Moduli spaces of weighted pointed stable curves

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1 Introduction

It has long been understood that a moduli space may admit a plethora of different compactifications, each corresponding to a choice of combinatorial data. Two outstanding examples are the toroidal compactifications of quotients of bounded symmetric domains [AMRT] and the theory of variation of geometric invariant theory (GIT) quotients [BP] [DH] [Th]. However, in both of these situations a modular interpretation of the points added at the boundary can be elusive. By a modular interpretation, we mean the description of a moduli functor whose points are represented by the compactification. Such moduli functors should naturally incorporate the combinatorial data associated with the compactification.

The purpose of this paper is to explore in depth one case where functorial interpretations are readily available: configurations of nonsingular points on a curve. Our standpoint is to consider pointed curves as 'log varieties', pairs (X, D) where X is a variety and $D = \sum_i a_i D_i$ is an effective $\mathbb{Q}$-divisor on X. The minimal model program suggests a construction for the moduli space of such pairs provided they are *stable*, i.e., (X, D) should have relatively mild

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singularities and the divisor $K_X + D$ should be ample. Of course, when $\dim(X) = 1$ and D is reduced the resulting moduli space is the Mumford-Knudsen moduli space of pointed stable curves [KnMu][Kn][Kn2]. In §3 we give a construction for arbitrary D. When $\dim(X) = 2$, a proof for the existence of such moduli spaces was given by Kollár, Shepherd-Barron, and Alexeev [Ko1],[Al1] when $D = 0$, and was sketched by Alexeev[Al2] for $D \neq 0$. The case of higher dimensions is still open, but would follow from standard conjectures of the minimal model program [Karu].

We embark on a systematic study of the dependence of these moduli spaces on the coefficients of the divisor D.[1] We find natural transformations among the various moduli functors which induce birational reduction morphisms among the associated compactifications (see §4). These morphisms can often be made very explicit. We recover the alternate compactifications studied by Kapranov [Kap1] [Kap2], Keel[Ke], and Losev-Manin [LM] as special cases of our theory (see §6). The blow-up constructions they describe are closely intertwined with our functorial reduction maps. The resulting contractions may sometimes be understood as log minimal models of the moduli space itself, where the log divisor is supported in the boundary (see §7).

The moduli spaces we consider do not obviously admit a uniform construction as the quotients arising from varying the linearization of an invariant theory problem. However, ideas of Kapranov (see [Kap1] 0.4.10) suggest indirect GIT approaches to our spaces. Furthermore, we indicate how certain GIT quotients may be interpretted as 'small parameter limits' of our moduli spaces, and the flips between these GIT quotients factor naturally through our spaces (see §8).

One motivation for this work is the desire for a better understanding of compactifications of moduli spaces of log surfaces. These have been studied in special cases [Has] and it was found that the moduli space depends on the coefficients of the boundary in a complicated way. For example, in the case of quintic plane curves (i.e., $X = \mathbb{P}^2$ and $D = aC$ with C a plane quintic) even the irreducible component structure and dimension of the moduli space depends on a. For special values of a the moduli space sprouts superfluous irreducible components attached at infinity. This pathology is avoided when the coefficient is chosen generically. Furthermore, recent exciting work of Hacking [Hac] shows that for small values of the coefficient a the moduli space is often nonsingular and its boundary admits an explicit description.

[1] I. Morrison has pointed out that [GHP] also addressed this question

Roughly, Hacking considers the moduli space parametrizing pairs $(\mathbb{P}^2, aC)$ where C is a plane curve of degree d as $a \to 3/d$. In a future paper, we shall consider birational transformations of moduli spaces of log surfaces induced by varying a.

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2 The moduli problem

Fix nonnegative integers g and n and let B be a noetherian scheme. A family of nodal curves of genus g with n marked points over B consists of

1. a flat proper morphism $\pi : C \to B$ whose geometric fibers are nodal connected curves of arithmetic genus g; and

2. sections $s_1, s_2, \ldots, s_n$ of π.

A morphism of two such families

$$\phi : (C, s_1, \ldots, s_n) \longrightarrow (C', s'_1, \ldots, s'_n)$$

consists of a B-morphism $\phi : C \to C'$ such that $\phi(s_j) \subset s'_j$ for $j = 1, \ldots, n$. The set of isomorphisms of two such families is denoted

$$\mathrm{Isom}((C, s_1, \ldots, s_n), (C', s'_1, \ldots, s'_n)),$$

or simply $\mathrm{Isom}(C, C')$ when there is no risk of confusion.

A collection of input data $(g, \mathcal{A}) := (g, a_1, \ldots, a_n)$ consists of an integer $g \geq 0$ and the *weight data*, an element $(a_1, \ldots, a_n) \in \mathbb{Q}^n$ such that $0 < a_j \leq 1$ for $j = 1, \ldots, n$ and

$$2g - 2 + a_1 + a_2 + \ldots + a_n > 0.$$

A family of nodal curves with marked points $(C, s_1, \ldots, s_n) \stackrel{\pi}{\to} B$ is stable of type $(g, \mathcal{A})$ if

1. the sections $s_1, \ldots, s_n$ lie in the smooth locus of π, and for any subset $\{s_{i_1}, \ldots, s_{i_r}\}$ with nonempty intersection we have $a_{i_1} + \ldots + a_{i_r} \leq 1$;
2. $K_\pi + a_1 s_1 + a_2 s_2 + \ldots + a_n s_n$ is π-relatively ample.

This coincides with the traditional notion of pointed stable curves when $a_1 = a_2 = \ldots = a_n = 1$.

Theorem 2.1 *Let $(g, \mathcal{A})$ be a collection of input data. There exists a connected Deligne-Mumford stack $\overline{\mathcal{M}}_{g,\mathcal{A}}$, smooth and proper over $\mathbb{Z}$, representing the moduli problem of pointed stable curves of type $(g, \mathcal{A})$. The corresponding coarse moduli scheme $\overline{M}_{g,\mathcal{A}}$ is projective over $\mathbb{Z}$.*

The universal curve is denoted $\mathcal{C}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$. Theorem 2.1 is proved in §3.

2.1 Variations on the moduli problem

2.1.1 Zero weights

One natural variant on our moduli problem is to allow some of the sections to have weight zero. We consider $(g, \tilde{\mathcal{A}}) := (g, a_1, \ldots, a_n)$ where $(a_1, \ldots, a_n) \in \mathbb{Q}^n$ with $0 \leq a_j \leq 1$ and

$$2g - 2 + a_1 + a_2 + \ldots + a_n > 0.$$

A family of nodal curves with marked points $(C, s_1, \ldots, s_n) \stackrel{\pi}{\to} B$ is stable of type $(g, \tilde{\mathcal{A}})$ if

1. the sections s_i with positive weights lie in the smooth locus of π, and for any subset $\{s_{i_1}, \ldots, s_{i_r}\}$ with nonempty intersection we have $a_{i_1} + \ldots + a_{i_r} \leq 1$;
2. $K_\pi + a_1 s_1 + a_2 s_2 + \ldots + a_n s_n$ is π-relatively ample.

There is no difficulty making sense of the divisor $K_\pi + a_1 s_1 + a_2 s_2 + \ldots + a_n s_n$ as any section meeting the singularities has coefficient zero. We emphasize that the stability condition is the natural one arising from the log minimal model program (cf. the proof of Proposition 3.7).

The resulting moduli spaces $\overline{\mathcal{M}}_{g,\tilde{\mathcal{A}}}$ are easily described. Let $\mathcal{A}$ be the subsequence of $\tilde{\mathcal{A}}$ containing all the positive weights and assume that $|\mathcal{A}| + N = |\tilde{\mathcal{A}}|$. Each $\tilde{\mathcal{A}}$-stable pointed curve consists of a $\mathcal{A}$-stable curve with N additional arbitrary marked points, i.e., the points with weight zero. Hence we may identify

$$\overline{\mathcal{M}}_{g,\tilde{\mathcal{A}}} = \underbrace{\mathcal{C}_{g,\mathcal{A}} \times_{\overline{\mathcal{M}}_{g,\mathcal{A}}} \ldots \times_{\overline{\mathcal{M}}_{g,\mathcal{A}}} \mathcal{C}_{g,\mathcal{A}}}_{N \text{ times}},$$

so $\overline{\mathcal{M}}_{g,\tilde{\mathcal{A}}}$ is the N-fold fiber product of the universal curve over $\overline{\mathcal{M}}_{g,\mathcal{A}}$.

The moduli spaces with zero weights differ from the original spaces in one crucial respect: they are generally singular. For example, the local analytic equation of a generic one-parameter deformation of a nodal curve may be written $xy = t$, where t is the coordinate on the base. The second fiber product of this family takes the form

$$x_1 y_1 = x_2 y_2 = t,$$

which is a threefold with ordinary double point.

2.1.2 Weights summing to two

We restrict to the case $g = 0$ and consider weight data $\hat{A} = (a_1, \ldots, a_n)$ where the weights are positive rational numbers with $a_1 + \ldots + a_n = 2$. Weighted pointed curves of this type have previously been considered by Kawamata, Keel, and McKernan in the context of the codimension-two subadjunction formulas (see [Kaw2] and [KeMc2]). One can construct an explicit family of such weighted curves over the moduli space

$$\mathcal{C}(\hat{A}) \to \overline{\mathcal{M}}_{0,n};$$

this family is realized as an explicit blow-down of the universal curve over $\overline{\mathcal{M}}_{0,n}$.

In this paper we do not give a direct modular interpretation of spaces $\overline{\mathcal{M}}_{0,\hat{A}}$. However, when each $a_j < 1$ we may interpret the geometric invariant theory quotient

$$(\mathbb{P}^1)^n // \mathrm{SL}_2$$

with linearization $\mathcal{O}(a_1, a_2, \ldots, a_n)$ as $\overline{\mathcal{M}}_{0,\hat{A}}$ (see Theorems 8.2 and 8.3). These spaces are often singular (see Remark 8.5).

2.1.3 Weighted divisors

We can also consider curves with weighted divisors rather than weighted points. A stable curve with weighted divisors consists of a nodal connected curve C of genus g, a collection of effective divisors supported in its smooth locus

$$D_1, \ldots, D_m,$$

and positive weights $a_1, \ldots, a_m$, so that the sum $D := a_1 D_1 + \ldots + a_m D_m$ has coefficient ≤ 1 at each point and $K_C + D$ is ample. Writing $d_j = \deg(D_j)$, we can construct a moduli space

$$\overline{M}_{g,((a_1,d_1),\ldots,(a_m,d_m))}$$

as follows. We associate to this problem the weight data

$$\mathcal{B} := (\underbrace{a_1, \ldots, a_1}_{d_1 \text{ times}}, \ldots, \underbrace{a_m, \ldots, a_m}_{d_m \text{ times}})$$

and the corresponding coarse moduli scheme of weighted pointed curves $\overline{M}_{g,\mathcal{B}}$. We take

$$\overline{M}_{g,((a_1,d_1),\ldots,(a_m,d_m))} = \overline{M}_{g,\mathcal{B}}/(S_{d_1} \times \ldots \times S_{d_m}),$$

where the product of symmetric groups acts componentwise on the m sets of sections. We will not discuss the propriety of writing the moduli space as such a quotient, except to refer the reader to chapter 1 of [GIT].

3 Construction of the moduli space

3.1 Preliminaries on linear series

In this section we work over an algebraically closed field F. Given a curve C and a smooth point s, note that the ideal sheaf I_s is invertible. We write $L(s)$ for $L \otimes I_s^{-1}$.

Proposition 3.1 *Let C be a connected nodal proper curve. Let M be an invertible sheaf such that M^{-1} is nef. Then $h^0(M) \neq 0$ if and only if M is trivial.*

The nef assumption means $\deg(M|C_j) \leq 0$ for each irreducible component $C_j \subset C$.
proof: This is elementary if C is smooth. For the general case, consider the normalization $\nu : C^\nu \to C$, with irreducible components $C_1^\nu, \ldots, C_N^\nu$. We have the formula

$$p_a(C) = \sum_{j=1}^{N} p_a(C_j^\nu) + \Delta - N + 1$$

relating the arithmetic genera and Δ, the number of singularities of C. Recall the exact sequence

$$0 \to T \to \mathrm{Pic}(C) \xrightarrow{\nu^*} \mathrm{Pic}(C^\nu)$$

where T is a torus of rank $\Delta - N + 1$. To reconstruct M from $\nu^* M$, for each singular point $p \in C$ and points $p_1, p_2 \in C^\nu$ lying over p we specify an isomorphism $(\nu^* M)_{p_1} \simeq (\nu^* M)_{p_2}$, unique up to scalar multiplication on $\nu^* M$. In particular, we obtain an exact sequence

$$0 \to \mathbb{G}_m \to \mathbb{G}_m^N \to \mathbb{G}_m^\Delta \to T \to 0.$$

If M has a nontrivial section then $\nu^* M \simeq \mathcal{O}_{C^\nu}$ and the section pulls back to a section of $\mathcal{O}_{C^\nu}$ constant and nonzero on each component. Thus the corresponding element of T is trivial and $M \simeq \mathcal{O}_C$. $\square$

Proposition 3.2 *Let C be an irreducible nodal curve with arithmetic genus g, B and D effective divisor of degrees b and d supported in the smooth locus of C. Let M be an ample invertible sheaf that may be written*

$$M = \omega_C^k(kB + D) \quad k > 0,$$

and $\Sigma \subset C$ a subscheme of length σ contained in the smooth locus of C.

