1, we find that

$$H^i_0(\mathcal{O}_U(n\mathrm{C}one(B_2))) = H^{i-1}(\mathcal{O}_{\mathbb{P}(r-1,mr-s,1)}(nB_2))$$

for $i > 1$. By standard cohomology computations for weighted projective spaces [7] 1.4, we conclude that the local cohomology vanishes for $i \leq 3$. Consequently, reflexive rank-one sheaves on $\mathcal{S}^c$ have depth three and restrict to depth-two sheaves on the central fiber.

The toric representation of $\mathcal{S}^c$ shows that S_1 defines a Weil divisor generating the divisor class group of $\mathcal{S}^c$. Recall that $S_1 \cap S_2 = B_1 = B_2$. If C_0 is of toric type (p, q), then the line bundle $\mathcal{O}_{S^c_0}(C^c_0)$ is the restriction of $\mathcal{F} = \mathcal{O}_{\mathcal{S}^c}(\frac{pq}{b}S_1)$ to the central fiber. More generally, if C_0 is of quasitoric type and C_2 is a member of $|\mathcal{O}_{\mathbb{P}(p/b,q/b,1)}(bL + D_2)|$ (resp. $|\mathcal{O}_{\mathbb{P}(p/b,q/b,1)}(bL + D_1)|$, $|\mathcal{O}_{\mathbb{P}(p/b,q/b,1)}(bL + D_1 + D_2)|$) then $\mathcal{O}_{S^c_0}(C^c_0)$ is the restriction of $\mathcal{F} = \mathcal{O}_{\mathcal{S}^c}((\frac{pq+q}{b}S_1)$ (resp. $\mathcal{O}_{\mathcal{S}^c}((\frac{pq+p}{b}S_1)$, $\mathcal{O}_{\mathcal{S}^c}((\frac{(pq+p+q)}{b}S_1))$.

The divisors $\mathcal{O}_{S_2}(C_2)$ have no higher cohomology by [7] 1.4. This is also the case for $\mathcal{O}_{S_1}(C_1)$, as can be seen by applying a variant of Kawamata-Viehweg vanishing [14] 1.2.5. It follows that $\mathcal{O}_{S^c_0}(C^c_0)$ also has no higher cohomology. In particular, each section of this reflexive sheaf is the restriction of a section of a suitable $\mathcal{F}$. We choose $\mathcal{C}^c$ to be a general section of $\mathcal{F}$ that restricts to C^c_0 along the central fiber. The fibers C^c_t are smooth for $t \neq 0$.

We check that $(\mathcal{S}^c, \mathcal{C}^c)$ is a local stable reduction. We know that (S^c_0, C^c_0) is semilog canonical, $K_{S^c_0} + C^c_0$ is relatively ample, and $\mathcal{S}^c$ is $\mathbb{Q}$-factorial, so inversion of adjunction ([17] 17.12) gives that $(\mathcal{S}^c, \mathcal{C}^c + S^c_0)$ is log canonical. The uniqueness of log canonical models implies this is a local stable reduction. □

We apply this to describe certain smooth tails of singularities of toric type. The statement is simplest in the unibranch case.

Theorem 6.2 (Main Theorem 1) *Let C_0 be a singularity of type (p, q), where p and q are relatively prime and $p > q$. Let C_2 be a nodal curve in the linear series $\mathcal{O}_{\mathbb{P}(p,q,1)}(L)$ and set $p_1 = B_2 \cap C_2$. Then we have*

$$(\mathbb{P}(p, q, 1), C_2 + B_2) \in \mathcal{T}'_{C_0} \quad \textit{and} \quad (C_2, p_1) \in \mathcal{T}_{C_0}.$$

Furthermore, these curves satisfy the following properties:

1. *p_1 is a subcanonical point of C_2, i.e. $(pq-p-q-1)p_1 = K_{C_2}$.*
2. *C_2 is q-gonal, with g^1_q equal to $|H^0(\mathcal{O}_{C_2}(qp_1))|$.*
3. *More generally, for $n = 1, \dots, pq-p-q-1$*

$$h^0(\mathcal{O}_{C_2}(np_1)) = \{(i,j) : i,j \geq 0 \text{ and } qi + pj \leq n\}.$$

