Solutions to Exam 2, Math 212 Spring 2012

1. (a) For $h(x,y) = 100 - x^6 - 4y^2$, $\nabla h(x,y) = (h_x, h_y) = (-6x^5, -8y)$.

(b) The *horizontal* direction of the initial downhill roll is given by the *direction* of $-\nabla h(-1,1) = -(6,-8) = (-6,8)$. Moreover, the *length* of the gradient, $\|\nabla h(-1,1)\| = \sqrt{(-6)^2+8^2} = 10$, gives the ratio between the vertical and horizontal components of the initial downhill direction. Also the vertical component is negative because it is going downhill. We conclude that the unit vector in $\mathbf{R}^3$ in the initial direction of the roll is

$$(-6,8,10\cdot 10)/\sqrt{6^2+8^2+100^2} = (-6,8,100)/\sqrt{10100} = (-3,4,50)/\sqrt{25025}\ .$$

A second solution involves defining the curve on the hill (i.e. on the graph of h) above a line through $(-1,1)$ in the gradient direction:

$$x(t) = -1-6t,\ \ y(t) = 1+8t, z(t) = h(x(t),y(t)) = 100-(-1-6t)^6-4(1+8t)^2\ .$$

Noting that the coefficient of t in $z(t)$ is $-6(-1)^5(-6)-4(2)(8) = -100$, the initial velocity of this curve is $\frac{d}{dt}\big|_{t=0}(x(t),y(t),z(t)) = (-6,8,-100)$ which again gives the normalized vector $(-3,4,50)/\sqrt{25025}$.

(c) $\nabla h(-1,1)\cdot(1,0) = (6,-8)\cdot(1,0) = 6$ and $\nabla h(-1,1)\cdot(0,-1) = (6,-8)\cdot(0,-1) = 8$. So the 2nd path is steeper.

2. Note that S is the level 8 surface of the function $f(x,y,z) = 2x^2+3y^2+4z^2-xy$ and that

$$f_x = 4x-y\ ,\ f_y = 6y-x\ ,\ f_z = 8z\ , \nabla f(1,1,1) = (4-1,6-1,8) = (3,5,8)\ .$$

So the tangent plane to S at $(1,1,1)$has the form $3x+5y+8z = c$ where $c = 3(1)+5(1)+8(1) = 16.$, that is, $3x+5y+8z = 16$.

(b) A parallel tangent plane at another point (x,y,z) of S would have a parallel normal vector, hence, for some nonzero λ,

$$4x-y = 3\lambda\ ,\ 6y-x = 5\lambda\ ,\ 8z = 8\lambda$$

so that $z=\lambda$. We can multiply the second equation by 4 and add it to the first to see that $23y = 23z$ or $y = z$. Then $x = 6y-5z = y$. Plugging into the equation for S, we have $8 = 2y^2+3y^2+4y^2-y^2 = 8y^2$. Thus $x = y = z = \pm 1$. We already had the point $(1,1,1)$, and we find the new point $(-1,-1,-1)$ with the parallel tangent plane.

Another way to find this other point is to notice that S is symmetric under the central reflection $(x,y,z)\to(-x,-y,-z)$ which takes the tangent plane at $(1,1,1)$ to a parallel tangent plane at $(-1,-1,-1)$.

3. (a) $x = r\cos\theta$ and $y = r\sin\theta$.

(b) From (a) we have

$$\frac{\partial r}{\partial r} = \cos\theta\ ,\quad \frac{\partial x}{\partial \theta} = -r\sin\theta\ ,\quad \frac{\partial y}{\partial r} = \sin\theta\ ,\quad \frac{\partial y}{\partial \theta} = r\cos\theta\ .$$

The chain rule the gives

$$\frac{\partial v}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\ ,$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\frac{\partial u}{\partial x}\sin\theta + r\frac{\partial u}{\partial y}\cos\theta\ ,$$

$$\Big(\frac{\partial v}{\partial r}\Big)^2 + r^{-2}\Big(\frac{\partial v}{\partial \theta}\Big)^2 = \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2\Big]\big(\cos^2\theta + \sin^2\theta\big)$$

$$= \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2\ .$$

4. For $f(x,y) = 6y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 - x^2$, we readily find that $f_x = -2x$ and $f_y = 12y + y^2 - y^3$. Setting these both equal to 0, we find that

$$x = 0 \ \text{ and } \ -y^3 + y^2 + 12 = 0\ .$$

We can factor this cubic as $y(-y+4)(y+3)$. So $y = 0, 4$, or -3, and there are 3 critical points $(0,0)$, $(0,4)$, $(0,-3)$.

