Math 567 Lecture Notes

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2 Linear subspaces in projective space

We shall see that the space of rational curves on a projective variety has a natural structure as an algebraic variety. The prototypical example is the space of lines $\ell \subset \mathbb{P}^n$.

2.1 Grassmannians

For each positive $m \leq n$, let $\mathbb{G}(m,n)$ denote the set of all m-dimensional linear subspaces of $\mathbb{P}^n$. Each such subspace is obtained as the projectivization of an $(m+1)$-dimensional vector subspace $\Lambda \subset \mathbb{C}^{n+1}$. Let $\mathrm{Gr}(m+1, n+1)$ denote the collection of all such subspaces; we obtain a natural identification

$$\begin{array}{rcl} \mathrm{Gr}(m+1,n+1) & = & \mathbb{G}(m,n) \\ \Lambda & \mapsto & \mathbb{P}(\Lambda). \end{array}$$

Both these sets are called the *Grassmannian*. The choice of notation is dictated by the context, i.e., whether we want to emphasize the projective geometry or the linear algebra.

2.2 Algebraic variety structure

Let $e_0, \ldots, e_{n+1}$ denote the standard basis vectors in $\mathbb{C}^{n+1}$. Each $(m+1)$-dimensional subspace $\Lambda \subset \mathbb{C}^{n+1}$ can be expressed

$$\Lambda = \mathrm{span}(v_0, \ldots, v_m)$$

where $v_0, \ldots, v_m \in \mathbb{C}^{n+1}$ are linearly independent vectors. Let V denote the $(m+1) \times (n+1)$ matrix with rows equal to $v_0, \ldots, v_m$; it has rank $m+1$ and thus some maximal minor is invertible. Note that the span of the rows of V does not change if we multiply it on the left be an invertible $(m+1) \times (m+1)$ matrix.

For notational simplicity, assume the first $m+1$ columns of V is an invertible matrix, denoted C. Consider the matrix

$$C^{-1}V = (I_{m+1}|B)$$

where I_{m+1} is the identity and $B = (b_{ij})_{i=0,\ldots,m, j=m+1,\ldots,n}$. Thus we can write

$$\Lambda = \operatorname{span}(e_i + \sum_{j=m+1}^{n} b_{ij}, i = 0, \ldots, m).$$

The subspace Λ determines the matrix B uniquely.

Exercise 1 Let B and B' be $(m+1) \times (n-m)$ matrices. Show that

$$\operatorname{span}(e_i + \sum_{j=m+1}^{n} b_{ij}, i = 0, \ldots, m) = \operatorname{span}(e_i + \sum_{j=m+1}^{n} b'_{ij}, i = 0, \ldots, m)$$

implies that $B = B'$.

Regard the entries of B as the coordinates functions of affine space $\mathbb{A}^{(m+1)(n-m)}$. We obtain a distinguished subset

$$\mathbb{A}^{(m+1)(n-m)} \simeq U_{0\ldots m} \subset \operatorname{Gr}(m+1, n+1)$$

corresponding to subspaces spanned by rows of $(I_{m+1}|B)$.

More generally, for each partition

$$\{0, \ldots, n\} = I \cup J, \quad I = \{i_0, \ldots, i_m\}, J = \{j_1, \ldots, j_{n-m}\}$$

with $i_0 < i_1 < \ldots < i_m$ and $j_1 < \ldots < j_{n-m}$, we have the distinguished subset

$$\mathbb{A}^{(m+1)(n-m)} \simeq U_I \subset \operatorname{Gr}(m+1, n+1)$$

parametrizing the subspaces

$$\operatorname{span}(e_i + \sum_{j \in J} b_{ij}, i \in I).$$

Exercise 2 Show that

$$\mathrm{Gr}(m+1, n+1) = \cup_{|I|=m+1} U_I, \tag{1}$$

i.e., each $\Lambda \in \mathrm{Gr}(m+1, n+1)$ lies in at least one U_I.

