(CRASH COURSE ON) PARA-DIFFERENTIAL OPERATORS

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We refer to [1] for an excellent introduction to pseudo-differential operators and to [2] for a complete treatment. We also refer to `https://math.berkeley.edu/~evans/semiclassical.pdf` for comprehensive lecture notes online.

1. Pseudo-differential operators

1.1. **The algebra of differential operators.** Our goal is to understand the Algebra of differential operators

$$\mathcal{A} := \{\text{differential operators}\} = \{\sum_\alpha a_\alpha(x)\partial_x^\alpha, \qquad \alpha \in \mathbb{N}^d, \quad a_\alpha \in C_b^\infty(\mathbb{R}^d)\}.$$

In fact, our primary goal is to find the invertible elements and how to invert them.

We can already isolate 2 remarkable (commutative) subalgebras: the algebra of operators of order 0:

$$\mathcal{A}^0 := \{a(Q); \qquad a \in C_b^\infty(\mathbb{R}^d\}, \quad (a(Q)f)(x) = a(x)f(x)$$

and the algebra of operators with constant coefficients

$$\mathcal{A}_P := \{a(P); a \in \mathbb{C}[X_1, \dots, X_d]\}; \quad \mathcal{F}(a(P)) = a(Q)\mathcal{F}.$$

From this, we can make two remarks:

- In $\mathcal{A}^0$, it is easy to find the invertible elements: they are the functions that never vanish. Their inverse is simple: $a^{-1} = 1/a$.
- it is fairly natural to extend $\mathcal{A}_P$ to $\mathcal{A}_c = \mathcal{F}^{-1}\mathcal{A}^0\mathcal{F}$, the algebra of Fourier multipliers. In particular, this way, we can easily find the inverse of operators in $\mathcal{A}_c$.
- $\mathcal{A}^0$ is invariantly defined, but $\mathcal{A}_c$ is not. Considering Pseudo-differential operators on manifold requires some care.

It is remarkable that $\mathcal{A}^0$ and $\mathcal{A}_c$ are so simple, seem to "generate" $\mathcal{A}$, yet $\mathcal{A}$ is much more subtle. This is mainly due to the fact that $\mathcal{A}$ is *noncommutative*.

1.2. **Vector fields.** There is a simple way to incrementally generate $\mathcal{A}$ from $\mathcal{A}^0$. Indeed, we may consider $\mathcal{A}^j$ the set of operators of order j or less. This is a module over $\mathcal{A}^0$. After operators of degree 0, one can form the $\mathcal{A}^1$ by adjoining the set of vector fields, namely of operators of the form

$$\chi: \ (\chi f)(x) = \sum_{j=1}^d \chi_j(x)\partial_j f(x), \qquad \chi_j \in C_b^\infty(\mathbb{R}^d).$$

Denoting $\mathcal{X}$ the set of vector fields, which is invariantly defined, we have that $\mathcal{A}^1 = \mathcal{A}^0 + \mathcal{X}$ can also be invariantly defined.

One can then build $\mathcal{A}^2 = \mathcal{A}^1 + \mathcal{X}\mathcal{A}^1$, e.g. choosing an orthonormal basis E_i, one has

$$\Delta f = -\sum_{j=1}^d E_j E_j f$$

and one can then construct $\mathcal{A}^3$,...

We note however that this algebra is noncommutative since $\mathcal{X}$ is:

$$[\sum_j a_j\partial_j, \sum_k b_k\partial_k] = \sum_k \left(\sum_{j=1}^d (a_j\partial_j b_k - b_j\partial_j a_k)\right)\partial_k \in \mathcal{X}$$

However, we note a fortunate fact: $\mathcal{A}$ is commutative to main order: the product of two vector fields is a differential operator of order 2, yet their commutator is only of order 1: the term of order exactly 2 does not depend of the order of the product.

1.3. **Quantization rule.** In fact, by considering the different coordinates (or by considering the actions of a vector on the coordinate functions x_1, x_2,..., x_d), one can write any vector field $\chi = (\chi_1, \dots, \chi_d)$ as

{Vector}
$$\chi = \sum_{j=1}^d \chi_j(x)\partial_j. \tag{1.1}$$

We want a way to assign a *symbol* to a vector. By the considerations of the previous subsection, one we specify a rule for vectors, this will automatically give a rule for $\mathcal{A}$.

A natural choice would be to associate to each vector χ as in (1.1) the symbol[1]

$$\chi(x,\xi) = \sum_{j=1}^d \chi_j(x)(i\xi_j)$$

and this is the *Kohn-Nirenberg* quantization. This leads to the quantization for elements in $\mathcal{A}^1$:

{KNQuant}
$$L := a_j(x)\partial_j + b(x) \to \sigma_{KN}(L)(x,\zeta) := ia_j(x)\xi_j + b(x) \tag{1.2}$$

This gives a natural identification of differential operators and symbols:

$$\sum_\alpha a_\alpha(x)\partial^\alpha \to \sum_\alpha a_\alpha(x) i^{|\alpha|}\xi^\alpha,$$

and we could very well work with this rule.

