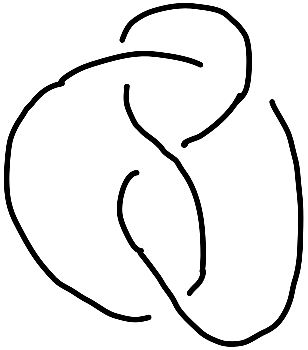
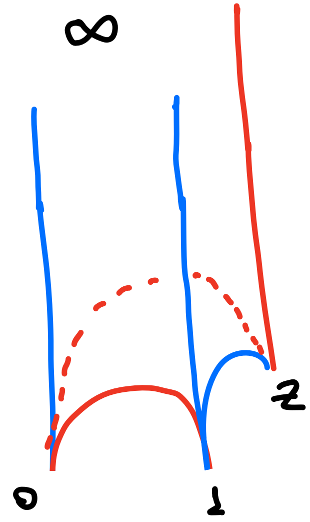
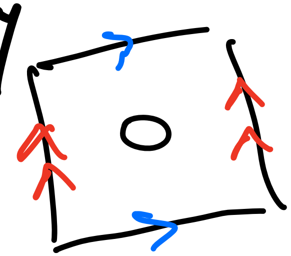
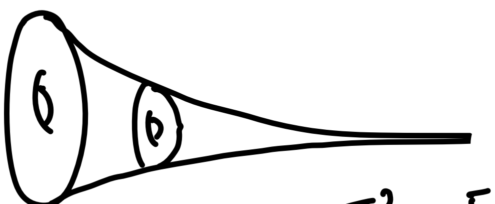




Projective Rigidity of Once-Punctured Torus Bundles

Charles Daly   Brown University





$T^2 \times [0, \infty)$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\Gamma \xhookrightarrow{\rho} PSL(2, \mathbb{C})$$

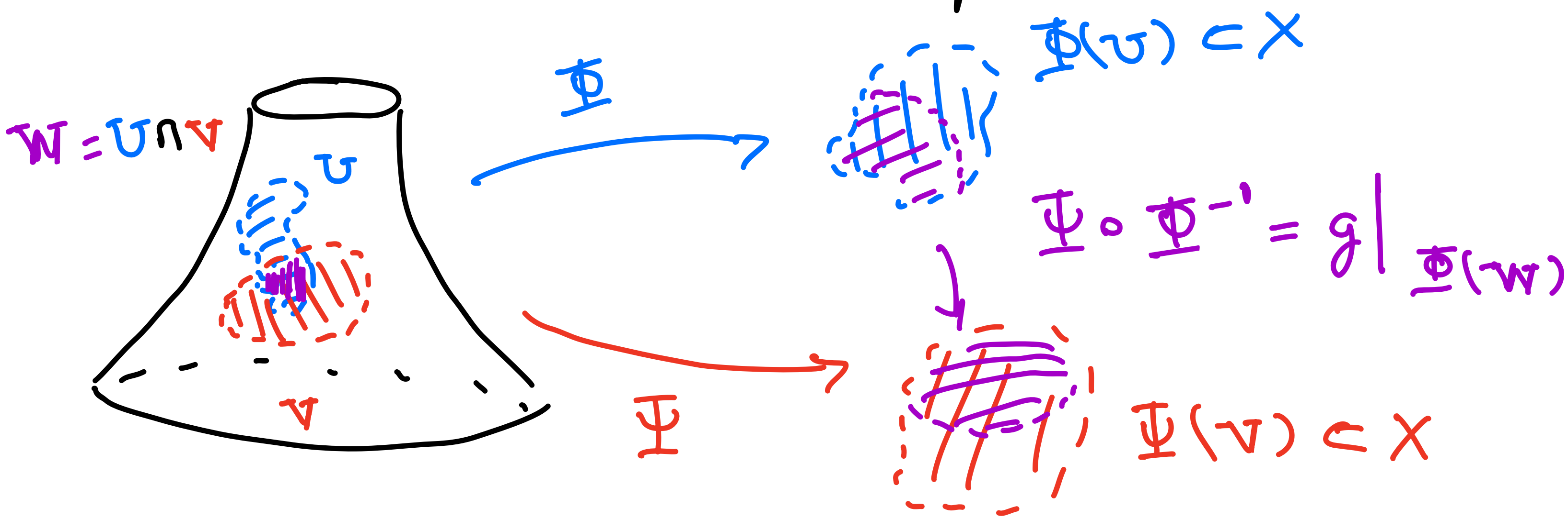
Overview of Talk

- What is a geometric structure?
- Deforming projective structures.
- Say some things in context of hyperbolic once-punctured torus bundles

What is a Geometric Structure?