Theorem 3 *The covering (1) endows* $\mathrm{Gr}(m+1, n+1)$ *with the structure of an algebraic variety. It is rational of dimension* $(m+1)(n-m)$.

The kernel of the argument is to show that the affine variety structures on the open sets U_I are compatible. For more details see [1, ch.11]. Here is a special case:

Exercise 4 Suppose that $m = 1$ and $n = 3$. Given

$$I, I' \subset \{0, 1, 2, 3\}, |I| = |I'| = 2$$

show that the inclusions $U_I, U_{I'} \subset \mathrm{Gr}(2, 4)$ induce a birational map

$$\mathbb{A}^4 \dashrightarrow \mathbb{A}^4.$$

2.3 Digression on vector bundles and sections

Let X be a variety (connected separated scheme) over a field k and $\pi_V : V \to X$ an algebraic vector bundle over X. Recall this means there exists an affine covering $\{U_i\}$ of X and *local trivializations* of V over each U_i

$$\phi_i : V|U_i \xrightarrow{\sim} k^r \times U_i$$

such that the induced transition maps

$$\tau_{ij} := \phi_i \circ \phi_j^{-1} : k^r \times (U_i \cap U_j) \xrightarrow{\sim} k^r \times (U_i \cap U_j)$$

are linear in each fiber. In particular, we get morphisms

$$\tau_{ij} : U_i \cap U_j \to \mathrm{GL}_r$$

satisfying the cocycle condition

$$\tau_{ij} \circ \tau_{jl} = \tau_{il}$$

over $U_i \cap U_j \cap U_l$.

Let $\Gamma(X, V)$ denote *sections* of V over X, i.e., morphisms

$$s : X \to V$$

with $\pi_V \circ s = \mathrm{Id}_X$. These can be realized locally as morphisms $s_i : U_i \to k^r$ satisfying the compatibility condition

$$s_i = \tau_{ij} s_j$$

over $U_i \cap U_j$.

Exercise 5 Verify that $\Gamma(X, V)$ is a k-vector space.

Exercise 6 Compute $\Gamma(\mathbb{P}^2, T\mathbb{P}^2)$ using the definition. *Hint:* $T\mathbb{P}^2$ can be trivialized over the $\{U_i = \{x_i \neq 0\}\}$.

2.4 Tautological bundles

We have the trivial bundle on the Grassmannian

$$\begin{array}{cl} E & = \mathbb{C}^{n+1} \times \mathrm{Gr}(m+1, n+1) \\ \downarrow \pi_E & \\ \mathrm{Gr}(m+1, n+1) & \end{array}$$

where π_E is projection onto the second factor.

We have the *universal subbundle*

$$\begin{array}{ccc} S & \subset & E \\ \downarrow \pi_S & \swarrow & \\ \mathrm{Gr}(m+1, n+1) & & \end{array}$$

where

$$S_\Lambda = \pi_S^{-1}([\Lambda]) = \Lambda.$$

Thus we have

$$\begin{array}{ccc} S_\Lambda & \subset & E_\Lambda \\ || & & || \\ \Lambda & \subset & \mathbb{C}^{n+1} \end{array}$$

via the tautological inclusion.

We also have the *universal quotient bundle*

$$\begin{array}{ccc} E & \twoheadrightarrow & Q \\ & \searrow & \downarrow \pi_Q \\ & & \mathrm{Gr}(m+1,n+1) \end{array}$$

where

$$Q_\Lambda = \pi_Q^{-1}([\Lambda]) = \mathbb{C}^{n+1}/\Lambda.$$

We thus have a natural exact sequence of vector bundles over the Grassmannian

$$0 \to S \to E \to Q \to 0.$$

2.5 Tangent bundle

Theorem 7 *The tangent bundle to the Grassmannian is*

$$T\mathrm{Gr}(m+1,n+1) = \mathrm{Hom}(S,Q),$$

i.e., for each $[\Lambda] \in \mathrm{Gr}(m+1,n+1)$ *we have a natural identification*

$$T_{[\Lambda]}\mathrm{Gr}(m+1,n+1) = \mathrm{Hom}(\Lambda, \mathbb{C}^{n+1}/\Lambda).$$

