Computer-assisted proofs in PDEs: the dispersive case

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Takeaway message

- New strategy to construct singular solutions to PDE at endpoints of bifurcation branches, and to develop uniqueness even without maximum principles.
- Simple ideas, qualitative information of the solution.
- Computer-assisted proof.
- Applicable to other "bad" situations: low regularity problems, even in unstable/ill-posed regimes.
- Special functions are your friend, not the enemy.
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Consider the KdV equation:

\[ v_t - 6vv_x + v_{xxx} = 0 \]

“Issues”:

- Local.
- Does not capture many phenomena: wave breaking, sharp crests, non-smooth solutions, etc.
Interested in (singular) solutions of greatest height:

- **Corners:**

- **Cusps:**
KdV features a 2nd order approximation of the full dispersion relation of gravity water waves on finite depth:

\[
\left( \frac{\tanh(\xi)}{\xi} \right)^{\frac{1}{2}} \sim 1 - \frac{1}{6} \xi^2
\]

“Better approximation”: change the linear part in KdV using the full dispersion relation.
The Whitham equation

Whitham proposed

\[ \partial_t \nu + 2\nu \nu_x + L \nu_x = 0, \]

\[ \tilde{L} \nu(\xi) = \left( \frac{\tanh(\xi)}{\xi} \right)^{\frac{1}{2}} \hat{\nu}(\xi) \]

as a shallow water approximation.
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as a shallow water approximation.

For small \( \xi \):

\[ \left( \frac{\tanh(\xi)}{\xi} \right)^{\frac{1}{2}} \sim 1 - \frac{1}{6} \xi^2 \]

\[ \Rightarrow \text{Whitham} \sim \text{KdV for small frequencies and small times, different for large times.} \]
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For small $\xi$:

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$\Rightarrow$ Whitham $\sim$ KdV for small frequencies and small times, different for large times.

For large $\xi$:

$$\left(\frac{\tanh(\xi)}{\xi}\right)^{\frac{1}{2}} \sim \frac{1}{\xi^{\frac{1}{2}}}$$

$\Rightarrow$ Whitham is a very weakly dispersive perturbation of Burgers.
Features of the Whitham equation

- Nonlocal, fractional, inhomogeneous.
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- Solitary waves (Ehrnström-Groves-Wahlén, 2013).
- Numerics, asymptotics (Klein-Saut, 2013).
- Wave breaking (Hur, 2015).
Conjecture (Whitham, 1967)

*There exists a limiting traveling wave of $C^{\frac{1}{2}}$ regularity.*
Whitham’s conjecture

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Theorem (Ehrnström-Wahlén, 2016)
The conjecture is true. The proof uses careful global bifurcation arguments. Not constructive.
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Conjecture (Ehrnström-Wahlén, 2016)

Whitham’s highest wave is everywhere convex and its asymptotic behavior at 0 is

$$v(x, t) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}} |x - \mu t|^{1/2} + o(|x - \mu t|).$$

Here, $\mu$ is part of the problem and needs to be found.
Water waves & Whitham

Water waves

Existence
Amick-Fraenkel-Toland, Plotnikov-Toland, 80’s.

Convexity
Plotnikov-Toland, 2004

Local uniqueness
Fraenkel, 2007

Uniqueness
Kobayashi, 2010

Whitham

Existence
Ehrnström-Wahlén, 2016

Convexity
Enciso-JGS-Vergara, 2018

???

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## Water waves & Whitham

<table>
<thead>
<tr>
<th>Water waves</th>
<th>Whitham</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Existence</strong></td>
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</tr>
<tr>
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<td>Ehrnström-Wahlén, 2016</td>
</tr>
<tr>
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</tr>
<tr>
<td>Plotnikov-Toland, 2004</td>
<td>Enciso-JGS-Vergara, 2018</td>
</tr>
<tr>
<td><strong>Local uniqueness</strong></td>
<td></td>
</tr>
<tr>
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<td></td>
</tr>
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<td></td>
<td>Enciso-JGS-Vergara, 2021</td>
</tr>
</tbody>
</table>
Theorem (Enciso–JGS–Vergara, 2018)

There exists a $2\pi$-periodic highest cusped traveling wave of the Whitham equation which is a convex, $C^{1/2}$ function and behaves asymptotically as

$$v(x, t) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}}|x - \mu t|^{1/2} + O(|x - \mu t|^{1+\eta})$$

for some $\eta > 0$.