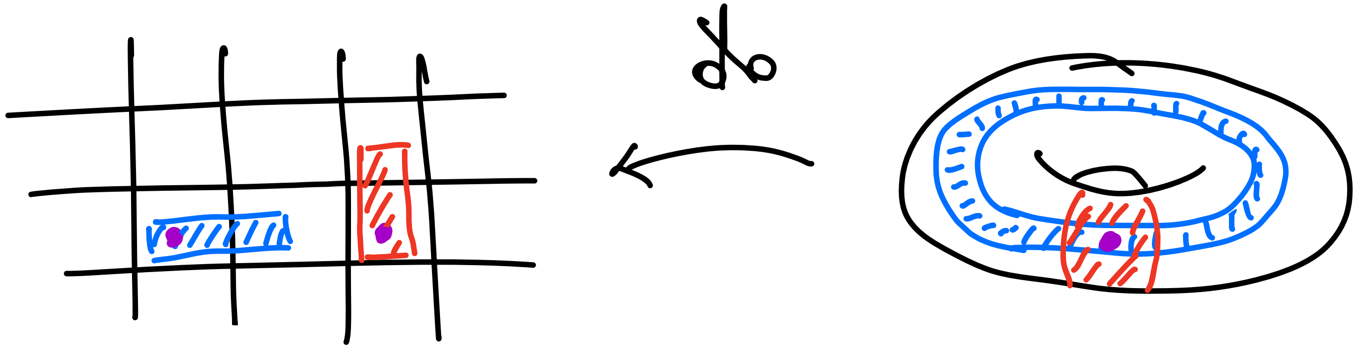
(G,X)-manifold. G group acting on space X transitively. Also want G to act strongly effectively, i.e. if $g|_U = id$ then $g = 1$.

A (G,X)-structure on a manifold M is atlas of charts into X so that the transition functions are locally restrictions of elements of G acting on X.

