**Final Exam**
**Math 102 – Spring 2009**

Solutions.

**Instructions**: This is a 3 hour exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem. Attach extra paper if you need more space.

**Name**:

**Honor Pledge**: On my honor, I have neither received nor given any unauthorized aid on this exam.

**Signature**:

| Problem | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
| 11 | |
| **Total** | |

- **Problem 1**
  Compute the integral

$$\int \frac{x^2}{\sqrt{16-x^2}}dx$$

Use the trig. substitution $x = 4\sin\theta$ ; $dx = 4\cos\theta\, d\theta$

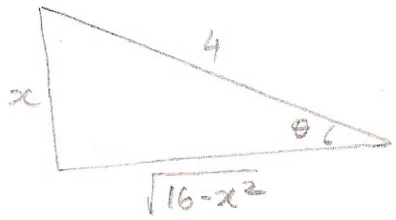
$\theta = \sin^{-1}\left(\frac{x}{4}\right)$

$$\int \frac{x^2}{\sqrt{16-x^2}}dx = \int \frac{16\sin^2\theta}{4\cos\theta} 4\cos\theta\, d\theta = \int 16\sin^2\theta\, d\theta = \int 8(1-\cos 2\theta)\, d\theta$$

$$= 8\theta - 4\sin 2\theta + C = 8\theta - 8\sin\theta\cos\theta + C =$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - 8 \cdot \frac{x}{4} \cdot \cos\left(\sin^{-1}\frac{x}{4}\right) + C$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C.$$




- **Problem 2**
  Compute the integral

$$\int \frac{x^2}{(x+2)^3}dx$$

Use partial fractions.

$$\frac{x^2}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \quad \Big| \cdot (x+2)^3$$

$$x^2 = A(x+2)^2 + B(x+2) + C$$

$$x^2 = Ax^2 + 4Ax + 4A + Bx + 2B + C$$

$x^2$: $1 = A$     $A = 1$

$x^1$: $0 = 4A + B$     $B = -4$

$x^0$: $0 = 4A + 2B + C$     $C = 4$

$$\int \frac{x^2}{(x+2)^3}dx = \int \frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{4}{(x+2)^3}\, dx = \qquad (u = x+2,\ du = dx).$$

$$= \ln|x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + C$$

- **Problem 3**

Decide if the following improper integral is convergent or divergent. If it converges, give the value to which it converges.

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$ Not continuous at $x = 0$.

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \lim_{b\to 0^+}\int_b^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \quad \left( u = \sqrt{x} \quad du = \frac{1}{2\sqrt{x}}dx \right)$$

$$= \lim_{b\to 0^+}\int_b^1 2e^u du = \lim_{b\to 0^+} 2e^{\sqrt{x}}\Big]_b^1 = \lim_{b\to 0^+}\left(2e - 2e^b\right) = 2e-2$$

The integral is convergent to $2e-2$.

- **Problem 4**

Find the third degree Taylor polynomial of $f(x) = \sqrt{x}$ , at $a = 1$.

$f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}}$ ; $f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4x^{3/2}}$ $f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8x^{5/2}}$ $f'''(1) = \frac{3}{8}$

$$T_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

- **Problem 5**

A. Consider the function

$$f(x) = \frac{\ln x}{2x}, \text{ for } x \geq 3$$

Is $f(x)$ increasing, decreasing, or neither on the interval $[3, \infty)$? Justify your answer.

$$f'(x) = \frac{\frac{1}{x} \cdot 2x - 2\ln x}{(2x)^2} = \frac{2 - 2\ln x}{(2x)^2} = \frac{2(1-\ln x)}{(2x)^2}$$

$x \geq 3 \Rightarrow \ln x \geq \ln 3 > 1 \Rightarrow 1 - \ln x < 0 \Rightarrow$

$f'(x) < 0$ for $x \geq 3$ and so $f$ is decreasing on $[3, +\infty)$

B. Is the following series convergent or divergent? Justify your answer.

$$\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{2n}$$

Use AST ; $a_n = \frac{\ln n}{2n}$

1) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{\ln n}{2n} \underset{\frac{\infty}{\infty}}{=} \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{2} = 0$ ✓

2) $a_{n+1} \leq a_n$ (by A) ✓

AST $\Rightarrow$ the series is convergent

C. Is the series from B absolutely convergent? Justify your answer.

We look at $\sum_{n=3}^{\infty} \frac{\ln n}{2n}$

$a_n = \frac{\ln n}{2n} > 0$. Use DCT. ~~For~~ $n \geq 3$ $\frac{\ln n}{2n} > \frac{1}{2n}$

$$\left.\begin{array}{l} \frac{\ln n}{2n} > \frac{1}{2n} \\ \sum_{n=3}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=3}^{\infty} \frac{1}{n} \text{ (divergent)} \end{array}\right\} \xrightarrow{DCT}$$

$\sum_{n=3}^{\infty} \frac{\ln n}{2n}$ divergent ; so $\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{2n}$ is not absolutely convergent



(the integral test also works)

- **Problem 6**

Decide if the following series are convergent or divergent? Be sure to show that you've checked the hypotheses for any tests you use.