The limiting wave is at the end of the branch.
Proof

- Travelling wave ansatz: \( v(x, t) = \varphi(x - \mu t) \), where the positive constant \( \mu \) represents the wave speed.
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- The Whitham equation becomes

\[
L\varphi - \mu \varphi + \varphi^2 = 0, \quad \widehat{L\varphi} = \left( \frac{\tanh(\xi)}{\xi} \right)^{1/2} \hat{\varphi}(\xi).
\]

- Whitham's heuristic argument: crest cusped with \( \varphi(x) \sim \frac{\mu}{2} - c|x|^{1/2} \).
Proof

Imposing $u(x) = \frac{\mu}{2} - \varphi(x - \mu t)$ and through the symmetries of the equation we can get rid of $\mu$. In particular, $u(x)$ satisfies the reduced equation:

$$u^2 = \mathcal{L}u,$$

Step 0: We reduced the problem to only find $u$. 
Proof

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$$u^2 = \mathcal{L}u,$$

with

$$\mathcal{L}u = \int_{-\pi}^{\pi} \left( K(x - y) + K(x + y) - 2K(y) \right) u(y) dy$$

and $K$ is related to the dispersive multiplier.
Proof

Imposing \( u(x) = \frac{\mu}{2} - \varphi(x - \mu t) \) and through the symmetries of the equation we can get rid of \( \mu \). In particular, \( u(x) \) satisfies the reduced equation:

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with

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  \mathcal{L}u = \int_{-\pi}^{\pi} \left( K(x - y) + K(x + y) - 2K(y) \right) u(y) dy
\]

and \( K \) is related to the dispersive multiplier. Once \( u \) is known we can recover \( \mu \) via

\[
  \mu \left( 1 - \frac{\mu}{2} \right) = 4 \int_{0}^{\pi} K(y) u(y).
\]

Step 0: We reduced the problem to only find \( u \).
Main Idea

- Construct $u_0$ ("sufficiently good" approximation) by hand.
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- Write $u = u_0 + \tilde{u}$, where $\tilde{u}$ is expected to be very small: $O(\varepsilon)$. Then:

\[
2u_0\tilde{u} - \mathcal{L}\tilde{u} = -\tilde{u}^2 - (u_0^2 - \mathcal{L}u_0)
\]

\[
(l - \frac{1}{2u_0}\mathcal{L})\tilde{u} = \frac{1}{2u_0}(-\tilde{u}^2 - (u_0^2 - \mathcal{L}u_0))
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If we can invert $(l - \frac{1}{2u_0}\mathcal{L})$:

- First term of RHS: $O(\varepsilon^2)$
- Second term of RHS: "sufficiently small"

Close using a fixed point argument $\Rightarrow$ Explicit estimates of $\|\bar{u}\|$ (small).
Main Idea

- Construct $u_0$ ("sufficiently good" approximation) by hand.
- Write $u = u_0 + \bar{u}$, where $\bar{u}$ is expected to be very small: $O(\varepsilon)$. Then:

\[
2u_0\bar{u} - L\bar{u} = -\bar{u}^2 - (u_0^2 - Lu_0)
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(I - \frac{1}{2u_0}L)\bar{u} = \frac{1}{2u_0}(-\bar{u}^2 - (u_0^2 - Lu_0))
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Expected: $u_0$ strictly convex $\Rightarrow u_0 + \bar{u}$ strictly convex
1. Construct a good $u_0$.