Assume that $\sigma \leq 2$ and $N \geq 4$ (resp. $\sigma \leq 1$ and $N \geq 3$). Then $H^0(\omega_C(B + \Sigma) \otimes M^{-N}) = 0$. This holds for $N = 3$ (resp. $N = 2$) except in the cases

1. *$d = 0, k = 1, g = 0,$ and $b = 3$; or*

2. *$d = 0, k = 1, g = 1, b = 1,$ and $\mathcal{O}_C(\Sigma) \simeq \mathcal{O}_C(\sigma B)$.*

In these cases, all the sections are constant. Finally, $H^0(\omega_C(B) \otimes M^{-N}) = 0$ when $N \geq 2$.

proof: Setting $F = \omega_C(B + \Sigma) \otimes M^{-N}$, we compute

$$\deg(F) = (2g - 2 + b) + \sigma - N \deg(M) = (1 - Nk)(2g - 2 + b) + \sigma - Nd.$$

We determine when these are nonnegative. First assume $N \geq 3$ and $\sigma \leq 2$. Using the first expression for $\deg(F)$ and $\deg(M) \geq 1$, we obtain $2g-2+b \geq 1$. The second expression implies that $2g - 2 + b \leq 1$. Thus $2g - 2 + b = 1$, and the first expression gives $N = 3, \sigma = 2$, and $\deg(M) = 1$; the second expression yields $k = 1$ and $d = 0$. These are the exceptional cases above.

Now assume $N \geq 2$ and $\sigma \leq 1$. Repeating the argument above, we find $2g - 2 + b = 1$ and therefore $N = 2, \sigma = 1, \deg(M) = 1, k = 1$, and $d = 0$. Again, we are in one of the two exceptional cases. Finally, if $N \geq 2$ and $\sigma = 0$, we obtain $(2g - 2 + b) > 0$ from the first expression and $(2g - 2 + b) < 0$ from the second. This proves the final assertion. □

Proposition 3.3 *Let C be a connected nodal curve of genus g, D an effective divisor supported in the smooth locus of C, L an invertible sheaf with $L \simeq \omega_C^k(D)$ for $k > 0$.*

1. *If L is nef and $L \neq \omega_C$ then L has vanishing higher cohomology.*
2. *If L is nef and has positive degree then L^N is basepoint free for $N \geq 2$.*
3. *If L is ample then L^N is very ample when $N \geq 3$.*
4. *Assume L is nef and has positive degree and let C' denote the image of C under L^N with $N \geq 3$. Then C' is a nodal curve with the same arithmetic genus as C, obtained by collapsing the irreducible components of C on which L has degree zero. Components on which L has positive degree are mapped birationally onto their images.*

Our argument owes a debt to Deligne and Mumford ([DM] §1).
proof: For the first statement, we use Serre duality $h^1(L) = h^0(\omega_C \otimes L^{-1})$ and Proposition 3.1 applied to $M = \omega_C \otimes L^{-1}$. One verifies easily that

$$M^{-1} = L \otimes \omega_C^{-1} = \omega_C^{k-1}(D)$$

is the sum of a nef and an effective divisor.

We prove the basepoint freeness statement. Decompose

$$C = Z \cup_T C_+$$

where Z contains the components on which L has degree zero, C_+ the components on which L is ample, and T is their intersection. Each connected component $Z_j \subset Z$ is a chain of $\mathbb{P}^1$'s and has arithmetic genus zero. A component Z_j is *type I* (resp. *type II*) if it contains one point $t_j \in T$ (resp. two points $t'_j, t''_j \in T$).

It suffices to show that for each $p \in C$

$$h^0(L^N \otimes I_p) = h^0(L^N) - 1$$

where $N \geq 2$. The vanishing assertion guarantees that L^N has no higher cohomology. It suffices then to show that $L^N \otimes I_p$ has no higher cohomology, or dually, $\mathrm{Hom}(I_p, \omega_C \otimes L^{-N}) = 0$. If p is a smooth point then I_p is locally free and

$$\mathrm{Hom}(I_p, \omega_C \otimes L^{-N}) = H^0(\omega_C(p) \otimes L^{-N}).$$

We analyze the restriction of $F := \omega_C(p) \otimes L^{-N}$ to the components of C. We first restrict to Z:

$$h^0(Z_j, F|Z_j) = \begin{cases} 0 \text{ if } p \notin Z_j \text{ and } Z_j \text{ is of type I;} \\ 1 \text{ if } p \in Z_j \text{ and } Z_j \text{ is of type I;} \\ 1 \text{ if } p \notin Z_j \text{ and } Z_j \text{ is of type II;} \\ 2 \text{ if } p \in Z_j \text{ and } Z_j \text{ is of type II.} \end{cases}$$

In each case the sections are zero if they are zero at $T \cap Z_j$. For components in C_+ we apply Proposition 3.2, where M is restriction of L to some irreducible component, $\sigma = p$, and B is the conductor. (In what follows, on applying Proposition 3.2 we always assume B contains the conductor.) The Proposition gives that the restriction to each component has no nontrivial sections, except perhaps when p lies on a component E listed in the exceptional cases. Then the sections of $F|E$ are constant and $E \simeq \mathbb{P}^1$ because $p \notin B$. Thus provided p is not contained in a component $E \simeq \mathbb{P}^1 \subset C_+$ with $|B| = 3$, we obtain that $h^0(C, F) = 0$. Indeed, clearly $h^0(F|C_+) = 0$ and the analysis of cases above yields $h^0(C, F) = 0$. If p does sit on such a component E, then C contains a component of type I or a second irreducible component of C_+; the conductor $B \subset E$ has three elements, so there must be at least one other non-type II component. The restriction of F to such a component has only trivial sections, and since the restrictions to all the other components have at most constant sections, we conclude $h^0(C, F) = 0$.

If p is singular, let $\beta : \hat{C} \to C$ be the blow-up of C at p and $p_1, p_2 \in \hat{C}$ the points lying over p, so that

$$\mathrm{Hom}(I_p, \omega_C \otimes L^{-N}) = H^0(\beta^*(\omega_C \otimes L^{-N})) = H^0(\omega_{\hat{C}}(p_1 + p_2)\beta^* L^{-N}).$$

We write $\hat{F} = \omega_{\hat{C}}(p_1 + p_2)\beta^* L^{-N}$. As before, we decompose

$$\hat{C} = \hat{C}_+ \cup_{\hat{T}} \hat{Z} \tag{1}$$

where L is positive on $\hat{C}_+$, and use $\hat{Z}_j$ to denote a connected component of Z. Note that $\beta(\hat{Z}) = Z$ and the p_i are not both contained on some $\hat{Z}_j$; otherwise, $\beta(\hat{Z}_j)$ would have positive arithmetic genus. Similarly, neither of the p_i lie on a type II component (with respect to decomposition (1)). It follows that $h^0(\hat{F}|\hat{Z}_j)$ has dimension at most one, and any section vanishes if it vanishes along $\hat{T} \cap \hat{Z}_j$. For components of $\hat{C}_+$, we apply Proposition 3.2, with $p_1, p_2 \in B$ and $\sigma = 0$, to show that $h^0(\hat{F}|\hat{C}_+) = 0$. Again, we conclude that $h^0(\hat{C}, \hat{F}) = 0$.

For ampleness, take p and q to be points of C, not necessarily smooth or distinct, with ideal sheaves I_p and I_q. Again, it suffices to prove $h^0(I_p I_q L^N) = 0$ for $N \geq 3$, or dually, $\mathrm{Hom}(I_p I_q, \omega_C \otimes L^{-N}) = 0$. When p and q are smooth points the assertion follows as above from Proposition 3.2, again with B as the conductor. The only case requiring additional argument is when p and q both lie on a component corresponding to one of the exceptional cases. Again, the nontrivial sections on this component are constants, whereas we have only zero sections on the other components.

We may assume that p is singular and write $\beta : \hat{C} \to C$ for the blow-up at p. Then for each invertible sheaf R on C we have

$$\mathrm{Hom}(I_p^2, R) = H^0(\beta^* R(p_1 + p_2))$$

where p_1 and p_2 are the points of $\hat{C}$ lying over p. When q is smooth we obtain

$$\mathrm{Hom}(I_p I_q, \omega_C \otimes L^{-N}) = H^0(\hat{C}, \omega_{\hat{C}}(p_1 + p_2 + q) \otimes \beta^* L^{-N}).$$

Note that the restriction of $\beta^* L$ to each component still satisfies the hypothesis of Proposition 3.2; we take $p_1, p_2 \in B$ and $q = \Sigma$. When q is singular and disjoint from p we obtain

$$\mathrm{Hom}(I_p I_q, \omega_C \otimes L^{-N}) = H^0(\hat{C}, \omega_{\hat{C}}(p_1 + p_2 + q_1 + q_2) \otimes \beta^* L^{-N}),$$

where $\hat{C} \to C$ is the blow-up at p and q. We apply Proposition 3.2 with $p_1, p_2, q_1, q_2 \in B$ and $\Sigma = \emptyset$. When $p = q$ we obtain

$$\mathrm{Hom}(I_p^2, \omega_C \otimes L^{-N}) = H^0(\hat{C}, \omega_{\hat{C}}(2p_1 + 2p_2) \otimes \beta^* L^{-N}),$$

and we apply Proposition 3.2 with $p_1, p_2 \in B$ and $\Sigma = \{p_1, p_2\}$, accounting for the exceptional cases as before.

We prove the last assertion. We have an exact sequence

$$0 \to H^0(C, L^N) \to H^0(C_+, L^N|C_+) \oplus H^0(Z, L^N|Z) \to H^0(T, L^N|T).$$

Choose N large so that $(L^N|C_+)(-T)$ is very ample. Since $L^N|Z$ is trivial, the image of C under L^N is obtained from C_+ by identifying pairs of points in T corresponding to t'_j, t''_j in some Z_j,i.e., by 'collapsing' each Z_j to a point. Let C' denote the resulting curve, which has the same arithmetic genus as C, and $r : C \to C'$ the resulting map. Writing $D' = r(D)$, we have $r^*\omega^k_{C'}(D') = L$. Thus the sections of L^N $(N \geq 3)$ induce r, as $(\omega^k_{C'}(D'))^N$ is very ample on C'. $\square$

Remark 3.4 The adjunction formula gives precise information about the points of D lying in Z. Each connected component Z_j of type II is disjoint from D. Recall that components Z_j of type I are chains $Z_{j1} \cup \ldots \cup Z_{jm}$ of $\mathbb{P}^1$'s, intersecting C_+ in a point of one of the ends of the chain (say Z_{j1}). Then $D|Z_j$ is supported in the irreducible component Z_{jm} at the opposite end and $\deg(D|Z_{jm}) = k$.

Applying Proposition 3.3 and Remark 3.4 with $D = k(b_1 s_1 + \ldots + b_n s_n)$, we obtain the following:

Corollary 3.5 Let $(C, s_1, \ldots, s_n)$ be a nodal pointed curve of genus g, $b_1, \ldots, b_n$ nonnegative rational numbers, and k a positive integer such that each kb_i is integral. Assume that $L := \omega^k_C(b_1 s_1 + \ldots + b_n s_n)$ is nef and has positive degree.