A.

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

$a_n = \frac{n!}{e^{n^2}} > 0$. Apply the ratio test.

$$\lim_{n\to\infty} \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} = \lim_{n\to\infty} \frac{(n+1)\, e^{n^2}}{e^{n^2+2n+1}} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} \overset{\frac{\infty}{\infty}}{=}$$

$$= \lim_{n\to\infty} \frac{1}{2e^{2n+1}} = 0 < 1$$

Ratio test $\Rightarrow$ the series is convergent

B.

$$\sum_{n=1}^{\infty} n \cdot \sin \frac{1}{n}$$

$$\lim_{n\to\infty} n \cdot \sin \frac{1}{n} \overset{\infty \cdot 0}{=} \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} \underset{LH}{\overset{\frac{0}{0}}{=}} \lim_{n\to\infty} \frac{\cos \frac{1}{n} \cdot \left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} =$$

$$= \lim_{n\to\infty} \cos \frac{1}{n} = 1 \neq 0$$

$n^{th}$ term test $\Rightarrow$ the series diverges

C.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n^3}$$

$\frac{\sqrt{n}}{n^2+n^3} > 0$ Apply DCT.

$$\left.\begin{array}{l} \frac{\sqrt{n}}{n^2+n^3} < \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}} \\ \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \text{ convergent} \\ \quad (p\text{-series}, p>1) \end{array}\right\} \overset{\text{DCT}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n^3} \text{ also convergent}$$

LCT with $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ also works.

- **Problem 7**

Find the interval of convergence for the Taylor series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{3n-1}.$$

Don't forget to check the endpoints of the interval and to show that you've check the hypotheses for any tests you use.

Ratio test :

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{3n+2} \cdot \frac{3n-1}{(x-2)^n} \right| = \lim_{n\to\infty} \frac{3n-1}{3n+2} \cdot |x-2| = |x-2| < 1 \Rightarrow$$

$$\Rightarrow \quad -1 < x-2 < 1 \quad \Rightarrow \quad 1 < x < 3.$$

$x=1$ : $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3n-1}$ Use AST

$a_n = \dfrac{1}{3n-1}$

1) $\displaystyle\lim_{n\to\infty} \frac{1}{3n-1} = 0$ ✓

2) $a_{n+1} = \dfrac{1}{3n+2} < \dfrac{1}{3n-1} = a_n$ ✓

Convergent by AST

$x=3$ : $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n-1}$

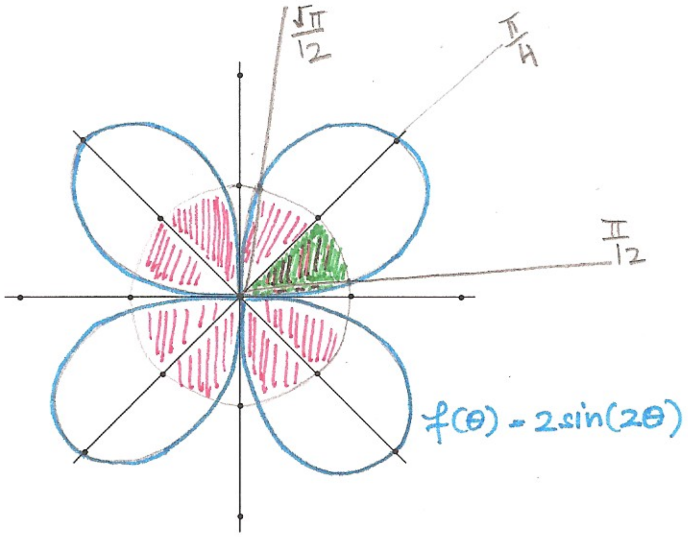
$\dfrac{1}{3n-1} > 0$. Use LCT.

$$\left.\begin{array}{l} \displaystyle\lim_{n\to\infty} \frac{\frac{1}{3n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3} > 0 \ (< +\infty) \\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent} \end{array}\right\} \xrightarrow{LCT} \sum_{n=1}^{\infty} \frac{1}{3n-1} \text{ is divergent}$$

$I = [1, 3).$

- **Problem 8**
  Let $f(\theta) = 2\sin(2\theta)$.

A. Sketch the graph of the polar equation $r = f(\theta)$.




B. Calculate the area of the region enclosed by the four loops of the graph of $r = f(\theta)$ which also lies inside the circle $r = 1$.