2. Prove that $(I - \frac{1}{2u_0} \mathcal{L})$ is invertible.

3. Check that the involved (explicit) constants are “small enough”.
Step 1: Construction of a good approximation

- Formal asymptotics: very good at $x \ll 1$, terrible at $x \gg 1$.
  Nontrivial exponents: $u_0 \sim c_1 \sqrt{x} + c_2 x^{1.11120\ldots} + \ldots$
Step 1: Construction of a good approximation

- Formal asymptotics: very good at $x \ll 1$, terrible at $x \gg 1$.
- Nontrivial exponents: $u_0 \sim c_1 \sqrt{x} + c_2 x^{1.1120} + \ldots + \ldots$

- Formal asymptotics + correction: we add $\sum_{n=1}^{N} b_n (\cos(nx) - 1)$ for some $N, b_n$. Better global control. In our case $N = 11$. 
Special functions save the day: (Clausen functions)

\[
C_z(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^z}, \quad S_z(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^z}.
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Approximate solution \( u_0 = \) combination of Clausen functions + trigonometric polynomials:

\[ u_0(x) = \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk} \left( \zeta \left( \frac{3}{2} + kp_0 + jp_1 \right) - C_{\frac{3}{2} + kp_0 + jp_1}(x) \right) \]
\[ + \sum_{n=1}^{N_2} b_n \left( \cos(nx) - 1 \right) , \]

where \( a_{jk} \) \( b_k \) are real, \( p_j \) solve the equation

\[ \frac{\Gamma(-1/2 - p_j)}{\Gamma(-1 - p_j)} \left( 1 - \cot \left( \frac{\pi}{2} p_j \right) \right) = \frac{2}{\sqrt{\pi}} , \]

for \( j = 0, 1 \) and \( N_j \) are fixed positive integers.
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for \( j = 0, 1 \) and \( N_j \) are fixed positive integers.
We choose the above coefficients so that the defect is small when measured in $L^\infty$:

$$u_0(x) = \sum_{k=0}^{N_j} \sum_{j=0}^{1} A_{jk} |x|^{\frac{1}{2}+kp_0+jp_1} + O(|x|^2)$$

$$\mathcal{L} u_0(x) = \sum_{k=0}^{N_j} \sum_{j=0}^{1} \tilde{A}_{jk} |x|^{1+kp_0+jp_1} + O(|x|^2),$$

with $A_{jk}$ and $\tilde{A}_{jk}$ real (combinations of the previous $a_{jk}$).

Nonlinear system of equations for the coefficients $A_{jk}$, $\tilde{A}_{jk}$:

$$u_0^2(x) - \mathcal{L} u_0(x) = O(|x|^p),$$

for a sufficiently large power $p$. 
Error measured in $L^\infty((0, \pi])$: 

![Graph showing error measurements]
A few words on the implementation

- No implementation of $C_z$ and $S_z$ available.

- One possibility is through the polylogarithm function (Arb):
  
  $$C_z(x) = \frac{1}{2} \left( \text{Li}_z(e^{ix}) + \text{Li}_z(e^{-ix}) \right)$$

  Not good enough, especially for large (interval) arguments since the polylog is not optimized for complex numbers on the unit circle. Good enough for singletons.

- Instead, use monotonicity (in $x$) of $C_z(x)$ and $C'_z(x) = -S_z - \frac{1}{x}$, $S'_z(x) = C_z - \frac{1}{x}$ together with a Taylor expansion.

- Multiprecision (∼100 bits) needed.
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Step 2: The linear part is invertible

Obs: \( \frac{1}{2u_0(x)} \mathcal{L} \) is compact, but it doesn't help to bootstrap.
If \( f(x) \sim x^n \), \( \frac{1}{2u_0(x)} \mathcal{L} f(x) \sim x^n \).
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$$\frac{1}{2u_0(x)}\mathcal{L}f(x) = \frac{1}{2u_0(x)} \int (K(x+y) + K(x-y) - 2K(y))f(y)dy$$
$$= \int K_0(x, y)f(y)dy.$$  

We work in (weighted) $L^\infty$-based spaces.
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$$\left\| \frac{1}{2u_0(x)} \mathcal{L} \right\|_\infty = \sup_x \int |K_0(x, y)|dy .$$

$$= 0.99736 \ldots$$

$\Rightarrow (I - \frac{1}{2u_0} \mathcal{L})$ is invertible and $\left\| (I - \frac{1}{2u_0} \mathcal{L})^{-1} \right\|_\infty \leq \frac{1}{1 - 0.99736 \ldots} \sim 380.$
To prove the red estimate:

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- Computer-assisted calculation of $\int |K_0(x, y)| dy$ for $\varepsilon \leq x \leq \pi$. 