However, from the operator viewpoint, this has a severe limitation: it is very difficult to know from the symbol when an operator is self-adjoint, whereas for $a \in \mathcal{A}^0$, $a(Q)$ is self-adjoint if and only if a is real. Since self-adjoint operators are so important (e.g. for energy-estimates), we will prefer the *Weyl quantization* which retains the property that an operator is self-adjoint if and only if its symbol is real. For differential operators of order 1, it is easy to see what the rule should be:

{weyl}
$$L := a_j(x)\partial_j + b(x) = \frac{1}{2}(a_j\partial_j + \partial_j\cdot a_j) + (b(x) - (\partial_j a_j)) \to \sigma^w(L)(x,\zeta) := ia_j(x)\xi_j + (b(x) - \mathrm{div}(\vec{a})). \tag{1.3}$$

We can then verify that (i) real valued symbols correspond to self-adjoint operators and (ii) the highest order term is the same for both quantizations (but lower order terms differ).

In fact, there is a natural 1-parameter choice of quantizations all simply forced once we choose the correspondance to $x\cdot\xi$:

$$x_i\cdot\xi_j = i(\theta x_i\partial_j + (1-\theta)\partial_j x_i)$$

and we see that the Kohn-Nirenberg quantization corresponds to $\theta = 1$, while the Weyl quantization corresponds to the case $\theta = 1/2$.

We note that there are other natural properties that one might want to preserve. Unfortunately, with neither of the quantization rules defined above can one have that an operator is positive if and only if its symbol is. When such a property is needed, one can consider the Bargmann transform. For other comparison, we refer to `http://www.fuw.edu.pl/~derezins/metz-slides.pdf`

[1]The additional factor of i is suggested by looking at the Fourier transform. However, it clearly does not make a difference.

1.4. **Beyond polynomial symbols.** Considering classical differential operators, we obtain polynomial symbols. If we are to find a framework where we can invert them, we need to consider rational symbols. We also want to be able to project, thus we would like to consider compactly supported symbols. We will consider general "polynomial like" symbols:

Definition 1.1. *A symbol of degree* 0 *is a smooth function* $a(x,\xi)$ *such that for all* $\alpha,\beta\in\mathbb{N}^d$

$$\sup_{(x,\xi)\in\mathbb{R}^{2d}} |\xi|^{|\beta|}|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}.$$

We note this $a\in S^0$*. More generally,* b *is a symbol of degree* p *(noted* $b\in S^p$*) if*

$$\underline{b}\in S^0, \qquad \underline{b}(x,\xi) := (1+|\xi|^2)^{-\frac{p}{2}} b(x,\xi).$$

Note that this does give a generalization of Rational functions and of compactly supported functions. As an example,

$$\sum_{|\alpha|\le p} a_\alpha \xi^\alpha \in S^p, \qquad \text{and} \qquad \varphi(\xi)a(x,\xi)\in S^{-q}$$

whenever $\varphi\in C_c^\infty(\mathbb{R}^d)$ and $a\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, $q\ge 0$.

From now on, whenever we talk about a "symbol", it will be a symbol according to the above definition.

Now that we have extended the notion of symbols, we need to find a way to transform them into operators that would agree with the map (1.3). This is easily done using the Fourier transform: if σ is a symbol, then

$$\{Op^w(\sigma)f\}(x) := \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^{2d}} e^{i\langle x-y,p\rangle}\sigma(\frac{x+y}{2},p)f(y)dydp.$$

A slightly simpler version is

$$\mathcal{F}\{Op^w(\sigma)f\}(\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^2} \widetilde{\sigma}(\xi-\eta,\frac{\xi+\eta}{2})\widehat{f}(\eta)d\eta \tag{1.4}$$ {Pseudo}

where $\widetilde{\sigma}$ denotes the Fourier transform in the first variable only.

Definition 1.2. *A pseudo-differential operator of order* p *is an operator of the form* $Op^w(\sigma)$ *for some* $\sigma\in S^p$*.*

Note that, using formula (1.4), one can also define symbols which are less regular and the corresponding pseudo-differential operators.

This already allows to define parametrix for the inverse of elliptic operators, see Subsection 1.6.

1.5. **Commutation properties.** Let S denote the set of symbols. We now have a correspondance $\mathcal{A}\to S$. This correspondance is obviously linear. But $\mathcal{A}$ is also equipped with a product. We can ask how to generalize this to S. Besides S is also naturally equipped with a notion of a product (the standard pointwise product of functions), but this is *commutative* so it cannot be the exact analog. However, we know that the product in $\mathcal{A}$ is commutative at top order, so we can hope that

(1) If $A\in\mathcal{A}^j$, $B\in\mathcal{A}^k$,

$$\sigma(AB)-\sigma(A)\sigma(B) \quad \text{is of degree j+k-1}$$

(2) If $a\in S^j$, $b\in S^k$ are polynomial, then

$$Op^w(ab)-Op^w(a)Op^w(b)\in\mathcal{A}^{j+k-1}$$

and this can be readily verified.

The exact formula for a product # satisfying

$$Op^w(ab) = Op^w(a)\#Op^w(b)$$

can then be computed for $a, b \in S$. For general pseudo-differential operators, however, this is only true in the sense that the difference is an operator with smooth kernel. We refer to the reference given at the beginning for more on this.