Find intersection points ( in quadrant I)

$$2\sin 2\theta = 1 \implies \sin 2\theta = \frac{1}{2} \implies 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

Area = 8· green area

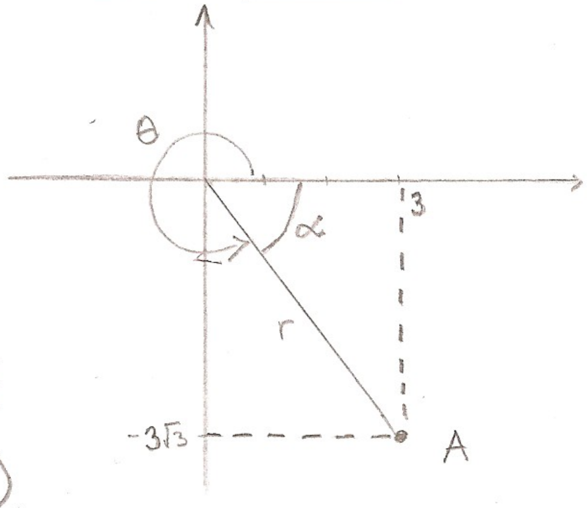
$$= 8\left[\int_0^{\frac{\pi}{12}} \frac{1}{2}\cdot(2\sin 2\theta)^2\, d\theta + \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{1}{2}\cdot 1^2\, d\theta\right] =$$

$$= 8\left[\int_0^{\frac{\pi}{12}} (1-\cos 4\theta)\, d\theta + \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{1}{2}\, d\theta\right] =$$

$$= 8\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{12}} + 4\theta\Big]_{\frac{\pi}{12}}^{\frac{\pi}{4}} = 8\cdot\frac{\pi}{12} - 2\cdot\frac{\sqrt{3}}{2} + 4\left(\frac{\pi}{4} - \frac{\pi}{12}\right)$$

$$= \frac{2\pi}{3} - \sqrt{3} + \frac{2\pi}{3} = \boxed{\frac{4\pi}{3} - \sqrt{3}}$$

- **Problem 9**
  Consider the point $A$ given in rectangular coordinates by $(x, y) = (3, -3\sqrt{3})$. Find the polar coordinates $(r, \theta)$ of $A$, where $r > 0$ and $0 \le \theta < 2\pi$.




$r^2 = x^2 + y^2$

$\left.\begin{array}{l} r^2 = 9 + 27 = 36 \\ r > 0 \end{array}\right\} \Rightarrow r = 6.$

$\sin\alpha = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} \Rightarrow \alpha = \frac{\pi}{3}$

$\theta = 2\pi - \alpha = \frac{5\pi}{3} \in [0, 2\pi)$

$A\left(6, \frac{5\pi}{3}\right).$

- **Problem 10**
  Consider the parametric curve $C$ given by $x(t) = e^t$, $y(t) = t^2$.

A. What is the slope of the line tangent to $C$ at the point $(x(1), y(1))$?

@ time $t$. Slope $= \frac{dy}{dx} \quad \frac{dy/dt}{dx/dt} = \frac{2t}{e^t}$

@ $t = 1$ Slope $= \frac{2}{e}$

B. Find the area between $C$ and the $x$-axis for $0 \le t \le 1$.

$x(0) < x(1)$

$A = \int_0^1 y(t)\, x'(t)\, dt = \int_0^1 t^2 e^t dt$ (int. by parts)

$u = t^2 \quad du = 2t\,dt$
$dv = e^t dt \quad v = e^t$

$A = t^2 e^t \Big]_0^1 - \int_0^1 e^t \cdot 2t\,dt =$

$u = 2t \quad du = 2dt$
$dv = e^t dt \quad v = e^t$

$= t^2 e^t - 2te^t \Big]_0^1 + \int_0^1 2e^t dt =$

$= t^2 e^t - 2te^t + 2e^t \Big]_0^1$

$= (e - 2e + 2e) - 2e^0$

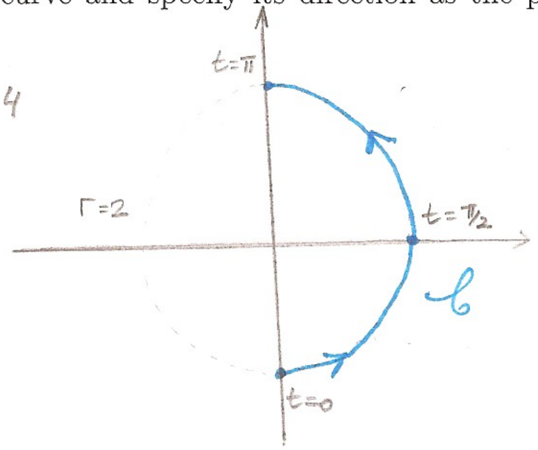
$\boxed{= e - 2}$

- **Problem 11**

Consider the parametric curve $\mathcal{C}$ given by $x(t) = 2\sin t$ , $y(t) = -2\cos t$, $0 \le t \le \pi$.

A. Graph this curve and specify its direction as the parameter $t$ goes from 0 to $\pi$.

$x^2+y^2=4$




$t=0 : (0,-2)$

$t=\frac{\pi}{2} : (2,0)$

$t=\pi : (0,2)$

B. Compute the length of the curve $\mathcal{C}$.

1) $L = \frac{1}{2}(2\pi r) = \pi r = 2\pi$

or.

2) $L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\pi} \sqrt{4\cos^2 t + 4\sin^2 t}\, dt =$

$= \int_0^{\pi} 2\, dt = 2\pi$