For $N \geq 3$ the sections of L^N induce a dominant morphism $r : C \to C'$ to a nodal curve of genus g. This morphism collapses irreducible components of C on which L has degree zero, and maps the remaining components birationally onto their images. If $\mathcal{B} = (b_{i_1}, \ldots, b_{i_r}), i_1 < i_2 < \ldots < i_r$, denotes the set of all nonzero weights and $s'_i = r(s_i)$, then $(C', s'_{i_1}, \ldots, s'_{i_r})$ is a stable pointed curve of type $(\mathcal{B}, g)$.

proof: The only claim left to verify is the singularity condition. Each s_i lying in a component of type II necessarily has weight $b_i = 0$. Thus no points with positive weight are mapped to singularities of the image C'. The points $\{s_{j_1}, \ldots, s_{j_a}\}$ lying on a single component of type I have weights summing to one, i.e., $b_{j_1} + \ldots + b_{j_a} = 1$. $\square$

We also obtain the following relative statement:

Theorem 3.6 *Let* $\pi : (C, s_1, \ldots, s_n) \to B$ *be a family of nodal pointed curves of genus* g, $b_1, \ldots, b_n$ *nonnegative rational numbers, and* k *a positive integer such that each* kb_i *is integral. Set* $L = \omega_\pi^k(kb_1 s_1 + \ldots + kb_n s_n)$ *and assume that* L *is* π*-nef and has positive degree. For* $N \geq 3$, $\mathrm{Proj}(\oplus_{m\geq 0}\pi_* L^{mN})$ *defines a flat family of nodal curves* C' *with sections* $s'_1, \ldots, s'_n$. *If* $\mathcal{B} = (b_{i_1}, \ldots, b_{i_r}), i_1 < i_2 < \ldots < i_r$, *denotes the set of all nonzero weights, then* $(C', s'_{i_1}, \ldots, s'_{i_r})$ *is a family of stable pointed curves of type* $(\mathcal{B}, g)$.

The new family may be considered as the *log canonical model* of C relative to $K_\pi + a_1 s_1 + \ldots + a_n s_n$.

proof: The vanishing assertion of Proposition 3.3 implies the formation of $\pi_* L^N$ commutes with base extensions $B' \to B$. Hence we may apply the fiberwise assertions of Corollary 3.5 to $(C', s'_1, \ldots, s'_n) \to B$. We therefore obtain a family of pointed stable curves of type $(\mathcal{B}, g)$. $\square$

3.2 The log minimal model program and the valuative criterion

To prove that our moduli problem is proper, we shall apply the valuative criterion for properness (cf. [LaMo] 7.5). The most important step is the following:

Proposition 3.7 *Let* R *be a DVR with quotient field* K, $\Delta = \mathrm{Spec}\ R$, $\Delta^* = \mathrm{Spec}\ K$, $(g, \mathcal{A})$ *a collection of input data,* $\pi^* : (C^*, s_1^*, \ldots, s_n^*) \to \Delta^*$ *a family of stable pointed curves of type* $(g, \mathcal{A})$. *Then there exists the spectrum of a DVR* $\tilde{\Delta}$, *a finite ramified morphism* $\tilde{\Delta} \to \Delta$, *and a family* $\pi^c : (\mathcal{C}^c, s_1^c, \ldots, s_n^c) \to \tilde{\Delta}$ *of stable pointed curves of type* $(g, \mathcal{A})$, *isomorphic to*

$$(C^* \times_\Delta \tilde{\Delta}, s_1^* \times_\Delta \tilde{\Delta}, \ldots, s_n^* \times_\Delta \tilde{\Delta})$$

over $\tilde{\Delta}^*$. *The family* $(\mathcal{C}^c, s_1^c, \ldots, s_n^c)$ *is unique with these properties.*

proof: We first reduce to the case where C^* is geometrically normal with disjoint sections. If sections $s_{i_1}, \ldots, s_{i_r}$ coincide over the generic point, we replace these by a single section with weight $a_{i_1} + \ldots + a_{i_r}$. Choose a finite extension of K over which each irreducible component of C^* is defined, as well as each singular point. Let $C^*(\nu)$ be the normalization, $s^*(\nu)_1, \ldots, s^*(\nu)_n$ the proper transforms of the sections, and $s^*(\nu)_{n+1}, \ldots, s^*(\nu)_{n+b}$ the points of the conductor. Then $(C^*(\nu), s^*(\nu)_1, \ldots, s^*(\nu)_{n+b})$ is stable with respect to the weights $(\mathcal{A}, 1, \ldots, 1)$. Once we have the stable reduction of $C^*(\nu)$, the stable reduction of C^* is obtained by identifying corresponding pairs of points of the conductor.

Applying the valuative criterion for properness for $\mathcal{M}_{g,n}$ (which might entail a base-change $\tilde{\Delta} \to \Delta$), we reduce to the case where $(C^*, s_1^*, \ldots, s_n^*)$ extends to a family $\pi : (C, s_1, \ldots, s_n) \to \Delta$ of stable curves in $\mathcal{M}_{g,n}$. If this family is stable with respect to the weight data $\mathcal{A}$ (i.e., $K_\pi + a_1 s_1 + \ldots a_n s_n$ is ample relative to π) then there is nothing to prove. We therefore assume this is not the case.

Our argument uses the log minimal model program to obtain a model on which our log canonical divisor is ample. This is well-known for surfaces over fields of arbitrary characteristic (see Theorem 1.4 of [Fu] or [KK]), but perhaps less well-known in the mixed characteristic case. For completeness, we sketch a proof.

Let λ be the largest number for which

$$D := \lambda(K_\pi + s_1 + \ldots + s_n) + (1 - \lambda)(K_\pi + a_1 s_1 + \ldots + a_n s_n)$$

fails to be ample. Our assumptions imply $0 \leq \lambda < 1$; λ is rational because there are only finitely many (integral projective) curves lying in fibers of π. Our argument is by induction on the number of such curves. The $\mathbb{Q}$-divisor D is nef and has positive degree, and we choose L to be the locally free sheaf associated to a suitable multiple of D. Applying Theorem 3.6, we obtain a new family of pointed curves $\pi' : (C', s_1' + \ldots + s_n') \to \tilde{\Delta}$, agreeing with the original family away from the central fiber, and stable with respect to the weight data $\mathcal{B}(1) := \lambda(1, 1, \ldots, 1) + (1 - \lambda)\mathcal{A}$. Note that in passing from C to C', we have necessarily contracted some curve in the central fiber π. Now if $\mathcal{B}(1) = \mathcal{A}$ (i.e., if $\lambda = 0$) the proof is complete. Otherwise, we repeat the procedure above using a suitable log divisor

$$D' := \lambda(K_{\pi'} + b_1(1) s_1' + \ldots + b_n(1) s_n') + (1 - \lambda)(K_{\pi'} + a_1 s_1' + \ldots + a_n s_n').$$

We continue in this way until $\mathcal{B}(j) = \mathcal{A}$; this process terminates because there are only finitely many curves in the central fiber to contract. $\square$

3.3 Deformation theory

Let $(C, s_1, \ldots, s_n)$ be a weighted pointed stable curve of genus g with weight data $\mathcal{A}$, defined over a field F. We compute its infinitesimal automorphisms and deformations. We regard the pointed curve as a map

$$s : \Sigma \to C,$$

where Σ consists of n points, each mapped to the corresponding $s_j \in C$.

The infinitesimal deformation theory of maps was analyzed by Ziv Ran [Ra]; he worked with holomorphic maps of reduced analytic spaces, but his approach also applies in an algebraic context. We recall the general formalism. Infinitesimal automorphisms, deformations, and obstructions of $s : \Sigma \to C$ are denoted by T^0_s, T^1_s, and T^2_s respectively. Similarly, we use $T^i_C = \mathrm{Ext}^i_C(\Omega^1_C, \mathcal{O}_C)$ and $T^i_\Sigma = \mathrm{Ext}^i_\Sigma(\Omega^1_\Sigma, \mathcal{O}_\Sigma)$ for the analogous groups associated to C and Σ. Finally, we consider the mixed group

$$\mathrm{Hom}_s(\Omega^1_C, \mathcal{O}_\Sigma) = \mathrm{Hom}_C(\Omega^1_C, s_*\mathcal{O}_\Sigma) = \mathrm{Hom}_\Sigma(s^*\Omega^1_C, \mathcal{O}_\Sigma)$$

and the associated Ext-groups, denoted $\mathrm{Ext}^i_s(\Omega^1_C, \mathcal{O}_\Sigma)$ and computed by either of the spectral sequences

$$E_2^{p,q} = \mathrm{Ext}^p_C(\Omega^1_C, \mathbb{R}^q s_*\mathcal{O}_\Sigma) \quad E_2^{p,q} = \mathrm{Ext}^p_\Sigma(\mathbb{L}^q s^*\Omega^1_C, \mathcal{O}_\Sigma).$$

We obtain long exact sequences

$$\begin{aligned} 0 \to T^0_s \to T^0_C \oplus T^0_\Sigma \to \mathrm{Hom}_s(\Omega^1_C, \mathcal{O}_\Sigma) \to T^1_s \to T^1_C \oplus T^1_\Sigma \\ \to \mathrm{Ext}^1_s(\Omega^1_C, \mathcal{O}_\Sigma) \to T^2_s \to T^2_C \oplus T^2_\Sigma \to \mathrm{Ext}^2_s(\Omega^1_C, \mathcal{O}_\Sigma). \end{aligned}$$

In our situation $T^i_\Sigma = 0$ (Σ is reduced zero-dimensional), $T^2_C = 0$ (C is a nodal curve), $\mathrm{Ext}^i_s(\Omega_C, \mathcal{O}_\Sigma) = \mathrm{Ext}^i_\Sigma(s^*\Omega_C, \mathcal{O}_\Sigma)$ (Ω^1_C is free along $s(\Sigma)$), and thus $\mathrm{Ext}^i_s(\Omega_C, \mathcal{O}_\Sigma) = 0$ for $i > 0$. Hence the exact sequence boils down to

$$0 \to T^0_s \to \mathrm{Hom}_C(\Omega^1_C, \mathcal{O}_C) \to \mathrm{Hom}_\Sigma(s^*\Omega^1_C, \mathcal{O}_\Sigma) \to T^1_s \to \mathrm{Ext}^1_C(\Omega^1_C, \mathcal{O}_C) \to 0$$

and $T^2_s = 0$.

Let $D \subset C$ denote the support of $a_1 s_1 + \ldots + a_n s_n$. Note that

$$D - (a_1 s_1 + \ldots + a_n s_n)$$

is an effective $\mathbb{Q}$-divisor and so the positivity condition guarantees that $\omega_C(D)$ is ample. The map s factors

$$s : \Sigma \to D \hookrightarrow C$$

which gives a factorization

$$\mathrm{Hom}_C(\Omega^1_C, \mathcal{O}_C) \to \mathrm{Hom}_C(\Omega^1_C, \mathcal{O}_D) \to \mathrm{Hom}_\Sigma(s^*\Omega^1_C, \mathcal{O}_\Sigma).$$

The second step is clearly injective. The kernel of the first step is

$$\mathrm{Hom}_C(\Omega^1_C, \mathcal{O}_C(-D)) \simeq H^0((\omega_C(D))^{-1}) = 0.$$

Thus $T^0_s = 0$ and T^1_s has dimension $3g - 3 + n$. We summarize this in the following proposition:

Proposition 3.8 *Let $(C, s_1, \ldots, s_n)$ be a weighted pointed stable curve of genus g with weight data $\mathcal{A}$. Then this curve admits no infinitesimal automorphisms and its infinitesimal deformation space is unobstructed of dimension $3g - 3 + n$.*

3.3.1 The canonical class

We digress to point out consequences of this analysis for the moduli stack. The tangent space to $\overline{\mathcal{M}}_{g,\mathcal{A}}$ at $(C, s_1, \ldots, s_n)$ sits in the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_C(\Omega^1_C, \mathcal{O}_C) &\to \oplus_{j=1}^n \mathrm{Hom}_C(\Omega^1_C, \mathcal{O}_{s_j}) \\ &\to T_{(C,s_1,\ldots,s_n)}\overline{\mathcal{M}}_{g,\mathcal{A}} \to \mathrm{Ext}^1_C(\Omega^1_C, \mathcal{O}_C) \to 0. \end{aligned}$$

The cotangent space sits in the dual exact sequence

$$0 \to H^0(\Omega^1_C \otimes \omega_C) \to T^*_{(C,s_1,\ldots,s_n)}\overline{\mathcal{M}}_{g,\mathcal{A}} \to \oplus_{j=1}^n \Omega^1_C|s_j \to H^1(\Omega^1_C \otimes \omega_C) \to 0.$$

Now let

$$\pi : \mathcal{C}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$$

be the universal curve and s_j the corresponding sections. The exact sequences above cannot be interpretted as exact sequences of vector bundles on the moduli stack because $h^0(\Omega^1_C \otimes \omega_C)$ and $h^1(\Omega^1_C \otimes \omega_C)$ are nonconstant. However, there is a relation in the derived category which, on combination with the Grothendieck-Riemann-Roch Theorem, yields a formula for the canonical class of the moduli stack (cf. [HM] pp. 159). This takes the form

$$K_{\overline{\mathcal{M}}_{g,\mathcal{A}}} = \frac{13}{12}\kappa'(\mathcal{A}) - \frac{11}{12}\nu(\mathcal{A}) + \sum_{j=1}^{n} \psi_j(\mathcal{A}),$$

where $\kappa'(\mathcal{A}) = \pi_*[c_1(\omega_\pi)^2]$, $\nu(\mathcal{A})$ the divisor parametrizing nodal curves, and $\psi_j(\mathcal{A}) = c_1(s_j^*\omega_\pi)$.