$K_0(x, y) = \sqrt{\frac{2}{\pi x}}$
To prove the red estimate:

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- Computer-assisted calculation of \( \int |K_0(x, y)|dy \) for \( \varepsilon \leq x \leq \pi \).

- To compute \( \int |K_0(x, y)|dy \) for small \( x \) we exploit the asymptotics

\[
K(x) = \frac{1}{\sqrt{2\pi|x|}} + K_{\text{reg}}(x),
\]

with \( K_{\text{reg}} \) real analytic.
Rigorous Singular integrals

\[ Hf(0) = -\frac{PV}{\pi} \int \frac{f(y)}{y} \, dy \]

\[ = \frac{PV}{\pi} \int_{|y|<\epsilon} \frac{f(0) - f(y)}{y} \, dy - \frac{PV}{\pi} \int_{|y|>\epsilon} \frac{f(y)}{y} \, dy \]

1. The second integral is not singular, we integrate as usual.
2. For the first integral, we expand around zero. For example, up to order 1:

\[ f(0) - f(y) \in -y f'([-\epsilon, \epsilon]) \]

Then we cancel factors of \( y \) in the integrand.

\[ PV \pi \int_{|y|<\epsilon} \frac{f(0) - f(y)}{y} \, dy \in PV \pi \int_{|y|<\epsilon} -f'([-\epsilon, \epsilon]) \, dy \in -2\epsilon f'([-\epsilon, \epsilon]) \pi \]

Obs: The boundary \( |y|<\epsilon \) can be optimized.
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\]
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\[ = \frac{PV}{\pi} \int_{|y|<\varepsilon} \frac{f(0) - f(y)}{y} dy - \frac{PV}{\pi} \int_{|y|>\varepsilon} \frac{f(y)}{y} dy \]

1. The second integral is not singular, we integrate as usual.
2. For the first integral, we expand around zero. For example, up to order 1:

\[ f(0) - f(y) \in -y f'(\pm \varepsilon) \]

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\[ \frac{PV}{\pi} \int_{|y|<\varepsilon} \frac{f(0) - f(y)}{y} dy \in \frac{PV}{\pi} \int_{|y|<\varepsilon} -f'(\pm \varepsilon) dy \in -\frac{2\varepsilon f'(\pm \varepsilon)}{\pi} \]

Obs: The boundary \( |y| < \varepsilon \) can be optimized.
For convenience, we write: $u = u_0 + |x|v_0$, where $v_0$ satisfies

$$(I - T_0)v_0 = \frac{1}{2|x|u_0} \left( (\mathcal{L}u_0 - u_0^2) - |x|^2 v_0^2 \right).$$
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v_0 \mapsto (1 - T_0)^{-1} \left( \frac{1}{2|x|u_0} \left( (\mathcal{L}u_0 - u_0^2) - |x|^2 v_0^2 \right) \right)
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- This only proves the existence of a solution with almost the conjectured asymptotic behavior. \( \Rightarrow \) Perturbation of the weight.
Completion of the proof

- $C^2$ estimates follow similar ideas but more calculations
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Completion of the proof

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- Very sensitive numbers (too delicate estimates / too small numbers to be done by hand)

- We are not using too much special structure of the equation.
New results

**Theorem (Dahne–JGS, forthcoming)**

There exists a $2\pi$-periodic highest cusped traveling wave of the Burgers-Hilbert equation

$$v_t + vv_x + H v = 0$$

which behaves asymptotically as

$$v(x, t) = \frac{\mu}{2} + C|x - \mu t| \log(|x - \mu t|) + O(|x - \mu t| \log(|x - \mu t|)^{\frac{1}{2}})$$

for some explicit $C$. 

*Main difficulties:*

▶ Much more careful bounds needed: we need to work with $x \sim 10^{-10^6}$.