Here natural has a technical definition: It does not depend on the choice of coordinates on $\mathbb{C}^{n+1}$ or generators for Λ.
proof: First, recall that elements of $T_{[\Lambda]}\mathrm{Gr}(m+1,n+1)$ correspond to diagrams:

$$\begin{array}{ccc} 0 & \in & \mathrm{Spec}\mathbb{C}[\epsilon]/\langle \epsilon^2 \rangle \\ f\downarrow & & \downarrow f \\ [\Lambda] & \in & \mathrm{Gr}(m+1,n+1) \end{array}$$

We construct a linear transformation

$$\begin{array}{rcl} T : \mathrm{Hom}(\Lambda, \mathbb{C}^{n+1}) & \to & T_{[\Lambda]}\mathrm{Gr}(m+1,n+1) \\ \phi & \mapsto & \Lambda_{\phi,\epsilon} \end{array}$$

as follows: If $v_0, \ldots, v_m \in \Lambda \subset \mathbb{C}^{n+1}$ are a basis for Λ then we set

$$\Lambda_{\phi,\epsilon} = \mathrm{span}(v_0 + \epsilon\phi(v_0), \ldots, v_m + \epsilon\phi(v_m)).$$

Note that the vectors $v_i + \epsilon\phi(v_i), i = 0, \ldots, m+1$ are still linearly independent: Small perturbations of a linearly independent set remain independent.

Exercise 8 Verify that the subspace $\Lambda_{\phi,\epsilon}$ is independent of the choice of basis for Λ.

We assert that T is surjective: Suppose for notational simplicity that $v_0, \ldots, v_m, e_{m+1}, \ldots, e_n$ form a basis for $\mathbb{C}^{n+1}$; equivalently, Λ is in the distinguished affine $U_{0\ldots m}$. Then each element of $\mathrm{Gr}(m+1, n+1)$ near Λ can be expressed in the form

$$\mathrm{span}(v_i + b_{im+1}e_{m+1} + \ldots + b_{in}e_n, i = 0, \ldots, m).$$

However, we can choose ϕ such that

$$\phi(v_i) = b_{im+1}e_{m+1} + \ldots + b_{in}e_n.$$

Next, we claim that

$$\mathrm{Hom}(\Lambda, \Lambda) \subset \mathrm{Hom}(\Lambda, \mathbb{C}^{n+1}/\Lambda)$$

is contained in $\ker(T)$. Since T is surjective, the rank-nullity theorem gives

$$\begin{aligned} \dim\ker(T) &= \dim \mathrm{Hom}(\Lambda, \mathbb{C}^{n+1}/\Lambda) - \dim \mathrm{Gr}(m+1, n+1) \\ &= (m+1)(n+1) - (m+1)(n-m) = (m+1)^2. \end{aligned}$$

It follows that $\mathrm{Hom}(\Lambda, \Lambda) = \ker(T)$ and

$$\bar{T} : \mathrm{Hom}(\Lambda, \mathbb{C}^{n+1}/\Lambda)/\mathrm{Hom}(\Lambda, \Lambda) \xrightarrow{\sim} T_\Lambda \mathrm{Gr}(m+1, n+1)$$

is an isomorphism. Since we generally have

$$\mathrm{Hom}(\Lambda, \mathbb{C}^{n+1}/\Lambda)/\mathrm{Hom}(\Lambda, \Lambda) = \mathrm{Hom}(\Lambda, \mathbb{C}^{n+1}/\Lambda)$$

our theorem follows.