3.3.2 An alternate formulation

We sketch an alternate formalism for the deformation theory of pointed stable curves. This was developed by Kawamata [Kaw1] in an analytic context for logarithmic pairs (X, D) consisting of a proper nonsingular variety X and a normal crossings divisor $D \subset X$. This approach is more appropriate when we regard the boundary as a *divisor* rather than the union of a sequence of sections. In particular, this approach should be useful for higher-dimensional generalizations of weighted pointed stable curves, like stable log surfaces.

We work over an algebraically closed field F. Let X be a scheme and $D_1, \ldots, D_n$ a sequence of distinct effective Cartier divisors (playing the role of the irreducible components of the normal crossings divisor). We define the sheaf $\Omega^1_X \langle D_1, \ldots, D_n \rangle$ of differentials on X with logarithmic poles along the collection $D_1, \ldots, D_n$. Choose an open affine subset $U \subset X$ so that each D_j is defined by an equation $f_j \in \mathcal{O}_U$. Consider the module

$$\Omega^1_U \langle D_1, \ldots, D_n \rangle := (\Omega^1_U \oplus \Omega^1_U e_{f_1} \oplus \ldots \oplus \Omega^1_U e_{f_n}) / \left\langle df_j - f_j e_{f_j} \right\rangle.$$

Up to isomorphism, this is independent of the choice of the f_j; indeed, if $f_j = u g_j$ with $u \in \mathcal{O}_U^*$ then we have the substitution $e_{g_j} = e_{f_j} - du/u$. The inclusion by the first factor induces a natural injection

$$\Omega^1_U \hookrightarrow \Omega^1_U \langle D_1, \ldots, D_n \rangle$$

with cokernel $\oplus_{j=1}^n \mathcal{O}_{D_j}$, where each summand is generated by the corresponding e_{f_j}. We therefore obtain the following natural exact sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^1_X \to \Omega^1_X \langle D_1, \ldots, D_n \rangle \to \oplus_{j=1}^n \mathcal{O}_{D_j} \to 0.$$

In the case where X is smooth and the D_j are smooth, reduced, and meet in normal crossings, we recover the standard definition of differentials with logarithmic poles and the exact sequence is the ordinary residuation exact sequence. When D_j has multiplicity > 1, the sheaf $\Omega^1_X \langle D_1, \ldots, D_n \rangle$ has torsion along D_j. Note that the residuation exact sequence is split when the multiplicities are divisible by the characteristic.

Assume for simplicity that X is projective and smooth along the support of the D_j. We claim that first-order deformations of $(X, D_1, \ldots, D_n)$ correspond to elements of

$$\mathrm{Ext}^1_X(\Omega^1_X \langle D_1, \ldots, D_n \rangle, \mathcal{O}_X).$$

The resolution

$$0 \to \mathcal{O}_X(-D_j) \to \mathcal{O}_X \to \mathcal{O}_{D_j} \to 0$$

implies $\mathrm{Ext}^1_X(\mathcal{O}_{D_j}, \mathcal{O}_X) = H^0(\mathcal{O}_{D_j}(D_j))$, so the residuation exact sequence yields

$$\oplus_{j=1}^n H^0(\mathcal{O}_{D_j}(D_j)) \to \mathrm{Ext}^1_X(\Omega^1_X \langle D_1, \ldots, D_n \rangle, \mathcal{O}_X) \to \mathrm{Ext}^1_X(\Omega^1_X, \mathcal{O}_X).$$

Of course, $H^0(\mathcal{O}_{D_j}(D_j))$ parametrizes first-order deformations of D_j in X and $\mathrm{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)$ parametrizes first-order deformations of the ambient variety X.

3.4 Construction of the moduli stack

In this section, we prove all the assertions of Theorem 2.1 except the projectivity of the coarse moduli space, which will be proved in §3.5. For a good general discussion of how moduli spaces are constructed, we refer the reader to [DM] or [Vi].

We refer to [LaMo], §3.1 and 4.1, for the definition of a stack. The existence of the moduli space as a stack follows from standard properties of descent: families of stable pointed curves of type $(g, \mathcal{A})$ satisfy effective descent and Isom is a sheaf in the étale topology.

We introduce an 'exhausting family' for our moduli problem, i.e., a scheme which is an atlas for our stack in the smooth topology. Set

$$L = \omega_\pi^k(ka_1 s_1 + \ldots + ka_n s_n)$$

where k is the smallest positive integer such that each ka_j is integral. Let $d = \deg(L^3) = 3k(2g - 2 + a_1 + \ldots + a_n)$ and consider the scheme H_0

parametrizing n-tuples $(s_1, \ldots, s_n)$ in $\mathbb{P}^d$, and the scheme H_1 parametrizing genus g, degree d curves $C \subset \mathbb{P}^d$. Let

$$U \subset H_0 \times H_1$$

be the locally closed subscheme satisfying the following conditions:

1. C is reduced and nodal;
2. $s_1, \ldots, s_n \subset C$ and is contained in the smooth locus;
3. $\mathcal{O}_C(+1) = L^3$.

We are using the fact that two line bundles (i.e., $\mathcal{O}_C(+1)$ and $L^{\otimes 3}$) coincide on a locally closed subset.

We shall now prove that our moduli stack is algebraic using Artin's criterion (see [LaMo] §10.) Proposition 3.3 implies that each curve in $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is represented in U. Furthermore, isomorphisms between pointed curves in U are induced by projective equivalences in $\mathbb{P}^d$. It follows that $U \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ is smooth and surjective, and that the diagonal $\overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}} \times \overline{\mathcal{M}}_{g,\mathcal{A}}$ is representable, quasi-compact, and separated. We conclude that $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is an algebraic stack of finite type.

To show that the stack is Deligne-Mumford, it suffices to show that our pointed curves have 'no infinitesimal automorphisms,' i.e., that the diagonal is unramified [LaMo], 8.1. This follows from Proposition 3.8. The moduli stack is proper over $\mathbb{Z}$ by the valuative criterion for properness (cf. [LaMo] 7.5) and Proposition 3.7. The moduli stack is smooth over $\mathbb{Z}$ if, for each curve defined over a field, the infinitesimal deformation space is unobstructed. This also follows from Proposition 3.8. $\square$

3.5 Existence of polarizations

We now construct polarizations for the moduli spaces of weighted pointed stable curves, following methods of Kollár [Ko1] (see also [KoMc]). We work over an algebraically closed field F. The first key concept is the notion of a semipositive sheaf. Given a scheme (or algebraic space) X and a vector bundle E on X, we say E is *semipositive* if for each complete curve C and map $f : C \to X$, any quotient bundle of f^*E has nonnegative degree. Second, we formulate precisely what it means to say that the 'classifying map is finite.'

Given an algebraic group G, a G-vector bundle W on X of rank w and a quotient vector bundle Q of rank q, the classifying map should take the form

$$u : X \to \mathrm{Gr}(w, q)/G$$

where the Grassmannian denotes the q-dimensional quotients of fixed w-dimensional space. Since the orbit space need not exist as a scheme, we regard u as a set-theoretic map on closed points $X(F) \to \mathrm{Gr}(w, q)(F)/G(F)$. We say that the classifying map u is finite when it has finite fibers and each point of the image has finite stabilizer.

The following result, a slight modification of the Ampleness Lemma of [Ko1], allows us to use semipositive sheaves to construct polarizations:

Proposition 3.9 *Let X be a proper algebraic space, W a semipositive vector bundle with structure group G and rank w. Let $Q_1, \ldots, Q_m$ be quotient vector bundles of W with ranks $q_1, \ldots, q_m$. Assume that*

1. *$W = \mathrm{Sym}^d(V)$ for some vector bundle V of rank v and $G = GL_v$;*
2. *the classifying map*

$$u : X \to (\prod_{j=1}^{m} \mathrm{Gr}(w, q_j))/G$$

   is finite.

Then for any positive integers $c_1, \ldots, c_m$ the line bundle

$$\det(Q_1)^{c_1} \otimes \det(Q_2)^{c_2} \ldots \otimes \det(Q_m)^{c_m}$$

is ample.

In the original result $m = 1$, so the classifying map takes values in the quotient of a single Grassmannian rather than a product of Grassmannians. However, the argument of [Ko1] generalizes easily to our situation, so we refer to this paper for the details.

To apply this result we need to produce semipositive vector bundles on the moduli space $\overline{\mathcal{M}}_{g,\mathcal{A}}$. Choose an integer $k \geq 2$ so that each ka_i is an integer. Given a family $\pi : (C, s_1, \ldots, s_n) \to B$ we take

$$V := \pi_*[\omega_\pi^k(ka_1 s_1 + \ldots + ka_n s_n)],$$

which is locally free. Indeed, a degree computation (using the assumption $k \geq 2$) yields

$$\omega_{C_b}^k(ka_1s_1(b) + \ldots + ka_ns_n(b)) \neq \omega_{C_b},$$

so Proposition 3.3 guarantees the vanishing of higher cohomology. The sheaf V is semipositive on B by Proposition 4.7 of [Ko1]. It follows that each symmetric product $\mathrm{Sym}^d(V)$ is also semipositive ([Ko1] 3.2).

We may choose k uniformly large so that $\omega_\pi^k(ka_1s_1 + \ldots + ka_ns_n)$ is very ample relative to π for *any* family π (see Proposition 3.3). We shall consider the multiplication maps

$$\mu_d : \mathrm{Sym}^d(V) \to \pi_*[\omega_\pi^{dk}(dka_1s_1 + \ldots + dka_ns_n)]$$

and the induced restrictions

$$\mathrm{Sym}^d(V) \to Q_j, \quad Q_j := s_j^*[\omega_\pi^{dk}(dka_1s_1 + \ldots + dka_ns_n)].$$

These are necessarily surjective and each has kernel consisting of the polynomials vanishing at the corresponding section. We also choose d uniformly large so that

$$W := \mathrm{Sym}^d(V) \to Q_{n+1}, \quad Q_{n+1} := \pi_*[\omega_\pi^{kd}(kda_1s_1 + \ldots + kda_ns_n)],$$

is surjective and the fibers of π are cut out by elements of the kernel (i.e., they are defined by equations of degree d). In particular, the pointed curve $(C_b, s_1(b), \ldots, s_n(b))$ can be recovered from the image of b under the classifying map associated to W and its quotients $Q_1, \ldots, Q_{n+1}$. The stabilizer of the image corresponds to the automorphisms of

$$\mathbb{P}(H^0(\omega_{C_b}^k(ka_1s_1(b) + \ldots + ka_ns_n(b))))$$

preserving the equations of C_b and the sections. This is finite since our curve has no infinitesimal automorphisms (Proposition 3.8).

If $\overline{M}_{g,\mathcal{A}}$ admits a universal family the existence of a polarization follows from Proposition 3.9. In general, we obtain only a line bundle $\mathcal{L}$ on the moduli stack $\overline{\mathcal{M}}_{g,\mathcal{A}}$, but some positive power $\mathcal{L}^N$ descends to the scheme $\overline{M}_{g,\mathcal{A}}$. This power is ample. Indeed, there exists a family $\pi : (C, s_1, \ldots, s_n) \to B$ of curves in $\overline{\mathcal{M}}_{g,\mathcal{A}}$ so that the induced moduli map $B \to \overline{M}_{g,\mathcal{A}}$ is finite surjective (cf. [Ko1] §2.6-2.8). The bundle $\mathcal{L}^N$ is functorial in the sense that it pulls back to the corresponding product

$$\det(Q_1)^{Nc_1} \otimes \det(Q_2)^{Nc_2} \ldots \otimes \det(Q_{n+1})^{Nc_{n+1}}$$

associated with our family. This is ample by Proposition 3.9. Our proof of Theorem 2.1 is complete.□

4 Natural transformations

4.1 Reduction and forgetting morphisms

Theorem 4.1 (Reduction) *Fix g and n and let $\mathcal{A} = (a_1, \ldots, a_n)$ and $\mathcal{B} = (b_1, \ldots, b_n)$ be collections of weight data so that $b_j \leq a_j$ for each $j = 1, \ldots, n$. Then there exists a natural birational reduction morphism*

$$\rho_{\mathcal{B},\mathcal{A}} : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{B}}.$$

Given an element $(C, s_1, \ldots, s_n) \in \overline{\mathcal{M}}_{g,\mathcal{A}}$, $\rho_{\mathcal{B},\mathcal{A}}(C, s_1, \ldots, s_n)$ is obtained by successively collapsing components of C along which $K_C + b_1 s_1 + \ldots + b_n s_n$ fails to be ample.