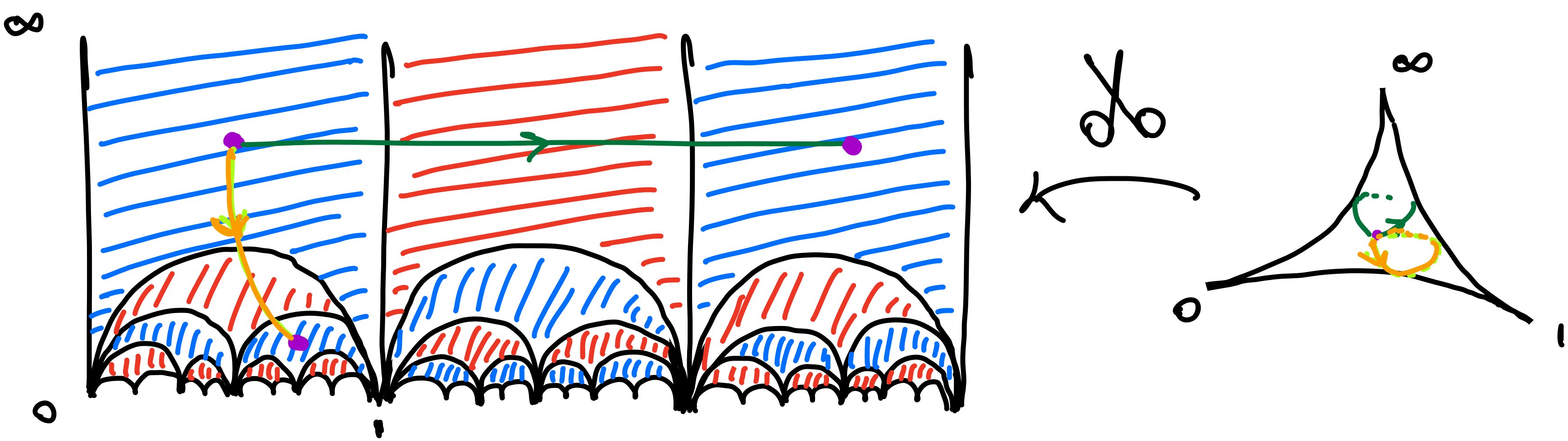


Examples

$G = \mathrm{Isom}(\mathbb{E}^2)$, $X = \mathbb{E}^2$, (G, X) euclidean surface

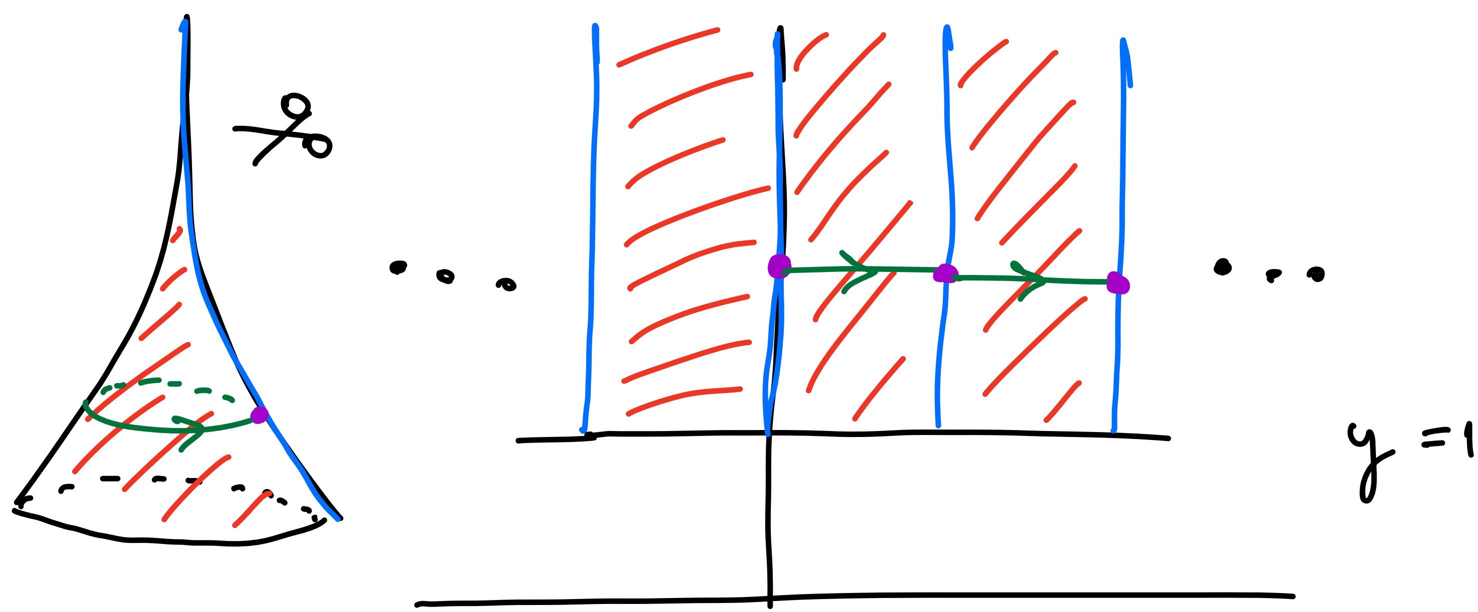


$G = \mathrm{Isom}(\mathbb{H}^2)$, $X = \mathbb{H}^2$, (G, X) hyperbolic surface.



From above for each such structure, you get a holonomy representation $\rho : \Gamma \longrightarrow G$

$\Gamma = \pi_1(M)$. In above examples can think of unfolding the manifold along loops.



$\rho(\text{loop})$ = translation by $+1$ to the right

Ehresmann-Weil-Thurston Principle

$$\left\{ \begin{matrix} (G,X)\text{-structure on } M \\ \text{up to equivalence} \end{matrix} \right\} \xrightarrow{\text{hol}} \{ \rho : \Gamma \to G \} / \text{conjugation}$$

is in nicest cases a local homeomorphism

where spaces are given appropriate topologies.