To prove the claim, suppose that $\phi(\Lambda) \subset \Lambda$ so that

$$v + \epsilon\phi(v) \in \Lambda$$

for each $v \in \Lambda$. Then the subspace

$$\Lambda_{\phi,\epsilon} = \mathrm{span}(v_0 + \epsilon\phi(v_0), \ldots, v_m + \epsilon\phi(v_m)) = \Lambda,$$

i.e., the corresponding tangent vector is 0. □

2.6 Digression on tensor operations

Let A be a commutative ring and V an A-module. For each $d \geq 0$, the *dth tensor power* is defined

$$V^{\otimes d} = \underbrace{V \otimes \ldots \otimes V}_{d \text{ times}}$$

where the tensor product is taken over A. By convention, $V^{\otimes 0} = A$. If V is a free A-module with generators $e_0, \ldots, e_n$ then

$$e_{i_1} \otimes \ldots \otimes e_{i_d}, \quad 0 \leq i_1, \ldots, i_d \leq n$$

freely generate $V^{\otimes d}$. The *tensor power algebra* is defined

$$T(V) = \oplus_{d \geq 0} V^{\otimes d}$$

with the associative product $\otimes$.

The tensor product algebra is not commutative, but we can impose commutativity by fiat by imposing relations

$$v \otimes w \equiv w \otimes v.$$

These generate a two-sided ideal $I \subset T(V)$. When V is freely generated by $e_0, \ldots, e_n$ then $I \cap V^{\otimes d}$ is generated by relations

$$e_{i_1} \otimes \ldots \otimes e_{i_d} - e_{i_{\sigma(1)}} \otimes \ldots \ \otimes e_{i_{\sigma(d)}}$$

where $\sigma \in \mathfrak{S}_d$, the symmetric group. The *dth symmetric power* is defined

$$\mathrm{Sym}^d V = V^{\otimes d} / I \cap V^{\otimes d}$$

and the *symmetric power algebra* is defined

$$\mathrm{Sym} V = \oplus_{d \geq 0} \mathrm{Sym}^d V.$$

Exercise 9 Consider $\mathbb{C}[x_0, \ldots, x_n]_1$ is a $\mathbb{C}$-vector space. Show that $\mathrm{Sym}^d \mathbb{C}[x_0, \ldots, x_n]_1 = \mathbb{C}[x_0, \ldots, x_n]_d$.

Tensor operations also apply to vector bundles: If $\pi_V : V \to X$ is a vector bundle then $V^{\otimes d} \to X$ and $\mathrm{Sym}^d V \to X$ are also well-defined vector bundles.

2.7 Incidence geometry

Consider the incidence correspondence

$$\mathcal{Z} \subset \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d) \times \mathbb{G}(m, n)$$

defined by

$$\{([F], \Lambda) : F|\Lambda = 0\}.$$

The projection

$$\pi_2 : \mathcal{Z} \to \mathbb{G}(m, n)$$

has fibers which are projective spaces of codimension

$$\binom{m+d}{d}.$$

Indeed, the quotient maps

$$q : \mathbb{C}[x_0, \ldots, x_n]_1 = (\mathbb{C}^{n+1})^* \twoheadrightarrow \Lambda^*$$

induce surjections

$$\mathrm{Sym}^d q : \mathbb{C}[x_0, \ldots, x_n]_d = \mathrm{Sym}^d (\mathbb{C}^{n+1})^* \twoheadrightarrow \mathrm{Sym}^d \Lambda^*.$$

The kernel consists of the degree-d forms vanishing on Λ.

Exercise 10 Let $X \subset \mathbb{P}^n$ be a generic hypersurface of degree $d \geq 2n - 2$. Show that X contains no lines.

Sketch: It suffices to show that

$$\dim \mathcal{Z} < \dim \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d) = \binom{n+d}{d} - 1$$

so π_1 cannot be dominant. The analysis of the incidence correspondence above shows that for each $\ell \in \mathbb{G}(1, n)$

$$\dim \pi_2^{-1}(\ell) = \binom{n+d}{d} - 1 - (d+1).$$

However,

$$\dim \mathcal{Z} = \dim \text{ fibers of } \pi_2 + \dim \mathbb{G}(1, n)$$

which is equal to

$$\binom{n+d}{d} - 1 - (d+1) + 2(n-1) = \binom{n+d}{d} - 1 + (2n - d - 3).$$

References

[1] Brendan Hassett. *Introduction to algebraic geometry*. Cambridge University Press, Cambridge, 2007.