Remark 4.2 The proof of Theorem 4.1 also applies when some of the weights of $\mathcal{B}$ are zero (see §2.1.1).

Theorem 4.3 (Forgetting) *Fix g and let $\mathcal{A}$ be a collection of weight data and $\mathcal{A}' := \{a_{i_1}, \ldots, a_{i_r}\} \subset \mathcal{A}$ a subset so that $2g - 2 + a_{i_1} + \ldots + a_{i_r} > 0$. Then there exists a natural forgetting morphism*

$$\phi_{\mathcal{A},\mathcal{A}'} : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}'}.$$

Given an element $(C, s_1, \ldots, s_n) \in \overline{\mathcal{M}}_{g,\mathcal{A}}$, $\phi_{\mathcal{A},\mathcal{A}'}(C, s_1, \ldots, s_n)$ is obtained by successively collapsing components of C along which $K_C + a_{i_1} s_{i_1} + \ldots + a_{i_r} s_{i_r}$ fails to be ample.

We refer the reader to Knudsen and Mumford [KnMu][Kn][Kn2] for the original results on the moduli space of unweighted pointed stable curves.
proof: We shall prove these theorems simultaneously, using $\psi_{\mathcal{A},\mathcal{B}}$ to denote either a reduction or a forgetting map, depending on the context. Let $(g, \mathcal{A})$ be a collection of input data, $\hat{\mathcal{B}} = (b_1, \ldots, b_n) \in \mathbb{Q}^n$ so that $0 \leq b_j \leq a_j$ for each j and $2g - 2 + b_1 + \ldots + b_n > 0$. Let $\mathcal{B} = (b_{i_1}, \ldots, b_{i_r}), i_1 < i_2 < \ldots < i_r$, be obtained by removing the entries of $\hat{\mathcal{B}}$ which are zero. We shall define a natural transformation of functors

$$\psi_{\mathcal{B},\mathcal{A}} : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{B}}.$$

Our construction will yield a morphism in the category of stacks, which therefore induces a morphism of the underlying coarse moduli schemes.

Consider $\hat{\mathcal{B}}(\lambda) = \lambda\mathcal{A} + (1-\lambda)\hat{\mathcal{B}}$ for $\lambda \in \mathbb{Q}$, $0 < \lambda < 1$, and write $\hat{\mathcal{B}}(\lambda) = (b_1(\lambda), \ldots, b_n(\lambda))$. We may assume there exists no subset $\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ such that $b_{i_1}(\lambda) + \ldots + b_{i_r}(\lambda) = 1$. If $\mathcal{A}$ and $\mathcal{B}$ do not satisfy this assumption, then there is a finite sequence $1 > \lambda_0 > \lambda_1 > \ldots > \lambda_N = 0$ so that each $(\hat{\mathcal{B}}(\lambda_j), \hat{\mathcal{B}}(\lambda_{j+1}))$ does satisfy our assumption. Then we may inductively define

$$\psi_{\mathcal{B},\mathcal{A}} = \psi_{\mathcal{B}(\lambda_N),\mathcal{B}(\lambda_{N-1})} \circ \ldots \circ \psi_{\mathcal{B}(\lambda_1),\mathcal{B}(\lambda_0)}.$$

Let $(C, s_1, \ldots, s_n) \xrightarrow{\pi} B$ be a family of stable curves of type $(g, \mathcal{A})$. Under our simplifying assumption, we define $\psi_{\mathcal{B},\mathcal{A}}(C, s_1, \ldots, s_n)$ as follows. Consider the $\mathbb{Q}$-divisors $K_\pi(\mathcal{A}) := K_\pi + a_1 s_1 + \ldots + a_n s_n$, $K_\pi(\mathcal{B}) := K_\pi + b_1 s_1 + \ldots + b_n s_n$, and $K_\pi(\mathcal{B}(\lambda)) := \lambda K_\pi(\mathcal{A}) + (1-\lambda) K_\pi(\mathcal{B})$ for $\lambda \in \mathbb{Q}, 0 \leq \lambda \leq 1$. We claim this is ample for each $\lambda \neq 0$. This follows from Remark 3.4. If L is nef but not ample, then there either exist sections with weight zero (on type II components) or sets of sections with weights summing to one (on type I components), both of which are excluded by our assumptions.

We now apply Corollary 3.5 and Theorem 3.6 to obtain a new family of pointed nodal curves $\pi' : C' \to B$ with smooth sections $s'_{i_1}, \ldots, s'_{i_r}$ corresponding to the nonzero weights of $\hat{\mathcal{B}}$. We define $\psi_{\mathcal{B},\mathcal{A}}(C, s_1, \ldots, s_n)$ to be the family $\pi' : C' \to B$, with image sections $s'_{i_1}, \ldots, s'_{i_r}$ and weights $b_{i_1}, \ldots, b_{i_r}$, a family of weighted pointed curves with fibers in $\overline{\mathcal{M}}_{g,\mathcal{B}}$. The vanishing statement in Proposition 3.3 guarantees our construction commutes with base extension. Thus we obtain a natural transformation of moduli functors

$$\psi_{\mathcal{B},\mathcal{A}} : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{B}}.$$

Assume that $\mathcal{B} = \hat{\mathcal{B}}$, so that $\psi_{\mathcal{B},\mathcal{A}}$ is interpretted as a reduction map $\rho_{\mathcal{B},\mathcal{A}}$. Then $\rho_{\mathcal{B},\mathcal{A}}$ is an isomorphism over the locus of configurations of points, it is a birational morphism. $\square$

Reduction satisfies the following compatibility condition:

Proposition 4.4 *Fix g and let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be collections of weight data so that the reductions $\rho_{\mathcal{B},\mathcal{A}}, \rho_{\mathcal{C},\mathcal{B}}$, and $\rho_{\mathcal{C},\mathcal{A}}$ are well defined. Then*

$$\rho_{\mathcal{C},\mathcal{A}} = \rho_{\mathcal{C},\mathcal{B}} \circ \rho_{\mathcal{B},\mathcal{A}}.$$

proof: Since the maps are birational morphisms of nonsingular varieties, it suffices to check that the maps coincide set-theoretically. $\square$

4.2 Exceptional loci

The exceptional locus of the reduction morphism is easily computed using Corollary 3.5:

Proposition 4.5 *The reduction morphism $\rho_{\mathcal{B},\mathcal{A}}$ contracts the boundary divisors*

$$D_{I,J} := \overline{M}_{0,\mathcal{A}'_I} \times \overline{M}_{g,\mathcal{A}'_J} \quad \mathcal{A}'_I = (a_{i_1}, \ldots, a_{i_r}, 1) \quad \mathcal{A}'_J = (a_{j_1}, \ldots, a_{j_{n-r}}, 1)$$

corresponding to partitions

$$\{1, \ldots, n\} = I \cup J \quad I = \{i_1, \ldots, i_r\} \quad J = \{j_1, \ldots, j_{n-r}\}$$

with $b_I := b_{i_1} + \ldots + b_{i_r} \leq 1$ and $2 < r \leq n$. We have a factorization of $\rho_{\mathcal{B},\mathcal{A}}|D_{I,J}$:

$$\overline{M}_{0,\mathcal{A}'_I} \times \overline{M}_{g,\mathcal{A}'_J} \xrightarrow{\pi} \overline{M}_{g,\mathcal{A}'_J} \xrightarrow{\rho} \overline{M}_{g,\mathcal{B}'_J} \quad \mathcal{B}'_J = (b_{j_1}, \ldots, b_{j_{n-r}}, b_I)$$

where $\rho = \rho_{\mathcal{B}'_J,\mathcal{A}'_J}$ and π is the projection.

Remark 4.6 Consider a set of weights $(a_{i_1}, \ldots, a_{i_r})$ so that

$$a_{i_1} + \ldots + a_{i_r} > 1$$

but any proper subset has sum at most one. Then $\overline{M}_{0,\mathcal{A}'_I}$ is isomorphic to $\mathbb{P}^{r-2}$ (cf. §6.2) and $\rho_{\mathcal{B},\mathcal{A}}$ is the blow-up of $\overline{M}_{g,\mathcal{B}}$ along the image of $D_{I,J}$.

Corollary 4.7 Retain the notation and assumptions of Proposition 4.5. Assume in addition that for each $I \subset \{1, \ldots, n\}$ such that

$$a_{i_1} + \ldots + a_{i_r} > 1 \text{ and } b_{i_1} + \ldots + b_{i_r} \leq 1$$

we have $r = 2$. Then $\rho_{\mathcal{A},\mathcal{B}}$ is an isomorphism.

5 Chambers and walls

Let $\mathcal{D}_{g,n}$ denote the domain of admissible weight data

$$\mathcal{D}_{g,n} := \{(a_1, \ldots, a_n) \in \mathbb{R}^n : 0 < a_j \leq 1 \text{ and } a_1 + a_2 + \ldots + a_n > 2 - 2g\}.$$

A *chamber decomposition* of $\mathcal{D}_{g,n}$ consists of a finite set $\mathcal{W}$ of hyperplanes $w_S \subset \mathcal{D}_{g,n}$, the *walls* of the chamber decomposition; the connected components of the complement to the union of the walls $\cup_{S\in\mathcal{W}} w_S$ are called the *open chambers.*

There are two natural chamber decompositions for $\mathcal{D}_{g,n}$[2]. The *coarse chamber decomposition* is

$$\mathcal{W}_c = \{\sum_{j\in S} a_j = 1 : S \subset \{1,\ldots,n\}, 2 < |S| \leq n\}$$

and the *fine chamber decomposition* is

$$\mathcal{W}_f = \{\sum_{j\in S} a_j = 1 : S \subset \{1,\ldots,n\}, 2 \leq |S| \leq n\}.$$

When $g = 0$ there are no walls with $|S| = n-1$ or n. Given a nonempty wall w_S, the set $\mathcal{D}_{g,n} \setminus w_S$ has two connected components defined by the inequalities $\sum_{j\in S} a_j > 1$ and $\sum_{j\in S} a_j < 1$.

Proposition 5.1 *The coarse chamber decomposition is the coarsest decomposition of $\mathcal{D}_{g,n}$ such that $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is constant on each chamber. The fine chamber decomposition is the coarsest decomposition of $\mathcal{D}_{g,n}$ such that $\mathcal{C}_{g,\mathcal{A}}$ is constant on each chamber.*

proof: It is clear that $\overline{\mathcal{M}}_{g,\mathcal{A}}$ (resp. $\mathcal{C}_{g,\mathcal{A}}$) changes as we pass from one coarse (resp. fine) chamber to another. It suffices then to show that $\overline{\mathcal{M}}_{g,\mathcal{A}}$ (resp. $\mathcal{C}_{g,\mathcal{A}}$) is constant on each chamber.

Let $\mathcal{A}$ and $\mathcal{B}$ be contained in the interior of a fine chamber and let $\pi : (C, s_1, \ldots, s_n) \to B$ be a family in $\overline{\mathcal{M}}_{g,\mathcal{A}}(B)$. Repeating the argument for Theorem 4.1, we find that $K_\pi + b_1 s_1 + \ldots + b_n s_n$ is ample. An application of Theorem 3.6 implies that $(C, s_1, \ldots, s_n)$ is also $\mathcal{B}$-stable. Thus we get an identification of $\overline{\mathcal{M}}_{g,\mathcal{A}}(B)$ and $\overline{\mathcal{M}}_{g,\mathcal{B}}(B)$ so $\mathcal{C}_{g,\mathcal{A}} \simeq \mathcal{C}_{g,\mathcal{B}}$.