Recall that L is the effective generator of the Picard group of $\mathbb{P}(p,q,1)$. The first assertion follows immediately from Theorem 6.1 and the definitions of $\mathcal{T}_{C_0}$ and $\mathcal{T}'_{C_0}$. The geometric statements about C_2 are consequences of the results on weighted projective spaces in § 5.1. The first statement follows from the fact that the divisor class group of $\mathbb{P}(p,q,1)$ is generated by B_2, which meets C_2 at the point p_1. The second statement follows from our description of the sections of $\mathcal{O}_{\mathbb{P}(p,q,1)}(qB_2)$, which restrict to give the g^1_q. Geometrically, the ruling of $\mathbb{P}(p,q,1)$ restricts to the g^1_q on C_2. As for the third statement, we first observe that the restriction

$$H^0(\mathcal{O}_{\mathbb{P}(p,q,1)}(nB_2)) \to H^0(\mathcal{O}_{C_2}(np_2))$$

is surjective for all n [7] 1.4. The isomorphism of graded rings in equation 1 gives the desired formula. □

Remark: In the special case where the singularity is analytically isomorphic to $y^q = x^p$, this result can be deduced from Pinkham's work (see [21] §1.16 and 13). More generally, he considers certain smoothings of curve singularities with $\mathbb{C}^*$-action, and shows that curves C_2 with distinguished Weierstrass point p_1 naturally arise as tails. Weierstrass points also play an important role in the theory of 'limit linear series' of Eisenbud and Harris [9].

In the case where the singularity is not unibranch, we have to keep track of a compatibility condition along the intersection of surfaces S_1 and S_2. This makes our statements a bit more complicated.

Theorem 6.3 (Main Theorem 2) *Let C_0 be a singularity of type (p,q) with $b = \gcd(p,q)$. Let $(S_1, C_1 + B_1)$ be the surface obtained from Theorem 4.1, and let S_0^c be the surface obtained by gluing S_1 and S_2 as in Theorem 5.2. Let C_2 be a nodal curve in the linear series $\mathcal{O}_{\mathbb{P}(p/b,q/b,1)}(bL)$, such that $C_2 \cap B_2 = \{p_1, p_2, \dots, p_b\} = C_1 \cap B_1$. Then we have*

$$(\mathbb{P}(p,q,1), C_2 + B_2) \in \mathcal{T}'_{C_0} \quad \text{and} \quad (C_2, p_1, p_2, \dots, p_b) \in \mathcal{T}_{C_0}.$$

Furthermore, these tails satisfy the following properties:

1. $p_1 + p_2 + \ldots + p_b$ *is a subcanonical divisor of* C_2, *i.e.*

$$\left(\frac{pq}{b} - \frac{p}{b} - \frac{q}{b} - 1\right)(p_1 + p_2 + \ldots + p_b) = K_{C_2}.$$

2. C_2 *is* q*-gonal, with* $g^1_q = H^0(\mathcal{O}_{C_2}(\frac{q}{b}(p_1 + p_2 + \ldots + p_b)))$.

3. *More generally, for* $n = 1, \ldots, \frac{pq}{b} - \frac{p}{b} - \frac{q}{b} - 1$

$$h^0(\mathcal{O}_{C_2}(n(p_1 + \ldots + p_b))) = \{(i, j) : i, j \geq 0 \text{ and } qi + pj \leq nb\}.$$

Again, this follows from Theorem 6.1 and the results on weighted projective spaces.

Proposition 6.4 *For each singularity* C_0 *of toric type, Theorem 6.2 (or 6.3) yields an irreducible component of* $\mathcal{T}'_{C_0}$.

We claim that the space of allowable deformations of S_0^c is smooth of dimension one, and thus consists only of smoothings to S_0. By definition, a deformation is *allowable* if it lifts locally to index-one covers. First, it is not hard to check that weighted projective spaces admit no equisingular deformations, hence S_0^c also admits no equisingular deformations. On the other hand, the results from §3.2 of [13] imply that the versal deformation space of the singularities of S_0^c is smooth of dimension one. □

Proposition 6.4 suggests that the tails described in Theorems 6.2 and 6.3 might also yield an irreducible component of $\mathcal{T}_{C_0}$. Proposition 7.3 provides further evidence for this contention. We therefore make the following conjecture:

Conjecture 6.5 *Let* C_0 *be a singularity of toric type. Then the tails described in Theorems 6.2 and 6.3 yield an irreducible component of* $\mathcal{T}_{C_0}$.