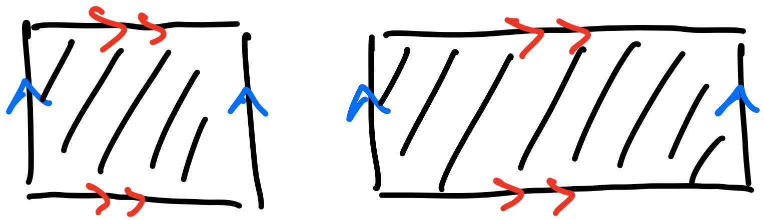
Hyperbolic : $(\mathrm{Isom}(\mathbb{H}^n), \mathbb{H}^n)$

Affine : $(\mathrm{Aff}(n,\mathbb{R}), \mathbb{A}^n)$

Projective : $(PGL(n+1,\mathbb{R}), \mathbb{RP}^n)$

Moving around in moduli space corresponds to <u>deforming</u> the geometry.

ex:



$$\rho_t : \Gamma \longrightarrow \mathrm{Isom}(\mathbb{R}^2)$$

$$\alpha \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+t \\ 0 \end{pmatrix}$$

$$\beta \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

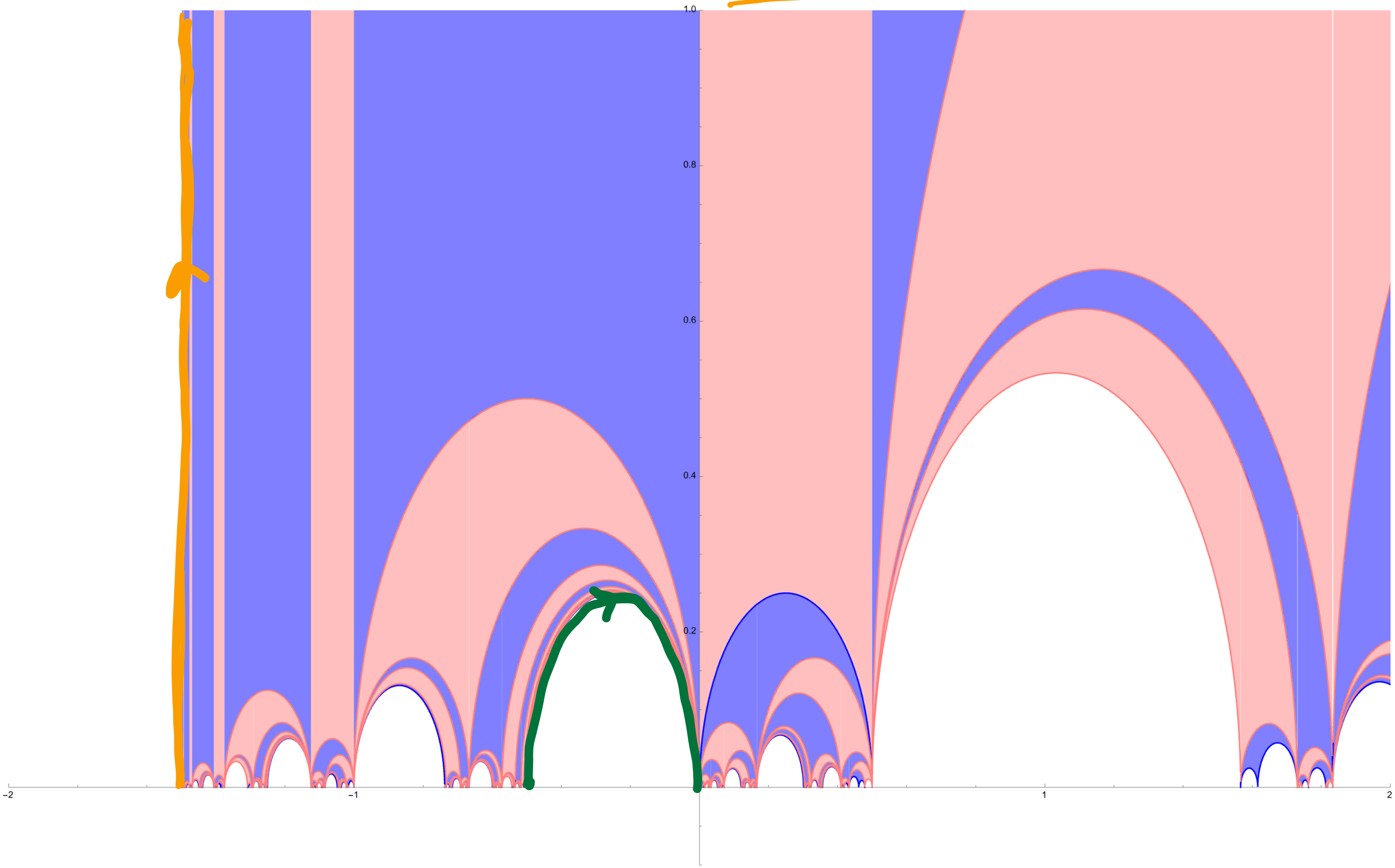
distinct euclidean structures on T^2!