Consider the fine chambers contained in a given coarse chamber Ch. We shall say that two such fine chambers are *adjacent* if there exists a wall w which is a codimension-one face of each. Any two fine chambers in Ch are related by a finite sequence of adjacencies, so it suffices to show that adjacent fine chambers yield the same moduli space. Fix two fine chambers in Ch adjacent along w, which we may assume is defined by $a_1 + a_2 = 1$. Let

[2] Corrected thanks to comments by K. Ascher

$(w_1, \ldots, w_n) \in w$ be an element contained in the closure of the chambers but not in any walls besides w. For small $\epsilon > 0$, the weight data

$$\mathcal{A} = (w_1 + \epsilon, w_2 + \epsilon, w_3, \ldots, w_n) \quad \mathcal{B} = (w_1 - \epsilon, w_2 - \epsilon, w_3, \ldots, w_n)$$

lie in our two fine chambers. Corollary 4.7 implies that $\rho_{\mathcal{A},\mathcal{B}}$ is an automorphism. $\square$

Problem 5.2 *Find formulas for the number of nonempty walls and chambers for $\mathcal{D}_{g,n}$.*

Proposition 5.3 *Let $\mathcal{A}$ be a collection of weight data. There exists a collection of weight data $\mathcal{B}$, contained in a fine open chamber, such that $\mathcal{C}_{g,\mathcal{A}} = \mathcal{C}_{g,\mathcal{B}}$.*

The fine open chamber produced in Proposition 5.3 is called the fine chamber associated to $\mathcal{A}$.
proof: Consider the collection T_1 (resp. $T_{<1}$, resp. $T_{>1}$) of all subsets $S \subset \{1, \ldots, n\}$ with $2 \leq |S| \leq n-2$ such that $\sum_{j \in S} a_j = 1$ (resp. $\sum_{j \in S} a_j < 1$, resp. $\sum_{j \in S} a_j > 1$.) There exists a positive constant ϵ such that for each $S \in T_{<1}$ we have $\sum_{j \in S} a_j < 1 - \epsilon$, for each $S \in T_{>1}$ we have $\sum_{j \in S} a_j > 1 + \epsilon$, $a_1 + \ldots + a_n > 2 + \epsilon$, and $a_n > \epsilon$. Setting

$$\mathcal{B} = (a_1 - \epsilon/n, a_2 - \epsilon/n, \ldots, a_n - \epsilon/n)$$

and using results of §4.2, we obtain the desired result. $\square$

Proposition 5.4 *Let $\mathcal{A}$ be weight data contained in a fine open chamber. Then $\mathcal{C}_{g,\mathcal{A}}$ is isomorphic to $\mathcal{M}_{g,\mathcal{A} \cup \{\epsilon\}}$ for some sufficiently small $\epsilon > 0$.*

proof: We retain the notation of the proof of Proposition 5.3. Under our assumptions the set T_1 is empty, so for each $T \subset \{1, \ldots, n\}$ with $2 \leq |S| \leq n-2$ we have

$$|\sum_{j \in S} a_j - 1| > \epsilon. \quad \square$$

6 Examples

6.1 Kapranov's approach to $\overline{M}_{0,n}$

The key classical observation is that through each set of n points of $\mathbb{P}^{n-3}$ in linear general position, there passes a unique rational normal curve of degree

$n-3$. It is therefore natural to realize elements of $M_{0,n}$ as rational normal curves in projective space. This motivates Kapranov's blow-up construction of $\overline{M}_{0,n}$ ([Kap1] §4.3, [Kap2]).

Start with $W_{1,1}[n] := \mathbb{P}^{n-3}$ and choose $n-1$ points $q_1, \dots, q_{n-1}$ in linear general position. These are unique up to a projectivity. We blow up as follows:

1: blow up the points $q_1, \dots, q_{n-2}$, then the lines passing through pairs of these points, followed by the planes passing through triples of triples of these points, etc.;

2: blow up the point q_{n-1}, then the lines spanned by the pairs of points including q_{n-1} but not q_{n-2}, then the planes spanned by triples including q_{n-1} but not q_{n-2}, etc.; ...

r: blow up the linear spaces spanned by subsets

$$\{q_{n-1}, q_{n-2} \dots, q_{n-r+1}\} \subset S \subset \{q_1, \dots, q_{n-r-1}, q_{n-r+1}, \dots, q_{n-1}\}$$

so that the order of the blow-ups is compatible with the partial order on the subsets by inclusion;...

Let $W_{r,1}[n], \dots, W_{r,n-r-2}[n] (:= W_r[n])$ denote the varieties produced at the rth step. Precisely, $W_{r,s}[n]$ is obtained once we finish blowing up subspaces spanned by subsets S with $|S| \leq s+r-2$. Kapranov proves $\overline{M}_{0,n} \simeq W_{n-3}[n]$.

This may be interpretted with the reduction formalism of §4. The exceptional divisors of the blow-downs $W_{r,s}[n] \to W_{r,s-1}[n]$ are proper transforms of the boundary divisors $D_{I,J}$ corresponding to partitions

$$\{1, \dots, n\} = I \cap J \text{ where } J = \{n\} \cup S.$$

Using the weight data

$$\begin{aligned} \mathcal{A}_{r,s}[n] \quad &:= \quad (\underbrace{1/(n-r-1), \dots, 1/(n-r-1)}_{n-r-1 \text{ times}}, s/(n-r-1), \underbrace{1, \dots, 1}_{r \text{ times}}), \\ & r = 1, \dots, n-3, \quad s = 1, \dots, n-r-2, \end{aligned}$$

we realize $W_{r,s}[n]$ as $\overline{M}_{0,\mathcal{A}_{r,s}[n]}$. The blow-downs defined above are the reduction morphisms

$$\begin{aligned} \rho_{\mathcal{A}_{r,s-1}[n],\mathcal{A}_{r,s}[n]} \quad &: \quad \overline{M}_{0,\mathcal{A}_{r,s}[n]} \longrightarrow \overline{M}_{0,\mathcal{A}_{r,s-1}[n]} \quad s = 2, \dots, n-r-2 \\ \rho_{\mathcal{A}_{r,n-r-2}[n],\mathcal{A}_{r+1,1}[n]} \quad &: \quad \overline{M}_{0,\mathcal{A}_{r+1,1}[n]} \longrightarrow \overline{M}_{0,\mathcal{A}_{r,n-r-2}[n]}. \end{aligned}$$

In particular, Kapranov's factorization of $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$ as a sequence of blow-downs is naturally a composition of reduction morphisms.

6.2 An alternate approach to Kapranov's moduli space

There are many factorizations of the map $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$ as a composition of reductions. We give another example here. First, we compute the fine chamber containing the weight data $\mathcal{A}_{1,1}[n]$ introduced in §6.1:

$$a_1 + \ldots + \widehat{a}_i + \ldots + a_{n-1} \leq 1 \text{ (and thus } a_i + a_n > 1\text{) for } i = 1, \ldots, n-1;$$

$$a_1 + \ldots + a_{n-1} > 1.$$

The corresponding moduli space $\overline{M}_{0,\mathcal{A}}$ is denoted $X_0[n]$ and is isomorphic to $\mathbb{P}^{n-3}$. Similarly, we define $X_k[n] := \overline{M}_{0,\mathcal{A}}$ provided $\mathcal{A}$ satisfies

$a_i + a_n > 1$ for $i = 1, \ldots, n-1$;

$a_{l_1} + \ldots + a_{l_r} \leq 1$ (resp. > 1) for each $\{l_1, \ldots, l_r\} \subset \{1, \ldots, n-1\}$ with $r \leq n-k-2$ (resp. $r > n-k-2$).

When $a_n = 1$ and $a_1 = \ldots = a_{n-1} = a(n,k)$ we have

$$1/(n-1-k) < a(n,k) \leq 1/(n-2-k)$$

and

$$\mathcal{A}(n,k) = (\underbrace{a(n,k), \ldots, a(n,k)}_{n-1 \text{ times}}, 1).$$

Thus there exist reduction maps

$$\begin{array}{cccccc} \rho_{\mathcal{A}(n,k-1),\mathcal{A}(n,k)}: & \overline{M}_{0,\mathcal{A}(n,k)} & \longrightarrow & \overline{M}_{0,\mathcal{A}(n,k-1)} & & \\ & || & & || & & k = 1, \ldots, n-4 \\ & X_k[n] & \longrightarrow & X_{k-1}[n] & & \end{array}$$

We interpret the exceptional divisors of the induced maps. For each partition

$$\{1, \ldots, n\} = I \cup J \quad I = \{i_1 = n, i_2, \ldots, i_r\}, J = \{j_1, \ldots, j_{n-r}\} \; 2 \leq r \leq n-2$$

consider the corresponding boundary divisor in $\overline{M}_{0,n}$

$$D_{I,J} \simeq \overline{M}_{0,r+1} \times \overline{M}_{0,n-r+1}.$$

The divisors with $|I| = r < n-2$ are the exceptional divisors of $X_{r-1}[n] \to X_{r-2}[n]$. The blow-down $X_1[n] \to X_0[n]$ maps $D_{I,J}$ with $I = \{n, i\}$ to the distinguished point $q_i \in \mathbb{P}^{n-3}$ mentioned in §6.1. Applying Proposition 4.5 and

$$(\underbrace{a(n,1),\dots,a(n,1)}_{n-2 \text{ times}},1) \simeq (\underbrace{a(n-1,0),\dots,a(n-1,0)}_{n-2 \text{ times}},1) = \mathcal{A}(n-1,0),$$

we obtain that the fiber over q_i is $\overline{M}_{0,\mathcal{A}(n-1,0)} \simeq \mathbb{P}^{n-4}$. More generally, $\overline{M}_{0,n} = X_{n-4}[n] \to X_0[n]$ maps $D_{I,J}$ to the linear subspace spanned by $q_{i_2},\dots,q_{i_r}$; the fibers are isomorphic to $\mathbb{P}^{n-r-2}$. The divisors $D_{I,J}$ with $|I| = n-2$ are proper transforms of the hyperplanes of $X_0[n] \simeq \mathbb{P}^{n-3}$ spanned by $q_{i_2},\dots,q_{i_{n-2}}$.

Thus our reduction maps give the following realization of $\overline{M}_{0,n}$ as a blow-up of $\mathbb{P}^{n-3}$:

1: blow up the points $q_1,\dots,q_{n-1}$ to obtain $X_1[n]$;

2: blow up proper transforms of lines spanned by pairs of the points $q_1,\dots,q_{n-1}$ to obtain $X_2[n]$;

3: blow up proper transforms of 2-planes spanned by triples of the points to obtain $X_3[n]$; ...

n-4: blow up proper transforms of $(n-5)$-planes spanned by $(n-4)$-tuples of the points to obtain $X_{n-4}[n]$.

6.3 Keel's approach to $\overline{M}_{0,n}$

Let $U \subset (\mathbb{P}^1)^n$ denote the set of all configurations of n *distinct* points in $\mathbb{P}^1$, and Q the resulting quotient under the diagonal action of PGL_2. For each configuration $(p_1,\dots,p_n)$, there exists a unique projectivity ϕ with

$$\phi : (p_1,p_2,p_3) \to (0,1,\infty).$$

The image of the configuration in Q is determined completely by the points $(\phi(p_4),\dots,\phi(p_n))$ and we obtain an imbedding

$$Q \hookrightarrow (\mathbb{P}^1)^{n-3}.$$

We may interpret $(\mathbb{P}^1)^{n-3}$ as $\overline{M}_{0,\mathcal{A}}$ where $\mathcal{A} = (a_1,\dots,a_n)$ satisfies the following inequalities:

$a_{i_1} + a_{i_2} > 1$ where $\{i_1,i_2\} \subset \{1,2,3\}$;

$a_i + a_{j_1} + \ldots + a_{j_r} \leq 1$ for $i = 1, 2, 3$ and $\{j_1, \ldots, j_r\} \subset \{4, 5, \ldots, n\}$ with $r > 0$.

These conditions guarantee that none of the first three sections coincide, but any of the subsequent sections may coincide with any of the first three or with one another. When $a_1 = a_2 = a_3 = a$ and $a_4 = a_5 = \ldots = a_n = \epsilon$ we have $1/2 < a \leq 1$ and $0 < (n-3)\epsilon \leq 1 - a$. Taking

$$\mathcal{A}(n) = (a, a, a, \underbrace{\epsilon, \ldots, \epsilon}_{n-3 \text{ times}}),$$

we obtain that the compactification of Q by $(\mathbb{P}^1)^{n-3}$ is isomorphic to $\overline{M}_{0,\mathcal{A}(n)}$.

We factor $\rho : \overline{M}_{0,n} \to (\mathbb{P}^1)^{n-3}$ as a product of reduction morphisms. Let Δ_d denote the union of the dimension d diagonals, i.e., the locus where at least $n - 2 - d$ of the points coincide. We will use this notation for both the locus in $(\mathbb{P}^1)^{n-3}$ and its proper transforms. Let

$$F_0 = \pi_1^{-1}(0) \cup \pi_2^{-1}(0) \cup \ldots \cup \pi_{n-3}^{-1}(0)$$

be the locus of points mapping to 0 under one of the projections π_j; we define F_1 and F_∞ analogously. Again, we use the same notation for proper transforms.

Write $Y_0[n] = (\mathbb{P}^1)^{n-3}$ and define the first sequence of blow-ups as follows:

1: $Y_1[n]$ is the blow-up along the intersection $\Delta_1 \cap (F_0 \cup F_1 \cup F_\infty)$;

2: $Y_2[n]$ is the blow-up along the intersection $\Delta_2 \cap (F_0 \cup F_1 \cup F_\infty)$; …

n-4: $Y_{n-4}[n]$ is the blow-up along the intersection $\Delta_{n-4} \cap (F_0 \cup F_1 \cup F_\infty)$.