(b) $f_{xx} = -2$, $f_{xy} = f_{yx} = 0$, and $f_{yy} = 12 + 2y - 3y^2$. At the 3 critical points,

$$f_{xx}(0,0) = f_{xx}(0,4) = f_{xx}(0,-3) = -2\ ,$$

$$f_{xy}(0,0) = f_{xy}(0,4) = f_{xy}(0,-3) = 0\ ,$$

$$f_{yy}(0,0) = 12,\ \ f_{yy}(0,4) = -28,\ \ f_{yy}(0,-3) = -21\ .$$

So $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$ has $D(0,0) = -24$, $D(0,4) = 56$, $D(0,-3) = 42$.

(c) $(0,0)$ is neither a local max nor local min, because the local behavior is like $-2x^2 + 12y^2$.

$(0,4)$ is a local max because the local behavior is like $-x^2 - 14y^2$.

$(0,-3)$ is also a local max because the local behavior is like $-x^2 - (21/2)y^2$.

5. (a)

$$\begin{aligned}
\int_0^{\pi/2}\int_0^{\pi/4} \sin(x+2y)\,dxdy &= \int_0^{\pi/2} -\cos(x+2y)|_{x=0}^{x=\pi/4}\,dy \\
&= \int_0^{\pi/2} \big[-\cos(\pi/4+2y)+\cos(2)y\big]\,dy \\
&= \frac{-\sin(\pi/4+2y)}{2}\Big|_0^{\pi/2} + \frac{\sin 2y}{2}\Big|_0^{\pi/2} \\
&= \frac{-\sin(5\pi/4)}{2} + \frac{\sin(\pi/4)}{2} \\
&= \frac{-(-1/\sqrt{2})+(1/\sqrt{2})}{2} = 1/\sqrt{2}\,.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^{\pi/4}\int_0^{\pi/2} \sin(x+2y)\,dydx &= \int_0^{\pi/4} -\frac{cos(x+2y)}{2}|_{y=0}^{y=\pi/2}\,dy \\
&= \int_0^{\pi/4} \Big[\frac{-\cos(x+\pi)}{2} + \frac{\cos x}{2}\Big]\,dx \\
&= \frac{-\sin(x+\pi)}{2}\Big|_0^{\pi/4} + \frac{\sin x}{2}\Big|_0^{\pi/4} \\
&= \frac{-\sin(5\pi/4)}{2} + \frac{\sin(\pi/4)}{2} \\
&= \frac{-(-1/\sqrt{2})+(1/\sqrt{2})}{2} = 1/\sqrt{2}\,.
\end{aligned}$$

6. The function $f(x,y,z) = x+y+z$ has no critical points because its gradient is $(1,1,1)$. So the minimum and maximum occurs on the boundary of D which is defined by $g(x,y,z) = x^2+y^2/4+z^2 = 1$. The equation $\nabla f = \lambda \nabla g$ gives

$$1 = 2\lambda x\ ,\ 1 = \frac{1}{2}\lambda y\ ,\ 1 = 2\lambda z$$

gives $\lambda \neq 0$, $y = 4x$, and $z = x$. So

$$1 = g(x,4x,x) = x^2+4x^2+x^2 = 6x^2\ ,\ \text{and}\quad x = \pm\frac{1}{\sqrt{6}}\ .$$

Since $f(\frac{1}{\sqrt{6}},\frac{4}{\sqrt{6}},\frac{1}{\sqrt{6}}) = \sqrt{6}$ and $f(-\frac{1}{\sqrt{6}},-\frac{4}{\sqrt{6}},-\frac{1}{\sqrt{6}}) = -\sqrt{6}$, we see that tne maximum value of $\sqrt{6}$ is obtained at $(\frac{1}{\sqrt{6}},\frac{4}{\sqrt{6}},\frac{1}{\sqrt{6}})$, and the minimum value of $-\sqrt{6}$ is obtained at $(-\frac{1}{\sqrt{6}},-\frac{4}{\sqrt{6}},-\frac{1}{\sqrt{6}})$.