Looking at vector fields, we can conjecture that

$$\sigma([A, B]) = i\{\sigma(A), \sigma(B)\}$$

where we introduce the Poisson bracket:

{PoissonB}
$$\{a, b\} = \nabla_x a \cdot \nabla_\xi b - \nabla_\xi a \cdot \nabla_x b. \tag{1.5}$$

1.6. **Elliptic operators.** Informally, a symbol is elliptic if it does not vanish outside of a neighborhood of 0. This is the natural generalization to the criterion for invertibility[2] on $\mathcal{A}^0$. A symbol σ is called elliptic if there exists C, $\delta > 0$ such that

$$|\sigma(x, \zeta)| \geq C^{-1}|\xi|^\delta, \qquad |\xi| \geq C.$$

The prototypical example is the symbol of the Laplacian: $\sigma^w(-\Delta) = |\xi|^2$, which has an inverse of symbol $1/|\xi|^2$. We say that a pseudo-differential operator is elliptic if its symbol is elliptic[3]. One can then find a first ansatz for an approximate inverse by setting

$$q(x, \xi) := \frac{1 - \chi(\xi)}{\sigma(x, \xi)}$$

where $\chi \equiv 1$ on the ball where we do not have control over σ. One can then verify that

$$Op^w(\sigma)Op^w(q) = Id + S_1, \qquad Op^w(q)Op^w(\sigma) = Id + S_2$$

where S_1 and S_2 are smoothing operators by one order.

2. Paradifferential operators

The purpose of this section is to make the previous qualitative discussion more quantitative.

2.1. **Paradifferential operators.** A small twist is that we want to consider operators and symbols which are not necessarily smooth. When considering products, in the case when a function is much smoother than the other, we want to keep track of this information (since the rougher one will force the regularity of the product). A simple but far-reaching remark by Bony is that we can always separate the smooth part from the rough part:

radiffIntro}
$$\begin{aligned} fg &= \sum_{k_1, k_2} P_{k_1} f \cdot P_{k_2} g \\ &= \sum_{k_1 \leq k_2 - 10} P_{k_1} f \cdot P_{k_2} g + \sum_{k_2 \leq k_1 - 10} P_{k_1} f \cdot P_{k_2} g + \sum_{|k_1 - k_2| < 10} P_{k_1} f \cdot P_{k_2} g \\ &= \sum_k (P_{\leq k-10} f) P_k g + \sum_k (P_{\leq k-10} g) P_k f + \sum_k \sum_{|j| < 10} P_k f \cdot P_{k+j} g \\ &= T_f g + T_g f + R(f, g) \end{aligned} \tag{2.1}$$

where (provided f and g have a reasonable amount of smoothness, say $f, g \in H^d$) $R(f, g)$ is smoother than f and g, $T_f g$ is as smooth as g and $T_g f$ is as smooth as f (see Lemma 2.2 for more

[2]Indeed, we want to allow elliptic operators to have a finite dimensional kernel. Otherwise, $-\Delta$ would not be elliptic on $\mathbb{T}^d$!

[3]Note that this is a property of its principal symbol.

precise estimates). This simple decomposition, called the "paraproduct decomposition" allows to better keep track of the regularity and we want to extend it for pseudo-differential operators (it amounts to "specifying the term where the derivative will fall").

In the following, for concreteness, we will specify to the case of dimension $d = 2$.

Recall that a symbol σ "if of degree p" if and only if $(1+|\zeta|^2)^{-\frac{p}{2}}\sigma(x,\zeta)$ is a symbol of degree 0. For this reason, we will in the next subsection concentrate on redefining a more quantified version of symbols of degree 0.

2.2. **Operator bounds.** A symbol denotes a function $\sigma(x,\zeta) \in C(\mathbb{R}^d \times \mathbb{R}^d : \mathbb{R})$. In this section, we will bound them using the following two norms[4]

$$\begin{aligned} \|\sigma\|_{\mathcal{S}^{a,b}_\infty} &:= \sup_{x,\zeta} \sum_{|\alpha|\le a,\, |\beta|\le b} ||\zeta|^{|\beta|}\partial^\beta_\zeta \partial^\alpha_x \sigma(x,\zeta)|, \\ \|\sigma\|_{\mathcal{S}^{a,b}_2} &:= \sup_{\zeta} \sum_{|\alpha|\le a,\, |\beta|\le b} \||\zeta|^{|\beta|}\partial^\beta_\zeta \partial^\alpha_x \sigma(\cdot,\zeta)\|_{L^2_x}. \end{aligned} \tag{2.2}$$ {SymNorm1}

These are the most important norms. However, we may more generally define

$$\|\sigma\|_{\mathcal{S}^{a,b}_p} := \sup_{\zeta} \sum_{|\alpha|\le a,\, |\beta|\le b} \||\zeta|^{|\beta|}\partial^\beta_\zeta \partial^\alpha_x \sigma(\cdot,\zeta)\|_{L^p_x}. \tag{2.3}$$ {SymNorm1Gen