We also raise the following wildly speculative conjecture:

Conjecture 6.6 *Let* C_0 *be a (simple) singularity of toric (or even quasitoric) type. Then* $\mathcal{T}_{C_0}$ *is irreducible.*

This is known for singularities of multiplicity two and some singularities of multiplicity three. The proof involves analyzing the allowable deformation spaces of each of the possible degenerate surface tails.

One can formulate analogous results for singularities of quasitoric type. Again, all the weighted plane curves in a given divisor class (satisfying certain compatibility conditions) appear as tails of stable reductions. We work out certain examples in subsection 6.3.

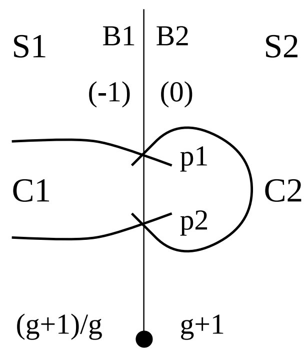

Figure 6: Some local stable reductions for $y^2 = x^{2g+2}$

6.2 Concrete examples

We illustrate these results by describing curves arising as tails for some simple singularities.

6.2.1 A_{2g+1} singularities

See figure 6 for a schematic representing this case; this figure uses the notational conventions of figure 3. Our general analysis implies that $(C_2, p_1 + p_2)$ is a hyperelliptic curve of genus g, where p_1 and p_2 are exchanged under the action of the involution. It is not difficult to verify that each such curve is obtained in this way. Indeed, let C be hyperelliptic of genus g with double cover $f : C \to \mathbb{P}^1$. Then $f_*\mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-g-1)$, which gives a natural imbedding

$$j : C \hookrightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-g-1)) = \mathbb{F}_{g+1}.$$

C is disjoint from the negative section of $\mathbb{F}_{g+1}$, so we may regard C as a curve in $S_2 = \mathbb{P}(g+1, 1, 1)$. The ruling of $\mathbb{F}_{g+1}$ cuts out the g^1_2 on C, so any conjugate points p_1 and p_2 lie on some ruling of S_2.

6.2.2 A_{2g} singularities

See figure 7 for a schematic representing this case. Our general analysis implies that (C_2, p_1) is a hyperelliptic curve of genus g, where p_1 is a fixed point of the hyperelliptic involution. Again, every such curve arises in this way. We imbed C into $\mathbb{F}_{g+1}$ as before. Each fixed point p_1 of the hyperelliptic involution corresponds to a ruling tangent R to C. Blow up $\mathbb{F}_{g+1}$ twice to

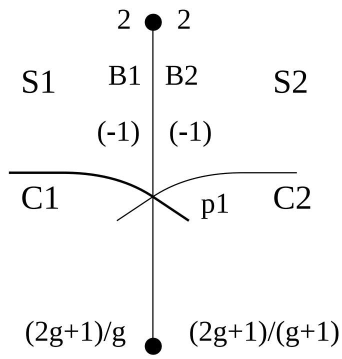

Figure 7: Some local stable reductions for $y^2 = x^{2g+1}$

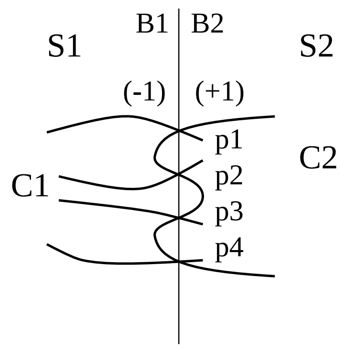

Figure 8: Some local stable reductions for $y^4 = x^4$

eliminate this tangency and let E_1 and E_2 denote the corresponding exceptional divisors. Then blow down all the curves with self-intersection < -1, i.e. the proper transforms of the negative section, R, and E_1. We obtain the weighted projective space $\mathbb{P}(2g+1, 2, 1)$ and E_2 corresponds to B_2.