▶ Unclear what is the next term in the asymptotic expansion (even formally).
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Back to Whitham: Uniqueness

*Is the convex travelling wave that we found before the only solution?*

- As in the case of Stokes wave, existence in the class of convex solutions does not imply uniqueness.
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▶ Thus, we will consider the problem in the class of even, monotone solutions $u$ which are increasing in $[0, \pi]$. 
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- Thus, we will consider the problem in the class of even, monotone solutions $u$ which are increasing in $[0, \pi]$.

- *Monotonicity* $+$ greatest height imply that solutions $u$ are positive.
Uniqueness

Theorem (Enciso–JGS–Vergara, 2021)

The Whitham equation admits a unique, even, $2\pi$-periodic traveling wave solution of greatest height between crest and trough that is non-increasing on $[0, \pi]$. 
Uniqueness

Main idea: obtain non-trivial upper and lower bounds $u_0^+, u_0^-$ which are iteratively refined until they converge to the unique solution $u$ of the equation:

$$u_0^- \leq u_1^- \leq \cdots \leq u_N^- \leq u \leq u_N^+ \leq \cdots \leq u_1^+ \leq u_0^+,$$

$$\|u_N^+ - u_N^-\| \to 0$$

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The proof relies on very fine bounds for the dispersive multiplier as well as computer assisted estimates.
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- Main idea: obtain non-trivial upper and lower bounds $u^+_0, u^-_0$ which are iteratively refined until they converge to the unique solution $u$ of the equation:

$$u^-_0 \leq u^-_1 \leq \cdots \leq u^-_N \leq u \leq u^+_N \leq \cdots \leq u^+_1 \leq u^+_0,$$

$$\|u^+_N - u^-_N\| \to 0$$

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- The proof relies on very fine bounds for the dispersive multiplier as well as computer assisted estimates.

- The operator is not monotone and does not satisfy a maximum principle. We get contractivity only when we are sufficiently close to the unique solution $u$. 
Uniqueness: setup

Derive estimates in $L^\infty(\mathbb{T})$ for the function $w(x) := |x|^{-1/2} u(x)$,

$$w^2 = \mathcal{F}w - \mathcal{G}w,$$

for positive (rather involved) linear operators $\mathcal{F}$ and $\mathcal{G}$:

$$\mathcal{F}(w)(x) : = \frac{1}{|x|} \int_{y^*(x)}^{\pi} \mathcal{K}_x(y) \sqrt{|y|} w(y) \, dy,$$

$$\mathcal{G}(w)(x) : = \frac{1}{|x|} \int_{0}^{y^*(x)} |\mathcal{K}_x(y)| \sqrt{|y|} w(y) \, dy,$$

where

$$\mathcal{K}_x(y) = K(x - y) + K(x + y) - 2K(y).$$
Uniqueness: setup

The kernel $K_x(y)$ is positive when $y^*(x) < y < \pi$ and negative for $0 < y < y^*(x)$, with $y^*(x)$ a curve on $[0, \pi]$:
The proof follows the next scheme:

1. First we prove rough initial bounds using fine estimates on the Whitham kernel.
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1. First we prove rough initial bounds using fine estimates on the Whitham kernel.

2. Then we iterate those bounds using the monotonicity assumption until we can truly exploit the structure of the equation.

3. Finally we reach the regime in which we can make automatic iterations for a discrete (but large) approximation of our nonlinear system, plus small errors.
Uniqueness: initial bounds

- We exploit this behavior to obtain initial estimates in $L^\infty$. For instance:

$$\|w\|_{L^\infty} \leq \|F(1)\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \int_0^{1/r} \left( \frac{1}{\sqrt{|1-t|}} + \frac{1}{\sqrt{1+t}} - 2 \right) \cdot \frac{1}{t^2} \, dt + \text{error},$$

where $r$ is the slope of the line tangent to the curve $y^*(x)$ at $x = 0$. 

- The curve $y^*(x)$ is enclosed through asymptotic analysis and computer assisted estimates.

- Asymptotic estimates yield $r = 0$, which in the end give us $w(x) \leq w^0(x) := 0$. For some explicit $C > 0$.