The variety $Y_k[n]$ is realized by $\overline{M}_{0,\mathcal{A}}$ where

$a_{i_1} + a_{i_2} > 1$ where $\{i_1, i_2\} \subset \{1, 2, 3\}$;

$a_i + a_{j_1} + \ldots + a_{j_r} \leq 1$ (resp. > 1) for $i = 1, 2, 3$ and $\{j_1, \ldots, j_r\} \subset \{4, 5, \ldots, n\}$ with $0 < r \leq n - 3 - k$ (resp. $r > n - 3 - k$).

The second sequence of blow-ups is

n-3: $Y_{n-3}[n]$ is the blow-up along Δ_1;

n-2: $Y_{n-2}[n]$ is the blow-up along Δ_2; …

2n-9: $Y_{2n-9}[n]$ is the blow-up along Δ_{n-5}.

The variety $Y_{k+n-4}[n]$ is realized by $\overline{M}_{0,\mathcal{A}}$ where

> $a_{i_1} + a_{i_2} > 1$ where $\{i_1, i_2\} \subset \{1,2,3\}$;
>
> $a_{j_1} + \ldots + a_{j_r} \leq 1$ (resp. > 1) for $\{j_1, \ldots, j_r\} \subset \{4,5,\ldots,n\}$ with $0 < r \leq n-3-k$ (resp. $r > n-3-k$).

Remark 6.1 Keel [Ke] has factored ρ as a sequence of blow-ups along smooth codimension-two centers in the course of computing the Chow groups of $\overline{M}_{0,n}$. However, the intermediate steps of his factorization do not admit interpretations as $\overline{M}_{0,\mathcal{A}}$. For example, consider

$$\overline{M}_{0,6} \to \overline{M}_{0,4} \times \overline{M}_{0,5} \to \overline{M}_{0,4} \times \overline{M}_{0,4} \times \overline{M}_{0,4} \simeq (\mathbb{P}^1)^3,$$

where the maps $\overline{M}_{0,N} \to \overline{M}_{0,4} \times \overline{M}_{0,N-1}$ are products of the forgetting morphisms arising from the subsets

$$\{1,2,3,N\}, \{1,2,3,\ldots,N-1\} \subset \{1,2,\ldots,N\}.$$

The intermediate space $\overline{M}_{0,4} \times \overline{M}_{0,5}$ is not of the form $\overline{M}_{0,\mathcal{A}}$ for any $\mathcal{A}$.

6.4 Losev-Manin moduli spaces

We refer to the paper [LM], where the following generalization of stable pointed curves is defined. This space was also studied by Kapranov ([Kap1], §4.3) as the closure of a generic orbit of $(\mathbb{C}^*)^{n-3}$ in the space of complete flags in $\mathbb{C}^{n-2}$.

Let S and T be two finite disjoint sets with $|S| = r$ and $|T| = n-r$, B a scheme, and $g \geq 0$. An (S,T)-pointed stable curve of genus g over B consists of the data:

1. a flat family $\pi : C \to B$ of nodal geometrically connected curves of arithmetic genus g;

2. sections $s_1, \ldots, s_r, t_1, \ldots, t_{n-r}$ of π contained in the smooth locus of π;

satisfying the following stability conditions:

1. $K_\pi + \mathrm{supp}(s_1 + \ldots + s_r + t_1 + \ldots + t_{n-r})$ is π-relatively ample;

2. each of the sections $s_1, \ldots, s_r$ are disjoint from all the other sections, but $t_1, \ldots, t_{n-r}$ may coincide.

Now assume that $r = 2$ and $g = 0$, and consider only pointed curves satisfying the following properties:

1. the dual graph is linear;

2. the sections s_1 and s_2 are contained in components corresponding to the endpoints of the graph.

Then there is a smooth, separated, irreducible, proper moduli space $\overline{L}_{n-2}$ representing such (S, T)-pointed stable curves (Theorem 2.2 of [LM]).

This space is isomorphic to $\overline{M}_{0,\mathcal{A}}$ where $\mathcal{A}$ satisfies the following conditions

$$a_1 + a_i > 1 \text{ and } a_2 + a_i > 1 \text{ for each } i = 1, \ldots, n;$$

$$a_{j_1} + \ldots + a_{j_r} \leq 1 \text{ for each } \{j_1, \ldots, j_r\} \subset \{3, \ldots, n\} \text{ with } r > 0.$$

We emphasize that these conditions force the dual graphs of the associated curves to have the properties postulated by Losev and Manin. Specializing the weights, we obtain that $\overline{L}_{n-2} \simeq \overline{\mathcal{M}}_{0,\mathcal{A}}$ where

$$\mathcal{A} = (1, 1, \underbrace{1/(n-2), \ldots, 1/(n-2)}_{n-2 \text{ times}}).$$

Using the reduction maps we obtain explicit blow-up realizations of the Losev-Manin moduli spaces. Setting $\mathcal{B} = (1, 1/(n-2), \ldots, 1/(n-2))$ we obtain a morphism

$$\rho_{\mathcal{B},\mathcal{A}} : \overline{M}_{0,\mathcal{A}} \to \overline{M}_{0,\mathcal{B}}$$

where $\overline{M}_{0,\mathcal{B}} \simeq \mathbb{P}^{n-3}$ is Kapranov's compactification. The Losev-Manin moduli space is the first step of the factorization described in §6.1.

7 Interpretations as log minimal models of moduli spaces

Recently, Keel and his collaborators [KeMc1],[GKM],[G],[Ru] have undertaken a study of the birational geometry of the moduli space of curves, with an emphasis on the geometry of the cones of effective curves and divisors. In many cases, they find natural modular interpretations for contractions and modifications arising from the minimal model program. Therefore, one might expect that natural birational modifications of a moduli space $\overline{M}$ should admit interpretations as log minimal models with respect to a boundary supported on natural divisors of $\overline{M}$. The most accessible divisors are those parametrizing degenerate curves, so we focus on these here.

Fix $\overline{M} = \overline{M}_{0,n}$, the moduli space of genus zero curves with n marked points. For each unordered partition

$$\{1, 2, \ldots, n\} = I \cup J \text{ where } |I|, |J| > 1$$

let $D_{I,J} \subset \overline{M}$ denote the divisor corresponding to the closure of the locus of pointed curves with two irreducible components, with the sections indexed by I on one component and by J on the other. Let δ be the sum of these divisors, with each $D_{I,J}$ appearing with multiplicity one. We can also describe degenerate curves on $\overline{M}_{0,\mathcal{A}}$ (see §4.2). There are two types to consider. First, consider a partition as above satisfying

$$a_{i_1} + \ldots + a_{i_r} > 1, a_{j_1} + \ldots + a_{j_{n-r}} > 1 \quad I = \{i_1, \ldots, i_r\}, J = \{j_1, \ldots, j_{n-r}\}.$$

Let $D_{I,J}(\mathcal{A})$ denote the image of $D_{I,J}$ in $\overline{M}_{0,\mathcal{A}}$ under the reduction map; this is a divisor and parametrizes nodal curves as above. The union of such divisors is denoted ν. Second, any partition with $I = \{i_1, i_2\}$ and $a_{i_1} + a_{i_2} \leq 1$ also corresponds to a divisor $D_{I,J}(\mathcal{A})$ in $\overline{M}_{0,\mathcal{A}}$, parametrizing curves where the sections s_{i_1} and s_{i_2} coincide. These curves need not be nodal. The union of such divisors is denoted γ and the union of γ and ν is denoted δ. The remaining partitions yield subvarieties of $\overline{M}_{0,\mathcal{A}}$ with codimension > 1.

Problem 7.1 *Consider the moduli space $\overline{M}_{0,\mathcal{A}}$ of weighted pointed stable curves of genus zero. Do there exist rational numbers $d_{I,J}$ so that*

$$K_{\overline{M}_{0,\mathcal{A}}} + \sum_{I,J} d_{I,J} D_{I,J}(\mathcal{A})$$

is ample and log canonical? The sum is taken over partitions

$$\{1, 2, \ldots, n\} = I \cup J \textit{ where } |I|, |J| \geq 2,$$

where either

$$a_{i_1} + \ldots + a_{i_r} > 1 \textit{ and } a_{j_1} + \ldots + a_{j_{n-r}} > 1$$

or

$$r = 2 \text{ and } a_{i_1} + a_{i_2} \leq 1.$$

The assertion that the singularities are log canonical implies that the coefficients are nonnegative and ≤ 1.

We shall verify the assertion of Problem 7.1 in examples by computing the discrepancies of the associated reduction morphisms. We also refer the reader to Remark 8.5 for another instance where it is verified.

7.1 Mumford-Knudsen moduli spaces

It is well known that $K_{\overline{M}_{0,n}} + \delta$ is ample and log canonical. We briefly sketch the proof. For ampleness, we use the identity

$$\kappa_{\log} := \pi_*[c_1(\omega_\pi(s_1 + \ldots + s_n))^2] = K_{\overline{M}_{0,n}} + \delta$$

where

$$\pi : (\mathcal{C}_{0,n}, s_1, \ldots, s_n) \to \overline{M}_{0,n}$$

is the universal curve. Fix pointed elliptic curves $(E_i, p_i), i = 1, \ldots, n$, which we attach to an n-pointed rational curve to obtain a stable curve of genus n. This yields an imbedding

$$j : \overline{M}_{0,n} \to \overline{M}_n.$$

The divisor

$$\kappa = u_*[c_1(\omega_u)^2],$$

where $u : \mathcal{C}_n \to \overline{M}_n$ is the universal curve, is ample ([HM] 3.110 and 6.40) and pulls back to $\kappa_{\log}$. As for the singularity condition, it suffices to observe that through each point of $\overline{M}_{0,n}$ there pass at most $n-3$ boundary divisors which intersect in normal crossings.

7.2 Kapranov's examples

We retain the notation of §6.2. The boundary divisors on

$$X_0[n] = \overline{M}_{0,\mathcal{A}} \simeq \mathbb{P}^{n-3}$$

are indexed by partitions

$$\{1, 2, \ldots, n\} = \{i_1, i_2\} \cup \{j_1, \ldots, j_{n-3}, n\}$$

and correspond to the hyperplanes spanned by the points $q_{j_1}, \ldots, q_{j_{n-3}}$. Consider the log canonical divisor

$$K_{X_0[n]} + \alpha D[n], \text{ where } D[n] := \sum_{I=\{i_1,i_2\}} D_{I,J}(\mathcal{A}),$$

which is ample if and only if

$$\alpha \binom{n-1}{2} > n-2, \text{ i.e., } \alpha > 2/(n-1). \tag{2}$$

Let $E_k[n]$ denote the exceptional divisor of the blow-up $X_k[n] \to X_{k-1}[n]$ and

$$\rho_k : X_k[n] \to X_0[n]$$

the induced birational morphism. Each component of $E_k[n]$ is obtained by blowing up a nonsingular subvariety of codimension $n-2-k$, so we obtain

$$K_{X_k[n]} = \rho_k^* K_{X_0[n]} + \sum_{r=1}^{k} (n-3-r) E_r[n].$$

Through each component of the center of $E_k[n]$, there are $\binom{n-1-k}{n-3-k}$ nonsingular irreducible components of $D[n]$. It follows that

$$\rho_k^* D[n] = D[n]_k + \sum_{r=1}^{k} \binom{n-1-r}{n-3-r} E_r[n]$$

where $D[n]_k$ is the proper transform of $D[n]$. The discrepancy equation takes the form

$$K_{X_k[n]} + \alpha D[n]_k = \rho_k^*(K_{X_0[n]} + \alpha D[n]) + \sum_{r=1}^{k} (n-3-r-\alpha \binom{n-1-r}{n-3-r}) E_r[n],$$

which is log canonical provided

$$-1 \geq (n-3-r) - \alpha \binom{n-1-r}{n-3-r} \quad r = 1, \ldots, k.$$

This yields the condition

$$\alpha \leq 2/(n-2),$$

which is compatible with inequality (2).

This computation yields a positive answer to Problem 7.1 for $X_0[n-4]$ but not necessarily for $X_k[n-4]$ with $k > 0$. The exceptional divisors $E_r[n]$ may have large positive discrepancies.