Note that there is a large variation of norms and notation for symbols. A few useful properties are[5]

$$\begin{aligned} \|ab\|_{\mathcal{S}^{\min\{r_1,r_2\},\min\{q_1,q_2\}}_p} &\lesssim \|a\|_{\mathcal{S}^{r_1,q_1}_{p_1}} \|b\|_{\mathcal{S}^{r_2,q_2}_{p_2}}, \qquad \{\infty,p\} = \{p_1,p_2\} \\ \|\{a,b\}\|_{\mathcal{S}^{\min\{r_1,r_2\}-1,\min\{q_1,q_2\}-1}_p} &\lesssim \|a\|_{\mathcal{S}^{r_1,q_1}_{p_1}} \|b\|_{\mathcal{S}^{r_2,q_2}_{p_2}}, \qquad \{\infty,p\} = \{p_1,p_2\} \\ \|P_k a\|_{\mathcal{S}^{r,q}_p} &\lesssim 2^{-sk}\|P_k a\|_{\mathcal{S}^{r+s,q}_p}, \qquad p \in \{2,\infty\}. \end{aligned} \tag{2.4}$$ {PropSymb1}

Given a symbol $\sigma(x,\zeta)$, we define the corresponding paradifferential operator (compare with (1.4)):

$$\mathcal{F}\left\{T_\sigma f\right\}(\xi) = \int_{\mathbb{R}^2} \chi(\frac{|\xi-\eta|}{|\xi+\eta|})\widetilde{\sigma}(\xi-\eta,\frac{\xi+\eta}{2})\widehat{f}(\eta)d\eta,$$

where $\widetilde{\sigma}$ denotes the Fourier transform in the first coordinate and $\chi \in C^\infty_c(\mathbb{R}:\mathbb{R})$ is a function satisfying

$$\mathbf{1}_{[-1/200,1/200]} \le \chi \le \mathbf{1}_{[-1/100,1/100]}.$$

It is clear that if σ is real, T_σ is a self-adjoint operator on L^2. But in fact T_σ also acts on L^p spaces. We first observe that paramultipliers define nice linear operators.

Proposition 2.1. *Let a be a symbol and $1 \le q \le \infty$, let[6] $\gamma > d/2$, $\gamma \in \mathbb{N}$ then*

$$\|P_k T_a f\|_{L^q} \lesssim \|a\|_{\mathcal{S}^{0,4\gamma}_\infty} \|P_{[k-2,k+2]} f\|_{L^q} \tag{2.5}$$ {LqBdTa}

and

$$\|P_k T_a f\|_{L^2} \lesssim \|a\|_{\mathcal{S}^{0,4\gamma}_2} \|P_{[k-2,k+2]} f\|_{L^\infty}. \tag{2.6}$$ {L2BdTa}

More generally, if

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \qquad 1 \le p,q,r \le \infty, \tag{2.7}$$ {Homogeneity

[4]These norms measure symbols of degree 0. Thanks to (2.5), the degree can then be trivially added. Another way to say this is that a symbol of degree p can be written $\sigma = (1-\Delta)^{p/2}\sigma^0$, where σ^0 is an operator of degree 0.

[5]Recall the Poisson bracket introduced in (1.5).

[6]as will be evident from the proof, we could take γ smaller when $p < \infty$, but we will not be concerned with the ζ-regularity which will always be automaitc.

then

$$\|P_k T_a f\|_{L^r} \lesssim \|a\|_{\mathcal{S}_p^{0,4\gamma}} \|P_{[k-2,k+2]} f\|_{L^q}. \tag{2.8}$$ {LrBdTa}

Proof. **Fact 1:** Inspecting the Fourier transform, we directly see that

$$P_k T_a f = P_k T_a P_{[k-2,k+2]} f,$$

and therefore, the Fourier localization of f is guaranteed and, up to rescaling, we may assume that $k = 0$, i.e.

Fact 2: It suffices to show (2.8) for $k = 0$. Indeed, introducing the rescaling operator $(\delta_h f)(x) = f(hx)$, we recall that

Homogeneity}
$$\|\delta_h f\|_{L^p} = h^{-\frac{d}{p}} \|f\|_{L^p}. \tag{2.9}$$

Besides, letting $h = 2^{-k}$ and writing down the Fourier transform, we find that

{Rescaling2}
$$\delta_h(P_k T_a f) = P_0 T_{a^h}(\delta_h f), \qquad a^h(x,\zeta) := a(hx, h^{-1}\zeta) \tag{2.10}$$

and observing that

$$\|a^h\|_{\mathcal{S}_p^{0,t}} = h^{-\frac{d}{p}} \|a\|_{\mathcal{S}_p^{0,t}}, \qquad \forall t \geq 0$$

and using (2.7) and (2.9), we see from (2.10) that the claim at $k = 0$ implies the claim for all k's.