6.2.3 $y^4 = x^4$

See figure 8 for a schematic representing this case. Let C_0 be the singularity $y^4 = x^4$. Then C_2 is a plane quartic curve and $B_2 \cap C_2 = \{p_1, p_2, p_3, p_4\}$ is a hyperplane section. There is a further compatibility condition to be satisfied. The branches of our singularity determine four distinct tangent directions, i.e. an element of $\mathcal{M}_{0,4}$, which coincides with the element determined by $(B_2, p_1, p_2, p_3, p_4)$.

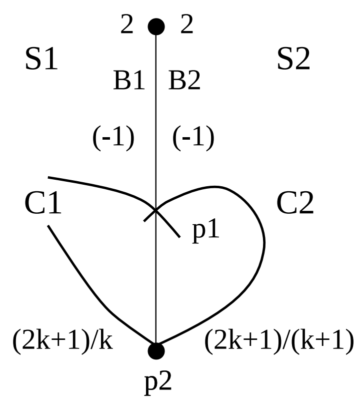

Figure 9: Some local stable reductions for D_{2k+3} singularities

6.3 Singularities of quasitoric type

Applying Theorem 6.1 and emulating Theorems 6.2 and 6.3, we can describe tails for certain singularities of quasitoric type. Again, the surface tail S_2 will be a weighted projective space of type $\mathbb{P}(p, q, 1)$, attached to S_1 along the same curve B_2. The only difference is that the tails C_2 are no longer Cartier on S_2. They pass through one (or both) of the singularities of S_2 and are smooth at these points. If necessary, we impose compatibility conditions like those of Theorem 6.3.

We illustrate this with the remaining ADE singularities. Note that the singularities A_n, D_4, E_6, and E_8 are of toric type.

6.3.1 D_{2k+3} singularities

See figure 9 for a schematic representing this case. Let C_0 be the singularity $x(y^2 - x^{2k+1})$, i.e. the singularity D_{2k+3}. The surface tail $S_2 = \mathbb{P}(2, 2k+1, 1)$. The curve C_2 is a Weil divisor with class $L + 2B_2 = 4(k+1)B_2$ and contains the index-$(2k+1)$ singularity $p_2 \in S_2$. It is hyperelliptic of genus $k+1$ and $C_2 \cap B_2 = \{p_1, p_2\}$, where p_1 is a Weierstrass point and p_2 is generic.

6.3.2 D_{2k+4} singularities, $k > 0$

See figure 10 for a schematic representing this case. Now let C_0 be the singularity $x(y^2 - x^{2k+2})$ for $k > 0$, i.e. the singularity D_{2k+4}. The surface tail $S_2 = \mathbb{P}(k+1, 1, 1)$. The curve C_2 is a Weil divisor with class $2L + B_2 = (2k+3)B_2$ and contains the index-$(k+1)$ singularity on $p_3 \in S_2$. It is

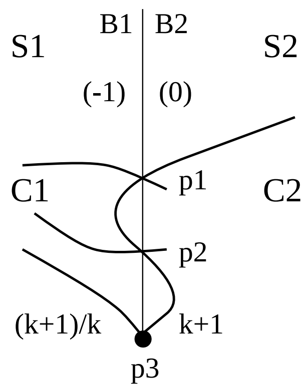

Figure 10: Some local stable reductions for D_{2k+4} singularities, $k > 0$

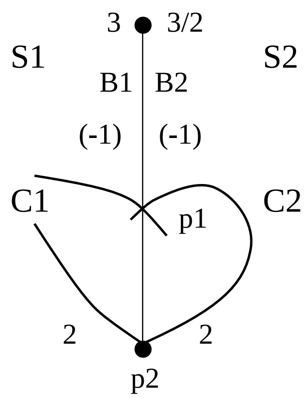

Figure 11: Some local stable reductions for E_7

hyperelliptic of genus $k+1$ and $C_2 \cap B_2 = \{p_1, p_2, p_3\}$, where p_1 and p_2 are conjugate under the hyperelliptic involution and p_3 is generic.

6.3.3 E_7

See figure 11 for a schematic representing this case. Let C_0 be the singularity $x(x^2 - y^3) = 0$, i.e. the singularity of type E_7. The surface tail $S_2 = \mathbb{P}(2,3,1)$. The curve C_2 is a Weil divisor with class $L + 3B_2 = 9B_2$ and contains the index-two singularity $p_2 \in S_2$. It is a non-hyperelliptic genus three curve and $C_2 \cap B_2 = \{p_1, p_2\}$, where $3p_1 + p_2 = K_{C_2}$. In other words, p_1 corresponds to a flex on the canonical image of C_2.