- Analogously, we have the lower bound $w(x) \geq \frac{1}{\sqrt{\pi}} \left( 2\sqrt{\delta} + \sqrt{2(1-\delta)} - \sqrt{2(1+\delta)} \right) \cdot \sqrt{|x| - c|x|},$ Optimizing in $\delta$: $w(x) \geq w^0(x) := 0.1940 - c|x|, c > 0$. 

**Uniqueness: initial bounds**

- We exploit this behavior to obtain initial estimates in $L^\infty$. For instance:

  $$\|w\|_{L^\infty} \leq \|F(1)\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \int_0^{1/r} \left( \frac{1}{\sqrt{|1-t|}} + \frac{1}{\sqrt{1+t}} - 2 \right) \cdot \frac{1}{t^2} \, dt + \text{error},$$

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where $r$ is the slope of the line tangent to the curve $y^*(x)$ at $x = 0$.

- The curve $y^*(x)$ is enclosed through asymptotic analysis and computer assisted estimates.

- Asymptotic estimates yield $r = 0.652 \ldots$ which in the end give us

$$w(x) \leq w^+_0(x) := 0.8425 + C|x|,$$

for some explicit $C > 0$. 
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 w(x) \geq \frac{1}{\sqrt{\pi}} \left( 2\sqrt{\delta} + \sqrt{2(1-\delta)} - \sqrt{2(1+\delta)} \right) \sqrt{|x|} - c|x|,
$$

Optimizing in $\delta$:

$$
 w(x) \geq w_0^-(x) := 0.1940 - c|x|, \quad c > 0.
$$
Uniqueness: self-improving bounds

- We have found then rough (but non-trivial!) bounds

\[ w_0^-(x) \leq w(x) \leq w_0^+(x). \]

- Define the operator \( \mathcal{J} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T}) \)

\[ \mathcal{J}(w^-, w^+)(x) := \left[ Fw^-(x) - Gw^+(x) \right]^{1/2}. \]

We would like to set up an iteration scheme that yields improved bounds

\[ w_{n+1}^-(x) := \max\{w_n^-(x), \mathcal{J}(w^-_n, w^+_n)(x)\}, \]
\[ w_{n+1}^+(x) := \min\{w_n^+(x), \mathcal{J}(w^+_n, w^-_n)(x)\}. \]

- However, \( \mathcal{J}(w_0^-, w_0^+)(x) \) is not well defined for our initial bound. We need to work more!
Uniqueness: self-improving bounds

- There are threshold bounds $w_{n_0}^-(x), w_{n_0}^+(x)$ for which the previous iteration scheme is well defined for all $n > n_0$.

- We introduce then a new operator $\tilde{J} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ that helps us to iterate lower bounds,

  $$w_{n+1}^-(x) \geq \tilde{J}(w^-_n, w^+_n)(x), \quad 0 \leq n \leq n_0.$$

- This operator is crafted so that one can exploit the monotonicity in a clever way. In particular, it is build upon integral estimates for

  $$K(\delta x - y) + K(\delta x + y) - K(x - y) - K(x + y), \quad 0 < \delta < 1.$$

- This procedure yields sharper bounds

  $$w_{n_0+1}^-(x) = 0.3373 - 0.1172 \sqrt{|x|} + 0.0023 |x|,$$

  $$w_{n_0+1}^+(x) = 0.7356 - 0.0194 \sqrt{|x|} - 0.0824 |x|.$$
Uniqueness: self-improving (automatized) bounds

▶ We are now in position to use $\mathcal{J}$ to improve our bounds.
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- This step only is similar to the iteration by Kobayashi for Euler (though with a much harder kernel)