7.3 Keel's example

We retain the notation of §6.3 and use $E_k[n]$ for the exceptional divisor of $Y_k[n] \to Y_{k-1}[n]$ and ρ_k for the birational morphism $Y_k[n] \to Y_0[n]$. Let $F[n] = F_0 + F_1 + F_\infty$ and $D[n]$ the union of the diagonals in $(\mathbb{P}^1)^{n-3}$. Their proper transforms are denoted $F[n]_k$ and $D[n]_k$. The divisor $K_{(\mathbb{P}^1)^{n-3}} + \alpha F[n] + \beta D[n]$ is ample if and only if

$$3\alpha + (n-4)\beta > 2. \tag{3}$$

We have the following discrepancy equations

$$\begin{aligned}
K_{Y_{2n-9}[n]} &= \rho_{2n-9}^* K_{Y_0[n]} + \sum_{r=1}^{n-4}(n-3-r)E_r[n] + \sum_{r=1}^{n-5}(n-4-r)E_{n-4+r}[n] \\
\rho_{2n-9}^* F[n] &= F[n]_{2n-9} + \sum_{r=1}^{n-4}(n-2-r)E_r[n] \\
\rho_{2n-9}^* D[n] &= D[n]_{2n-9} + \sum_{r=1}^{n-4}\binom{n-2-r}{2}E_r[n] + \sum_{r=1}^{n-5}\binom{n-2-r}{2}E_{n-4+r}[n],
\end{aligned}$$

which yield inequalities

$$\begin{aligned}
-1 &\leq (n-3-r) - \alpha(n-2-r) - \beta\binom{n-2-r}{2}, r = 1, \ldots, n-4 \\
-1 &\leq (n-4-r) - \beta\binom{n-2-r}{2}, r = 1, \ldots, n-5.
\end{aligned}$$

These in turn yield

$$\beta \leq 2/(n-3) \quad \alpha + \beta((n-4)/2) \leq 1.$$

To satisfy these conditions and inequality (3), we may choose

$$\alpha = 1/(n-3) \quad \beta = 2/(n-3).$$

8 Variations of GIT quotients of $(\mathbb{P}^1)^n$

In this section, we show how geometric invariant theory quotients of $(\mathbb{P}^1)^n$ may be interpretted as 'small-parameter limits' of the moduli schemes $\overline{M}_{0,\mathcal{A}}$ as $\sum_{j=1}^n a_j \to 2$.

We review the description of the stable locus for the diagonal action of PGL_2 on $(\mathbb{P}^1)^n$ (see [Th], §6 and [GIT], Ch. 3). The group PGL_2 admits no characters, so ample fractional linearizations correspond to line bundles $\mathcal{O}(t_1, \ldots, t_n)$ on $(\mathbb{P}^1)^n$, where the t_i are positive rational numbers. A point $(x_1, \ldots, x_n) \in (\mathbb{P}^1)^n$ is *stable* (resp. *semistable*) if, for each $x \in \mathbb{P}^1$,

$$\sum_{j=1}^{n} t_j \delta(x, x_j) < (\leq) \frac{1}{2} (\sum_{j=1}^{n} t_j),$$

where $\delta(x, x_j) = 1$ when $x = x_j$ and 0 otherwise.

Remark 8.1 The linearizations $\mathcal{T} = (t_1, \ldots, t_n)$ for which a given point of $(\mathbb{P}^1)^n$ is stable (resp. semistable) form an open (resp. closed) subset.

To strengthen the analogy with our moduli spaces, we renormalize so that $t_1 + t_2 + \ldots + t_n = 2$. Then the stability condition takes the following form: for any $\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$, $x_{i_1}, \ldots, x_{i_r}$ may coincide only when

$$t_{i_1} + \ldots + t_{i_r} < 1.$$

In particular, the stable locus is nonempty only when each $t_j < 1$. In this case, the corresponding GIT quotient is denoted $\mathcal{Q}(t_1, \ldots, t_n)$ or $\mathcal{Q}(\mathcal{T})$. We define the *boundary* of $\mathcal{D}_{0,n}$ as

$$\partial \mathcal{D}_{0,n} := \{(t_1, \ldots, t_n) : t_1 + \ldots + t_n = 2, 0 < t_i < 1 \text{ for each } i = 1, \ldots, n\}.$$

We shall say that $\mathcal{T}$ is *typical* if all semistable points are stable and *atypical* otherwise. Of course, $\mathcal{T}$ is typical exactly when $t_{i_1} + \ldots + t_{i_r} \neq 1$ for any $\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$.

Theorem 8.2 *Let $\mathcal{T}$ be a typical linearization in $\partial \mathcal{D}_{0,n}$. Then there exists an open neighborhood U of $\mathcal{T}$ so that $U \cap \mathcal{D}_{0,n}$ is contained in an open fine chamber of $\mathcal{D}_{0,n}$. For each set of weight data $\mathcal{A} \in U \cap \mathcal{D}_{0,n}$, there is a natural isomorphism*

$$\overline{M}_{0,\mathcal{A}} \xrightarrow{\simeq} \mathcal{Q}(\mathcal{T}).$$

proof: We choose

$$\begin{aligned} U &= \{(u_1, \ldots, u_n) \in \mathbb{Q}^n : 0 < u_i < 1 \text{ and} \\ &\quad u_{i_1} + \ldots + u_{i_r} \neq 1 \text{ for any } \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}\}. \end{aligned}$$

It follows that $U \cap \mathcal{D}_{0,n}$ is contained in a fine chamber. In particular, Proposition 5.1 implies that for any weight data $\mathcal{A}_1, \mathcal{A}_2 \in U \cap \mathcal{D}_{0,n}$, we have $\overline{M}_{0,\mathcal{A}_1} \simeq \overline{M}_{0,\mathcal{A}_2}$.

We construct the morphism $\mathcal{Q}(\mathcal{T}) \to \overline{M}_{0,\mathcal{A}}$, where $\mathcal{A} \in U \cap \mathcal{D}_{0,n}$. Points of $\mathcal{Q}(\mathcal{T})$ classify $(x_1, \ldots, x_n) \in (\mathbb{P}^1)^n$ up to projectivities, where $x_{i_1}, \ldots, x_{i_r}$ do not coincide unless $t_{i_1} + \ldots + t_{i_r} < 1$. In our situation, $t_{i_1} + \ldots + t_{i_r} < 1$ if and only if $a_{i_1} + \ldots + a_{i_r} < 1$. Since $K_{\mathbb{P}^1} + \sum a_j x_j$ has positive degree, we may conclude that $(\mathbb{P}^1, x_1, \ldots, x_n)$ represents a point of $\overline{M}_{0,\mathcal{A}}$. As $\mathcal{Q}(\mathcal{T})$ parametrizes a family of pointed curves in $\overline{M}_{0,\mathcal{A}}$, we obtain a natural morphism $\mathcal{Q}(\mathcal{T}) \to \overline{M}_{0,\mathcal{A}}$. This is a bijective birational projective morphism from a normal variety to a regular variety, and thus an isomorphism. $\square$

For an atypical point $\mathcal{T}$ of the boundary the description is more complicated. From a modular standpoint, each neighborhood of $\mathcal{T}$ intersects a number of fine chambers arising from different moduli problems. See Figure 1 for a crude schematic diagram. From an invariant-theoretic standpoint,

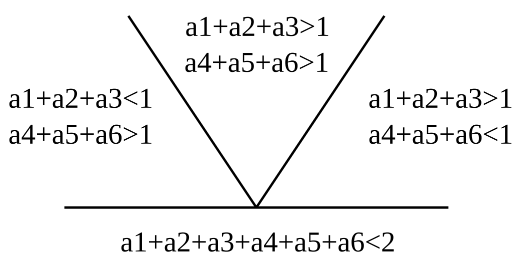

Figure 1: Chamber structure at the boundary

for each linearization $\mathcal{T}'$ in a sufficiently small neighborhood of $\mathcal{T}$ we have a birational morphism

$$\mathcal{Q}(\mathcal{T}') \to \mathcal{Q}(\mathcal{T}),$$

induced by the inclusion of the $\mathcal{T}'$-semistable points into the $\mathcal{T}$-semistable points (cf. Remark 8.1). A more general discussion of this morphism may be found in §2 of [Th].

Theorem 8.3 *Let $\mathcal{T} \in \partial\mathcal{D}_{0,n}$ be an atypical linearization. Suppose that $\mathcal{T}$ is in the closure of the coarse chamber associated with the weight data $\mathcal{A}$. Then there exists a natural birational morphism*

$$\rho : \overline{M}_{0,\mathcal{A}} \longrightarrow \mathcal{Q}(\mathcal{T}).$$

proof: Consider the map $\tau : \mathcal{D}'_{0,n} \to \partial\mathcal{D}_{0,n}$ given by the rule

$$\tau(b_1, \dots, b_n) = (b_1 B^{-1}, \dots, b_n B^{-1}),$$

where $B = (b_1 + \ldots + b_n)/2$ and

$$\mathcal{D}'_{0,n} = \{(b_1, \dots, b_n) : b_1 + \ldots + b_n \geq 2, 0 < b_i < 1\}.$$

For each $\mathcal{B} \in \mathcal{D}'_{0,n}$ such that $\tau(\mathcal{B})$ is typical, we obtain a birational morphism

$$\rho_{\tau(\mathcal{B}),\mathcal{B}} : \overline{M}_{0,\mathcal{B}} \to \mathcal{Q}(\tau(\mathcal{B})).$$

This is defined as the composition of the reduction morphism $\rho_{\mathcal{B}_1,\mathcal{B}} : \overline{M}_{0,\mathcal{B}} \to \overline{M}_{0,\mathcal{B}_1}$, where $\mathcal{B}_1 = \epsilon\mathcal{B} + (1-\epsilon)\tau(\mathcal{B})$ for small $\epsilon > 0$, and the isomorphism $\overline{M}_{0,\mathcal{B}_1} \to \mathcal{Q}(\tau(\mathcal{B}))$ given by Theorem 8.2.

The closure of each coarse chamber is the union of the closures of the finite collection of fine chambers contained in it. By Proposition 5.1, we may assume $\mathcal{A}$ is in a fine open chamber Ch with closure $Ch' \subset \mathcal{D}'_{0,n}$ containing $\mathcal{T}$. Clearly, $\tau(Ch')$ contains $\mathcal{T}$ and $\tau(Ch)$ contains typical points arbitrarily close to $\mathcal{T}$. Choose $\mathcal{A} \in Ch$ so that $\tau(\mathcal{A})$ is typical and close to $\mathcal{T}$, so there exists a generalized reduction morphism

$$\rho_{\tau(\mathcal{A}),\mathcal{A}} : \overline{M}_{0,\mathcal{A}} \to \mathcal{Q}(\tau(\mathcal{A}))$$

and an induced birational morphism of GIT quotients

$$\mathcal{Q}(\tau(\mathcal{A})) \to \mathcal{Q}(\mathcal{T}).$$

Composing, we obtain the birational morphism claimed in the theorem. $\square$

In light of Theorems 8.2 and 8.3, when $\mathcal{T} \in \partial\mathcal{D}_{0,n}$ we may reasonably interpret the GIT quotient $\mathcal{Q}(\mathcal{T})$ as $\overline{M}_{0,\mathcal{T}}$. This gives one possible definition for moduli spaces with weights summing to two (cf. §2.1.2.)

Remark 8.4 (Suggested by I. Dolgachev) Theorem 8.2 implies that $\overline{M}_{0,\mathcal{A}}$ is realized as a GIT quotient $Q(\mathcal{T})$ when the closure of the coarse chamber associated with $\mathcal{A}$ contains a typical linearization. For example, if

$$\mathcal{A} = (2/3, 2/3, 2/3, \underbrace{\epsilon, \dots, \epsilon}_{n-3 \text{ times}}), \quad \epsilon > 0 \text{ small},$$

then the moduli space $\overline{M}_{0,\mathcal{A}} \simeq (\mathbb{P}^1)^{n-3}$, studied in §6.3, arises as a GIT quotient (see [KLW]).

Remark 8.5 We explicitly construct

$$\overline{M}_{0,\mathcal{T}} := \mathcal{Q}(\mathcal{T}) \quad \mathcal{T} = (1/3, 1/3, 1/3, 1/3, 1/3, 1/3)$$

using a concrete description of the map

$$\rho : \overline{M}_{0,(1,1/3,1/3,1/3,1/3,1/3)} \to \overline{M}_{0,\mathcal{T}}$$

produced in Theorem 8.3. Recall that the first space is the space $X_1[6]$ (see §6.2), isomorphic to $\mathbb{P}^3$ blown-up at five points $p_1, \ldots, p_5$ in linear general position. The map ρ is obtained by contracting the proper transforms ℓ_{ij} of the ten lines joining pairs of the points. It follows that $\overline{M}_{0,\mathcal{T}}$ is singular at these ten points. Concretely, ρ is given by the linear series of quadrics on $\mathbb{P}^3$ passing through $p_1, \ldots, p_5$. Thus we obtain a realization of $\overline{M}_{0,\mathcal{T}}$ as a cubic hypersurface in $\mathbb{P}^4$ with ten ordinary double points. Finally, we observe that $K_{\overline{M}_{0,\mathcal{T}}} + \alpha\delta$ is ample and log canonical provided $2/5 < \alpha \leq 1/2$, thus yielding a positive answer to Problem 7.1 in this case.

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