7 Equisingularity and boundary divisors

7.1 Blowing up equisingular deformations

Using Theorem 2.11 we compute the codimension of the locus of singularities of type (p, q):

Corollary 7.1 *Assume that* $p \geq q$ *and* $b = \gcd(p, q)$. *Then the locus of singularities of type* (p, q) *has codimension*

$$c(p, q) = \frac{1}{2}(pq + p + q - b) - \lceil \frac{p}{q} \rceil - 1.$$

The versal deformation space of the singularity $x^p = y^q$ has dimension $pq - p - q + 1$. It remains to find a formula for the cardinality N of the set

$$\{(i, j) : qi + pj \geq pq, 0 \leq j \leq q - 2, 0 \leq i \leq p - 2\} \quad = \\ \{(i, j) : si + rj \geq brs, 0 \leq j \leq bs - 2, 0 \leq i \leq br - 2\}$$

where $p = br$ and $q = bs$. We have that

$$\begin{aligned} N &= \sum_{j=1}^{bs-2} (\lfloor \frac{jr}{s} \rfloor - 1) \\ &= -(bs - 2) - \lfloor \frac{(bs-1)r}{s} \rfloor + \sum_{j=0}^{bs-1} \lfloor \frac{jr}{s} \rfloor \\ &= -bs - br + 2 + \lceil \frac{r}{s} \rceil + \frac{b}{2}(brs - r - s + 1) \\ &= 2 + \lceil \frac{r}{s} \rceil + \frac{b}{2}(brs - 3r - 3s + 1) \end{aligned}$$

In the third step, we applied the summation formula

$$\sum_{j=0}^{bs-1} \lfloor \frac{jr}{s} \rfloor = \frac{b}{2}(brs - r - s + 1).$$

Therefore, the codimension equals

$$\frac{1}{2}(b^2rs + br + bs - b) - \lceil \frac{r}{s} \rceil - 1$$

which agrees with the formula. □

Now we can state the main result of this section.

Theorem 7.2 *Let C_0 be a singularity of type (p,q) and let $Z \subset \mathcal{T}_{C_0}$ denote the tails described in Theorems 6.2 and 6.3. Then*

$$\dim(Z) = c(p,q) - 1$$

where $c(p,q)$ is the codimension of the locus of curves of type (p,q).

The proof of the theorem is a parameter count: for each C_0 of type (p,q), the stable limits parametrized by Z depend on $c(p,q)-1$ parameters. It suffices to compute the dimension of the corresponding linear series on $\mathbb{P}(p,q,1)$, minus the dimension of the automorphism group and the constraints imposed by compatibility conditions. Indeed, any deformation of the imbedding $C_2 \hookrightarrow S_2$ is induced by an automorphism of S_2. This follows from the fact that $H^1(T_{S_2}(-C_2)) = 0$, which can be proved using Serre duality and the cohomology computations of [7] 2.3.2.

We first consider the case where $p = q = b$. In this case, $C_2 \subset \mathbb{P}^2$ is a plane curve of degree b, with a distinguished hyperplane section B_2. Furthermore, the compatibility condition $C_2 \cap B_2 \cong C_1 \cap B_1$ must be satisfied, which imposes $b-3$ conditions. Finally, we have to account for the automorphism group of $\mathbb{P}^2$. We obtain

$$\dim(Z) = \frac{1}{2}(b^2+3b) + 2 - (b-3) - 8 = \frac{1}{2}(b^2+b-6)$$

which equals $c(b,b)-1$.

In the case where $p = rq$ with $r > 1$, we have that $S_2 = \mathbb{F}_r$, $C_2 \in |q\xi_r|$ where ξ_r is a positive section, and B_2 is a distinguished ruling. The compatibility conditions are that $B_2 \cap C_2 = B_1 \cap C_1$ and the singularities of S_1 and S_2 coincide; this imposes $q-2$ conditions. Using the dimension counts for weighted projective spaces (§ 5.1)

$$\dim(Z) = \frac{q}{2}(rq+r+1+1) + 1 - (q-2) - (r+5) = \frac{1}{2}(rq^2+rq) - r - 2$$

which equals $c(rq,q)-1$.