Fact 3: The claim holds when $k = 0$. We compute that

$$\begin{aligned}
\langle g, P_0 T_a h\rangle &= \int_{\mathbb{R}^{2d}} \overline{g}(x) h(y) I(x,y) dx dy, \\
I(x,y) &= \int_{\mathbb{R}^{3d}} a(z, \frac{\xi+\eta}{2}) e^{i\langle \xi, x-z\rangle} e^{i\langle \eta, z-y\rangle} \chi(\frac{|\xi-\eta|}{|\xi+\eta|}) \varphi_0(\xi) d\eta d\xi dz \\
&= \int_{\mathbb{R}^{3d}} a(z, \eta + \frac{\theta}{2}) e^{i\langle \theta, x-z\rangle} e^{i\langle \eta, x-y\rangle} \chi(\frac{|\theta|}{|\eta+2\theta|}) \varphi_0(\eta+\theta) d\eta d\theta dz.
\end{aligned}$$

We now observe that

opSymEstimI}
$$\begin{aligned}
&(1+|x-y|^2)^\gamma I(x,y) \\
&= \int_{\mathbb{R}^{3d}} \frac{a(z,\eta+\frac{\theta}{2})}{(1+|x-z|^2)^\gamma} \chi(\frac{|\theta|}{|\eta+2\theta|}) \varphi_0(\eta+\theta) \left[(1-\Delta_\theta)^\gamma e^{i\langle\theta, x-z\rangle} (1-\Delta_\eta)^\gamma e^{i\langle \eta, x-y\rangle}\right] d\eta d\theta dz
\end{aligned} \tag{2.11}$$

so that

$$|(1+|x-y|^2)^\gamma I(x,y)| \lesssim \|a\|_{\mathcal{S}_p^{0,4\gamma}}$$

and we conclude that

$$\begin{aligned}
\langle g, P_0 T_a h\rangle &\leq \|a\|_{\mathcal{S}_p^{0,4\gamma}} \int_{\mathbb{R}^4} |\overline{g}(x)| \cdot |h(y)| \cdot \frac{1}{(1+|x-y|^2)^\gamma} dx dy \\
&\lesssim \|a\|_{\mathcal{S}_p^{0,4\gamma}} \|g\|_{L^{r'}} \|h\|_{L^q},
\end{aligned}$$

which is sufficient for our estimate.

□

One of the main interest of T_a is that it linearizes the product as observed in (2.1):

Lemma 2.2. *Let $f, g \in L^2$ we have*

$$fg = T_f g + T_g f + \mathcal{HH}(f,g)$$

where $\mathcal{HH}$ is smoothing in the sense that for $\gamma > 0$ and r, p, q as in (2.7), there holds that

$$\|P_k \mathcal{HH}(f,g)\|_{L^r} \lesssim 2^{-\gamma k} \sup_{l-20 \leq \mu, \nu \leq l+20} 2^{\gamma l} \|P_\mu f\|_{L^p} \|P_\nu g\|_{L^q}.$$

Proof. Starting from

$$\mathcal{H}\mathcal{H}(f,g) = fg - T_f g - T_g f$$

and expanding on the frequency projections, we see that

$$P_k \mathcal{H}\mathcal{H}(P_j f, P_l g) \equiv 0,$$

unless $|j-l| \leq 10$ and $k \leq j+10$. Therefore

$$\|P_k \mathcal{H}\mathcal{H}(f,g)\|_{L^r} \lesssim 2^{-\gamma k} \sum_{j \geq k-10} 2^{\gamma(k-j)} \sum_{0 \leq j' \leq 10} 2^{rj} \|P_j f P_{j+j'} g\|_{L^r}$$

and the result follows by Hölder's inequality.

□

Proposition 2.3. *Let $1 \leq q \leq \infty$ and $\gamma > d/2$, $\gamma \in \mathbb{N}$. Given a and b symbols, we may decompose*

$$T_a T_b = T_{ab} + \frac{i}{2} T_{\{a,b\}} + E(a,b) \tag{2.12}$$ {PropCompSym

where the error E obeys the following bounds

$$\|P_k E(a,b) f\|_{L^q} \lesssim (1+2^k)^{-2} \|a\|_{\mathcal{S}^{2,6\gamma+2}_\infty} \|b\|_{\mathcal{S}^{2,6\gamma+2}_\infty} \|P_{[k-5,k+5]} f\|_{L^q} \tag{2.13}$$ {RemRemBound

for $1 \leq q \leq \infty$ and more generally, letting

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}, \qquad 1 \leq r, p_1, p_2, p_3 \leq \infty \tag{2.14}$$ {TrilinearHo

there holds that

$$\|P_k E(a,b) f\|_{L^r} \lesssim (1+2^k)^{-2} \|a\|_{\mathcal{S}^{2,6\gamma+2}_{p_1}} \|b\|_{\mathcal{S}^{2,6\gamma+2}_{p_2}} \|P_{[k-5,k+5]} f\|_{L^{p_3}}. \tag{2.15}$$ {RemRemBound

Moreover $E(a,b) = 0$ if both a and b are independent of x.

