Rice University, Spring 2010

Final Exam

Math 212
Instructor: Dr. O'Donnol

04/28/2010

Pledge: ____

Name: ______________________ Signature: ______________________

Please read the following information

- This exam is closed book, closed notes. Calculators are NOT allowed.
- You have 3 hours to complete the exam.
- This test is worth 200 points total.
- Show all your work to earn full credit.
- Clearly indicate your final answers (box, circle, write "answer", etc.).
- Justify your answers whenever possible.
- No credit will be given for correct but unsupported answers.
- Points will be deducted for incorrect, irrelevant or incoherent statements.
- Good luck!

1. (*15 points*) Mark each of the following quantities as **CS** (constant scalar), **SF** (scalar function), **CV** (constant vector), **VF** (vector function), or **ND** (not defined). No reasons need be given. Assume:

   $\mathbf{u}$ is a fixed unit vector in $\mathbb{R}^3$

   $\mathbf{v}$ is a fixed vector in $\mathbb{R}^3$

   $g(x, y, z)$ is a fixed but arbitrary differentiable function whose domain is $\mathbb{R}^3$

   $\mathbf{F}(x, y, z)$ is a fixed but arbitrary differentiable vector field $\mathbb{R}^3 \to \mathbb{R}^3$

   C is the line from the point $(0, 0, 0)$ to $(5, 6, 7)$

   S is the surface of the unit sphere, centered at the origin and oriented outward

   B is the region inside the unit sphere S

   $\mathbf{u} \cdot \mathbf{v}$ ____________

   $\mathbf{u} \times \mathbf{v}$ ____________

   $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$ ____________

   The directional derivative of g in the $\mathbf{u}$ direction, based at the point $(1, 2, 3)$ ____________

   ∇g ____________

   $\nabla \mathbf{F}$ ____________

   div g ____________

   div $\mathbf{F}$ ____________

   curl g ____________

   curl $\mathbf{F}$ ____________

   curl $\mathbf{F} + \nabla g$ ____________

   $\int_C \nabla g \cdot d\mathbf{s}$ ____________

   $\iint_S \mathbf{F} \cdot d\mathbf{S}$ ____________

   $\iiint_B \text{div } \mathbf{F} \, dV$ ____________

   $\iiint_B \text{curl } \mathbf{F} \cdot d\mathbf{S}$ ____________

2. (*15 points*) Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = \left(\sqrt{x} + y^2, x^2 + \sqrt{y}\right)$ and where C is the curve $y = \sin(x)$ from $(0, 0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0, 0)$.

3. (*25 points*) Let $M(x, y, z) = (M_1(x, y, z), M_2(x, y, z))$ where $M_1(x, y, z) = xe^{y-z^2}$ and $M_2(x, y, z) = \cos(yz)$, and let $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$ where $x(u, v) = 2uv$, $y(u, v) = u - v$, and $z(u, v) = u + v$.

(a) Find $\frac{\partial M}{\partial(x,y,z)}$, the matrix of partial derivatives of M with respect to x, y, and z.

(b) Find $\frac{\partial \Phi}{\partial(u,v)}$, the matrix of partial derivatives of Φ with respect to u and v.

(c) Express $\frac{\partial M}{\partial(u,v)}$ in terms of $\frac{\partial M}{\partial(x,y,z)}$ and $\frac{\partial \Phi}{\partial(u,v)}$.

(d) Find $\frac{\partial M_1}{\partial v}(3, -1)$.

4. (*10 points*)

(a) Is $\mathbf{F}(x, y, z) = (x^3 - 3xy^2, y^3 - 3x^2y, z)$ a conservative vector field? **Explain**.

(b) Suppose $\mathbf{G}$ is a conservative vector field in the plane such that

$\int_{\mathbf{c}} \mathbf{G} \cdot d\mathbf{s} = 4$
$\int_{ABC} \mathbf{G} \cdot d\mathbf{s} = 6$
$\int_{AO} \mathbf{G} \cdot d\mathbf{s} = 2$
$\int_{OF} \mathbf{G} \cdot d\mathbf{s} = -1$

Find

(i) $\int_{\mathbf{d}} \mathbf{G} \cdot d\mathbf{s} =$ ____________

(ii) $\int_{CEF} \mathbf{G} \cdot d\mathbf{s} =$ ____________

(iii) $\int_{BC} \mathbf{G} \cdot d\mathbf{s} =$ ____________

5. (*15 points*) Let

$$f(x,y)=\frac{\sin^2(xy-2)}{xy-2}$$

(a) Find

$$\lim_{(x,y)\to(3,1)} f(x,y)$$

(b) Where is $f(x,y)$ defined?

(c) Find

$$\lim_{(x,y)\to(1,2)} f(x,y)$$

6. (*20 points*) Calculate $\iiint_E yz\ dV$, where E is the region above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 1$.

7. (*20 points*) Find the point(s) on the surface $z = 2xy + 4$ closest to the origin.

8. (*15 points*)

(a) Determine the maximum rate of change of the pressure function $f(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ at the point $P = (1, 2, -1)$, and the direction in which it occurs.

(b) At what point Q is the maximum rate of change of $g(x, y, z) = x^2yz$ in the direction $\mathbf{i} - \mathbf{j} - \mathbf{k}$?

9. (*25 points*)

(a) Find the linear transformation T that maps the unit square $[0,1] \times [0,1]$ to the parallelogram determined by $\mathbf{x} = (1,-3)$ and $\mathbf{y} = (2,-1)$.

(b) Let $S(u,v) = (v, 1-u^2)$. Sketch the image of $D = [-1,1] \times [-1,1]$ under S.

(c) Is S one-to-one on D? **Explain**.

(d) Is there a domain D^* such that $S : D^* \to D^*$ is onto? If so, find such a D^* and **explain**.

10. (*20 points*) Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \left(\frac{x}{2}, y, x^y\right)$ and S is the union of the half ellipsoid $\left(\frac{x}{2}\right)^2 + y^2 + \left(\frac{z-2}{3}\right)^2 = 1$ for $z \geq 2$ and the elliptical cylinder $\left(\frac{x}{2}\right)^2 + y^2 = 1$ for $0 \leq z \leq 2$, with outward pointing normal.

11. (*20 points*) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F}(x, y, z) = (2x + 3y)\mathbf{i} - (4y + 3z)\mathbf{j} + 4z\mathbf{k}$, where S consists of the paraboloid $z = x^2 + y^2$ with $0 \leq z \leq 1$ and the disk $0 \leq x^2 + y^2 \leq 1$ with $z = 1$, and outward pointing normal.

Scratch paper