Now we consider the case where q does not divide p, and set $b = \gcd(p,q)$, $p = br$, and $q = bs$. We have $S_2 = \mathbb{P}(r,s,1)$ and $C_2 \in |brsB_2|$. The compatibility conditions are that $B_2 \cap C_2 = B_1 \cap C_1$ and the singularities of S_1 and S_2 coincide; since S_1 and S_2 have two singularities, this imposes $b-1$

conditions. Again using the dimension counts for weighted projective spaces

$$\begin{aligned}\dim(Z) &= \frac{b}{2}(brs + r + s + 1) - (b-1) - \lfloor \frac{r}{s} \rfloor - 4 \\ &= \frac{1}{2}(b^2rs + br + bs - b) - \lceil \frac{r}{s} \rceil - 2\end{aligned}$$

which equals $c(p,q) - 1$. □

Theorem 7.2 is consistent with the following geometric picture of the 'map to moduli.' Let C_0 be an isolated plane curve singularity. Consider the rational map $\mu : \mathrm{Ver}_{C_0} \dashrightarrow \overline{\mathcal{M}}$ associating to each curve its corresponding stable limit. Of course, μ is well-defined only after choosing a linear series of projective curves representing Ver_{C_0} (cf. proof of Proposition 3.2). μ has indeterminacy along the locus where the curves have singularities worse than nodes. For each (p,q), the total transform of the curves of type (p,q) under μ has dimension $\dim(\mathrm{Ver}_{C_0}) - 1$, i.e. it forms a divisor in the image of μ. Note that this immediately implies the following:

Proposition 7.3 *Let C_0 be a general singularity of type (p,q). Then the closure of Z is an irreducible component of $\mathcal{T}_{C_0}$.*

Here 'general' means contained in an open subset of the parameter space for singularities of type (p,q); this parameter space is discussed at the end of § 2.

For a concrete linear series Λ parametrizing curves of genus g, the moduli map $\mu : \Lambda \dashrightarrow \overline{\mathcal{M}}_g$ does not necessarily map the curves with a singularitiy of type (p,q) to a divisor in the image. For example, if $\Lambda = |\mathcal{O}_{\mathbb{P}^2}(+4)|$ then μ sends the tacnodal locus to a codimension-two subvariety of $\overline{\mathcal{M}}_3$ [13].

7.2 Application to boundary divisors

In this section, we apply these results to analyze boundary components for certain moduli spaces of stable log surfaces. See [18],[16], and [1] for more information on the construction of these spaces. Consider pairs $(\mathbb{P}^2, C)$ where C is a smooth plane curve of degree $d \geq 4$; let $\mathcal{P}_d$ denote the isomorphism classes of such pairs. Let $\mathcal{M}$ be the connected moduli space of smoothable stable log surfaces containing $\mathcal{P}_d$ as an open subset, and let $\overline{\mathcal{P}}_d$ denote the closure of $\mathcal{P}_d$ is $\mathcal{M}$. This space is discussed in more detail in [13]. Our first result is

Proposition 7.4 *Assume there exists a plane curve D_0 of degree d with a single singularity, of type (p, q), such that*

1. *the plane curves of degree d map surjectively onto the versal deformation space of D_0;*

2. *$K_{S_1} + B_1 + D_1$ is ample.*

Then the stable limits of the curves of type (p, q) yield a boundary divisor of $\overline{\mathcal{P}}_d$.

By the first assumption, when we apply local stable reduction to degree d plane curves, we obtain all the tails described in Theorems 6.2 and 6.3. It also implies that the curves with a singularity of type (p, q) have the expected codimension $c(p, q)$. The second assumption means that local and global stable reductions coincide (see the end of § 3). Theorem 7.2 implies that the corresponding stable limits yield a divisor in $\overline{\mathcal{P}}_d$. □

For fixed (p, q), the assumptions of the proposition are satisfied for sufficiently large d. As a consequence, we obtain the following theorem:

Theorem 7.5 *Fix (p, q). For d sufficiently large, the stable reductions of curves with a singularity of type (p, q) yield a boundary divisor of $\overline{\mathcal{P}}_d$.*

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Department of Mathematics
University of Chicago
5734 University Avenue
Chicago, IL 60637
USA
email address: hassett@math.uchicago.edu