Proof. We prove (2.13). The bounds in (2.15) follow similarly. We begin by writing

$$\begin{aligned}
&T_a T_b - T_{ab} = E_s + E_r, \\
&\mathcal{F}\{E_s f\}(\xi) = \int_{\mathbb{R}^4} \left[\chi(\frac{|\xi-\theta|}{|\xi+\theta|}) \chi(\frac{|\theta-\eta|}{|\theta+\eta|}) - \chi(\frac{|\xi-\eta|}{|\xi+\eta|}) \right] \widetilde{a}(\xi-\theta, \frac{\xi+\theta}{2}) \widetilde{b}(\theta-\eta, \frac{\eta+\theta}{2}) \widehat{f}(\eta) d\eta d\theta, \\
&\mathcal{F}\{E_r f\}(\xi) = \int_{\mathbb{R}^4} \chi(\frac{|\xi-\eta|}{|\xi+\eta|}) \Big[\widetilde{a}(\xi-\theta, \frac{\xi+\theta}{2}) \widetilde{b}(\theta-\eta, \frac{\eta+\theta}{2}) \\
&\qquad\qquad - \widetilde{a}(\xi-\theta, \frac{\xi+\eta}{2}) \widetilde{b}(\theta-\eta, \frac{\xi+\eta}{2}) \Big] \widehat{f}(\eta) d\eta d\theta,
\end{aligned}$$

and observe directly that if $\widetilde{a}(x,\zeta)$ and $\widetilde{b}(y,\zeta)$ are supported on $\{x=0\}$ and $\{y=0\}$, both E_s and E_r vanish. We also observe that, using Lemma 2.1 together with (2.4),

$$\|P_k T_{P_j a P_l b} f\|_{L^q} + \|P_k T_{P_j a} T_{P_l b} f\|_{L^q} \lesssim 2^{-rj-sl} \|P_j a\|_{\mathcal{S}^{r,4\gamma}_\infty} \|P_l b\|_{\mathcal{S}^{s,4\gamma}_\infty} \|P_{[k-5,k+5]} f\|_{L^q}.$$

Thus in the following, we mays assume that $a = P_{\leq k-10} a$ and $b = P_{\leq k-10} b$. In this case, $E_s = 0$. Using this, it suffices to consider

$$\begin{aligned}
\mathcal{F}(E_r^k f)(\xi) = \int_{\mathbb{R}^{2d}} \underline{\varphi}_k(\xi,\eta,\theta) \chi(\frac{|\xi-\eta|}{|\xi+\eta|}) \Big[&\widetilde{a}(\xi-\theta, \frac{\xi+\theta}{2}) \widetilde{b}(\theta-\eta, \frac{\eta+\theta}{2}) \\
&- \widetilde{a}(\xi-\theta, \frac{\xi+\eta}{2}) \widetilde{b}(\theta-\eta, \frac{\xi+\eta}{2}) \Big] \widehat{f}(\eta) d\eta d\theta,
\end{aligned}$$

where we define

$$\underline{\varphi}_k(\xi,\eta,\theta) = \varphi_k(\xi) \varphi_{[k-2,k+2]}(\eta) \varphi_{[k-2,k+2]}(\theta). \tag{2.16}$$ {Underlineph

We note that $E_r^k = P_k E_r$ has kernel whose Fourier transform is

$$m(\xi,\eta) = \\ = \int_{\mathbb{R}^{3d}} e^{i\Phi} \underline{\varphi}_k(\xi,\eta,\theta)\chi(\frac{|\xi-\eta|}{|\xi+\eta|})\Big[a(y_1,\frac{\xi+\theta}{2})b(y_2,\frac{\eta+\theta}{2}) - a(y_1,\frac{\xi+\eta}{2})b(y_2,\frac{\xi+\eta}{2})\Big]dy_1dy_2d\theta$$

$$\Phi = \langle y_1, \theta-\xi\rangle + \langle y_2, \eta-\theta\rangle$$

where in particular,

{DerPh}
$$\nabla_{y_1}\Phi = \theta-\xi, \qquad \nabla_{y_2}\Phi = \eta-\theta. \tag{2.17}$$

Integrating by parts using (2.17), we may rewrite this kernel as

$$\begin{aligned} m(\xi,\eta) = \frac{i}{2}\int_{\mathbb{R}^{3d}} & e^{i\Phi}\underline{\varphi}_k(\xi,\eta,\theta)\chi(\frac{|\xi-\eta|}{|\xi+\eta|}) \\ & \times\left[\nabla_1 a(y_1,\frac{\xi+\eta}{2})\nabla_2 b(y_2,\frac{\xi+\eta}{2}) - \nabla_2 a(y_1,\frac{\xi+\eta}{2})\nabla_1 b(y_2,\frac{\xi+\eta}{2})\right]dy_1dy_2d\theta \\ + \frac{i}{2}\int_{s=0}^1\int_{\mathbb{R}^{3d}} & e^{i\Phi}\underline{\varphi}_k(\xi,\eta,\theta)\chi(\frac{|\xi-\eta|}{|\xi+\eta|}) \\ & \times\Big(\nabla_1 a(y_1,\frac{\xi+\eta}{2})\left[\nabla_2 b(y_2,\frac{\xi+\eta}{2}+s\frac{\theta-\xi}{2}) - \nabla_2 b(y_2,\frac{\xi+\eta}{2})\right] \\ & -\left[\nabla_2 a(y_1,\frac{\xi+\theta}{2}+s\frac{\eta-\theta}{2}) - \nabla_2 a(y_1,\frac{\xi+\eta}{2})\right]\nabla_1 b(y_2,\frac{\eta+\theta}{2}) \\ & -\nabla_2 a(y_1,\frac{\xi+\eta}{2})\left[\nabla_1 b(y_2,\frac{\eta+\theta}{2}) - \nabla_1 b(y_2,\frac{\xi+\eta}{2})\right]\Big)dy_1dy_2d\theta ds \end{aligned}$$