- Spatial discretization to approximate the operators $\mathcal{F}$ and $\mathcal{G}$ by $N \times N$ (interval) matrices

\[ \mathcal{F}_{ij} := \mathcal{F}1_{(x_{j-1}, x_j)}(x_i), \quad \mathcal{G}_{ij} := \mathcal{G}1_{(x_{j-1}, x_j)}(x_i), \]

and piecewise constant functions

\[ w_n^-(x) := \sum_{j=1}^{N} w_{n;j}^- 1_{(x_{j-1}, x_j)}(x), \quad w_n^+(x) = \sum_{j=1}^{N} w_{n;j-1}^+ 1_{(x_{j-1}, x_j)}(x), \]

where $n_0 \leq n < N_0$ and

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$$
Uniqueness: how to deal with so many integrals of a complicated kernel
  ▶ Special functions to the rescue (again):

\[
F_S(z) := \int_0^z \sin(\frac{\pi}{2}t^2) \, dt
\]

Then:

\[
\int_0^{x^2} K_x(y) \sqrt{y} dy = \frac{1}{\pi} (f_{x^2, x}^2 - f_{x^2, 0}^2) + \text{small error}
\]

where \( f_{x^2, x}^2 := \text{MF} \sum_{n=1}^\infty \cos(nx^3) n^2 \left[ \frac{\sqrt{2}}{\pi} (F_S(0) - F_S(\sqrt{2} \pi nx^2)) \right] + \sqrt{x^2} \left( S_{3/2} \frac{1}{x^2 + x^3} + S_{3/2} \frac{1}{x^2 - x^3} \right)
\]

\( \text{MF} \) is “medium-sized” (around 300) and \( S \) is a Clausen function.

▶ The Fresnel integral is implemented in Arb.

▶ Bonus: Other necessary integrals involving \( K_x \) use the \( \text{2F}_1 \) hypergeometric for a faster calculation.
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\[ f_{x_2,x_3} := \sum_{n=1}^{M_F} \frac{\cos(nx_3)}{n^2} \left[ \sqrt{2\pi} \left( F_S(0) - F_S\left( \sqrt{\frac{2}{\pi} nx_2} \right) \right) \right] \]

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Uniqueness

We need one last pass to go from red to pink, changing our operators again and using the monotonicity.
Uniqueness: fixed point argument

After one last refinement (that only uses the monotonicity), we get sharp bounds $w_{N_0}^-(x), w_{N_0}^+(x)$.
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Key: the nonlinear map $u \mapsto \sqrt{L}u$ is parity and monotonicity preserving in the space

$$X := \{ u \text{ even and monotone: } \sqrt{|x|} w_{N_0}^- (x) \leq u(x) \leq \sqrt{|x|} w_{N_0}^+(x) \}.$$
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- Our fine bounds allow us to prove the uniqueness of Whitham’s highest wave by using a fixed point argument in $(X, L^\infty(\mathbb{T}, |x|^{-1/2} \, dx))$: 

$$\| \sqrt{L}u - \sqrt{L}v \|_X \leq C \| u - v \|_X,$$

so that $\sqrt{L}$ is contractive provided that $C > 0$ is a constant less than 1.
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so that $\sqrt{L}$ is contractive provided that $C > 0$ is a constant less than 1.

- Life is hard: $C > 1$ if we work directly in $X$. 

Fortunately, there is room to circumvent this problem: $\sqrt{\mathcal{L}}$ becomes contractive in $X$ endowed with the norm

$$
\|u\|_X := \sup_{0<x<\pi} |x|^{-1/2} a^{-1}(x)|u(x)|, \quad a(x) = 1 + 2\sqrt{|x|}.
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The conclusion follows then from Banach's fixed point theorem: monotonically increasing solutions verify our bounds and so they belong automatically to $X$. This yields uniqueness in the class of even and monotone functions!
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Final remarks

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- There exists a unique function that is even, monotone in $[0, \pi]$ and convex such that $v(x, t) := \varphi(x - \mu t)$ satisfies

$$\partial_t v + \partial_x (Lv + v^2) = 0, \quad \hat{L}f(\xi) := \sqrt{\frac{\tanh(\xi)}{\xi}} \hat{f}(\xi).$$

This solution is the one we found in the existence part.
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  This solution is the one we found in the existence part.

- Moreover, this solution can be written as

  $$v(x, t) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}} |x - \mu t|^{1/2} + \text{l. o. t.}$$

  where $\mu = 0.768\ldots$
THANK YOU!