The first integral gives the Poisson bracket. After another integration by parts using (2.17), the second integral can be rewritten

$$\begin{aligned} m_r(\xi,\eta) = -\frac{1}{4}\int_{s=0}^1\int_{\mathbb{R}^{3d}} & e^{i\Phi}\underline{\varphi}_k(\xi,\eta,\theta)\chi(\frac{|\xi-\eta|}{|\xi+\eta|})\Big(\nabla_{11}a(y_1,\frac{\xi+\eta}{2})\nabla_{22}b(y_2,\frac{\xi+\eta}{2}+s\frac{\theta-\xi}{2})(1-s) \\ & -\nabla_{12}a(y_1,\frac{\xi+\eta}{2})\nabla_{21}b(y_2,\frac{\xi+\eta}{2}+s\frac{\theta-\eta}{2}) \\ & +\nabla_{22}a(y_1,\frac{\xi+\eta}{2}+s\frac{\theta-\eta}{2})\nabla_{11}b(y_2,\frac{\eta+\theta}{2})(1-s)\Big)dy_1dy_2d\theta ds. \end{aligned}$$

We may now use Lemma 2.6 and Lemma 2.4 to finish the proof. □

Note that we could go further in the expansion (2.12). Indeed, a look at the above formula shows that the main contribution will come from the second Poisson bracket:

$$\{\{a,b\}\}_2 := \nabla_{11}a : \nabla_{22}b - 2\nabla_{12}a : \nabla_{21}b + \nabla_{22}a : \nabla_{11}b.$$

Further contributions are determined by higher order Poisson brackets.

Lemma 2.4. *For $0 \le s \le 1$, define*

$$\begin{aligned} m_k(\xi,\eta) &= \int_{\mathbb{R}^{3d}} \underline{\varphi}_k(\xi,\eta,\theta)\chi(\frac{|\xi-\eta|}{|\xi+\eta|})e^{i\Phi}\{\{a,b\}\}dy_1dy_2d\theta \\ \{\{a,b\}\} &= \Big(\nabla_{11}a(y_1,\frac{\xi+\eta}{2})\nabla_{22}b(y_2,\frac{\xi+\eta}{2}+s\frac{\theta-\xi}{2})(1-s) \\ &\quad - \nabla_{12}a(y_1,\frac{\xi+\eta}{2})\nabla_{21}b(y_2,\frac{\xi+\eta}{2}+s\frac{\theta-\eta}{2}) \\ &\quad + \nabla_{22}a(y_1,\frac{\xi+\eta}{2}+s\frac{\theta-\eta}{2})\nabla_{11}b(y_2,\frac{\eta+\theta}{2})(1-s)\Big) \\ \Phi &= \langle y_1,\theta-\xi\rangle + \langle y_2,\eta-\theta\rangle \end{aligned}$$

where $\underline{\varphi}_k$ is defined in (2.16)*. Then*

$$|(\mathcal{F}^{-1}m_k)(x,z)|_{L^1_x} \lesssim (1+2^k)^{-2}\|a\|_{\mathcal{S}^{2,6\gamma+2}_\infty}\|b\|_{\mathcal{S}^{2,6\gamma+2}_\infty}\cdot\left[1+|x-z|^2\right]^{-\gamma}. \tag{2.18}$$ {EstEsFL1}

Proof. Assume first that $k \ge 0$. The Fourier transform is then

$$\begin{aligned} I_{k,s}(x,z) &:= \int_{\mathbb{R}^{2d}} J_{k,s}(y_1,y_2)dy_1dy_2 \\ J_{k,s}(y_1,y_2) &= \int_{\mathbb{R}^{3d}} e^{i\Psi}\mathfrak{A}_{k,s}d\eta d\xi d\theta \\ \mathfrak{A}_{k,s} &= \underline{\varphi}_k(\xi,\eta,\theta)\chi(\frac{|\xi-\eta|}{|\xi+\eta|})\{\{a,b\}\}, \\ \Psi &= \langle y_1,\theta-\xi\rangle + \langle y_2,\eta-\theta\rangle + \langle \xi,x\rangle - \langle \eta,z\rangle \end{aligned}$$

The main information we use is that

$$\begin{gathered} \left|\partial^\alpha_{\xi,\eta,\theta}\left[\mathfrak{A}_{k,s}\right]\right| \lesssim 2^{-|\alpha|k}2^{-2k}\cdot \mathbf{1}_{\{|\xi|+|\eta|+|\theta|\lesssim 2^k\}}\cdot\|a\|_{\mathcal{S}^{2,2+|\alpha|}_\infty}\|b\|_{\mathcal{S}^{2,2+|\alpha|}_\infty} \\ \nabla_\xi\Psi = x-y_1, \qquad \nabla_\eta\Psi = y_2-z, \qquad \nabla_\theta\Psi = y_1-y_2, \end{gathered}$$

so that, integrating by parts, we find that, for $0 \le q,r,t \le \gamma$,

$$\begin{aligned} |x-y_1|^{2q}|y_2-z|^{2r}|y_1-y_2|^{2t}|J_{k,s}(y_1,y_2)| &= \left|(-1)^{q+r+t}\int_{\mathbb{R}^6} e^{i\Psi}\Delta^q_\xi\Delta^r_\eta\Delta^t_\theta\mathfrak{A}_{k,s}d\eta d\xi d\theta\right| \\ &\lesssim 2^{-2k}2^{(3d-2(q+r+t))k}\|a\|_{\mathcal{S}^{2,6\gamma+2}_\infty}\|b\|_{\mathcal{S}^{2,6\gamma+2}_\infty} \end{aligned}$$

which is sufficient.

If $k \le 0$, we simply observe that, using the Fourier transform in y_1, y_2 in the integration, we may trade the derivatives in y_1, y_2 by small factors of size $O(2^k)$, namely we may replace $\{\{a,b\}\}$ by

$$\begin{aligned} \{\{a,b\}\} = \frac{1}{4}\Big(&- a(y_1,\frac{\xi+\eta}{2})\left((\theta-\xi)\cdot\nabla_2\right)^2 b(y_2,\frac{\xi+\eta}{2}+s\frac{\theta-\xi}{2})(1-s) \\ &+ ((\eta-\theta)\cdot\nabla_2)a(y_1,\frac{\xi+\eta}{2})((\theta-\xi)\cdot\nabla_2)b(y_2,\frac{\xi+\eta}{2}+s\frac{\theta-\eta}{2}) \\ &- \left((\eta-\theta)\cdot\nabla_2\right)^2 a(y_1,\frac{\xi+\eta}{2}+s\frac{\theta-\eta}{2})b(y_2,\frac{\eta+\theta}{2})(1-s)\Big), \end{aligned}$$

and since $|\eta|+|\theta|+|\xi| \lesssim 2^k$, we obtain that

$$\left|\partial^\alpha_{\xi,\eta,\theta}\left[\mathfrak{A}_{k,s}\right]\right| \lesssim 2^{-|\alpha|k}\cdot \mathbf{1}_{\{|\xi|+|\eta|+|\theta|\lesssim 2^k\}}\cdot\|a\|_{\mathcal{S}^{0,2+|\alpha|}_\infty}\|b\|_{\mathcal{S}^{0,2+|\alpha|}_\infty}$$

which again, is sufficient. □

Proposition 2.5. *Assume that $F(z) = z + h(z)$, where h is analytic for $|z| \leq 1/2$ and satisfies $|h(z)| \lesssim |z|^3$, then there holds that*

$$\begin{aligned} F(u) &= T_{F'(u)}u + E(u), \\ \|E(u)\|_{H^N} &\lesssim \|u\|^2_{W^{10,\infty}} \|u\|_{H^{N-5}} \end{aligned} \tag{2.19}$$

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so long as $\|u\|_{L^\infty} \leq 1/100$. This also holds if $z \in \mathbb{C}^2$, $F : \mathbb{C}^2 \to \mathbb{C}^2$, where $F'(u)$ denotes the differential of F.

Proof. Since F is analytic, it suffices to show this for $h(x) = x^n$, $n \geq 0$, but this as in the proof of Lemma 2.2. Indeed, we may decompose

$$\begin{aligned} P_k(u^n) = n &\sum_{|k_1 - k| \leq 2n,\, \max\{k_2,\dots,k_n\} \leq k-3n} (P_{k_2}u \dots P_{k_n}u) \cdot P_{k_1}u \\ &+ O(n^2) \sum_{k_1 \sim k_2 \geq \max\{k_3,\dots,k_n,k-3n\}} P_{k_1}u \cdot P_{k_2}u \cdot P_{k_3}u \dots P_{k_n}u. \end{aligned}$$

The first term in this expansion leads to the paraproduct up to smoothing terms, while the second leads to a smoothing terms. □

Recall the following simple Lemma for the operator

$$\mathcal{F}M(f_1, f_2, \dots, f_n)(\xi) = \int_{\mathbb{R}^d} \widehat{f}(\xi_1)\widehat{f}(\xi_2)\dots\widehat{f}(\xi_d) m(\xi_1, \dots, \xi_d)\delta(\xi - \xi_1 - \dots - \xi_d) d\xi_1 \dots d\xi_d.$$

This operator obeys simple bounds:

Lemma 2.6. *Assume that $1 \leq p_1, p_2, \dots, p_n, s \leq \infty$ and*

$$\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots \frac{1}{p_n}.$$

Then

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$$\|M(f_1, f_2, \dots, f_n)(x)\|_{L^s} \lesssim \|\mathcal{F}m\|_{L^1} \prod_{k=1}^n \|f_k\|_{L^{p_k}}. \tag{2.20}$$

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