(Equivalent under $\mathrm{Aff}(2,\mathbb{R})$ though)

An incomplete structure on $S^2 \setminus \{a, b, c\}$

generated by $\begin{pmatrix} 2 & 9/4 \\ 0 & 1/2 \end{pmatrix}$ and $\begin{pmatrix} 1/2 & 0 \\ 3 & 2 \end{pmatrix}$

$(-3/2, \infty)$ and $(-1/2, 0)$



How to study deformations of geometric structures?

Tons of work in low-dimensional setting i.e. $\Gamma = \pi_1$ of a surface or 3-manifold.

- Understand $\mathrm{Hom}(\Gamma, G)/G$ character variety
- If G is real semi-simple theory of Θ-positive representations / Anosov representations
- (Higher) Teichmüller theory
- Weil's Infinitesimal Rigidity theorem

Weil's Infinitesmal Rigidity Theorem

Deforming the geometric structure will deform the holonomy ρ

so $\rho_t : \Gamma \to G$ 1-parameter family.

Trivial deformations arise by $\rho_t = g_t \rho g_t^{-1}$ for some path g_t in G with $g_0 = 1$. A non-trivial ρ_t will determine a path in $\mathrm{Hom}(\Gamma, G)/G$ starting at $[\rho]$. Under the nicest possible circumstances $[\rho]$ is a smooth point of $\mathrm{Hom}(\Gamma, G)/G$,

and ρ_t determines a tangent vector.

$\{$Tangent vectors to $\mathrm{Hom}(\Gamma, G)$ at point $\rho\}$

$\leftrightarrow \{ f : \Gamma \to \mathfrak{g}_{\mathrm{Ad}\,\rho}$ crossed homomorphisms $\}$

$\{$ Subspace of tangent vectors to $\mathrm{Hom}(\Gamma, G)$ at point ρ coming from conj $\}$

$\leftrightarrow \{ f : \Gamma \to \mathfrak{g}_{\mathrm{Ad}\,\rho}$ is a principal derivation $\}$

Given a deformation get a crossed homomorphism via

$$f(a) = \frac{d}{dt}\Big|_{t=0} \rho_t(a)\,\rho(a)^{-1} \in \mathfrak{g}$$

Satisfies $f(ab) = f(a) + \mathrm{Ad}\,\rho(a)\,f(b) = f(a) + a f(b)$

$\Gamma \xrightarrow{\rho} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$

for all $a, b \in \Gamma$, hence $\mathfrak{g}_{\mathrm{Ad}\,\rho}$.

Principal derivations arise from $\rho_t = g_t \rho g_t^{-1}$

$f(a) = (1-a)X$ for some $X \in \mathfrak{g}$ and all $a \in \Gamma$

$$H^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \cong \left(\text{Crossed homomorphisms}\right) \Big/ \left(\text{principal derivations}\right)$$

So we can think of $H^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\,\rho})$ as

tangent space of character variety at $[\rho] \in \mathrm{Hom}(\Gamma, G)/G$

We say $\rho : \Gamma \to G$ is infinitesmally rigid if $H^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\,\rho}) = 0$.

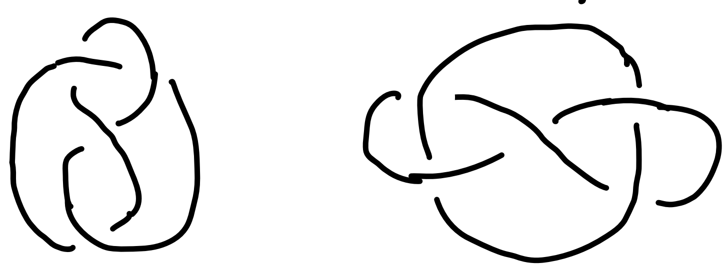
Infinitesmally rigid $\Rightarrow$ locally rigid

Mostow's Rigidity theorem (one form of it)

Let N be a compact 3-manifold. Any two hyperbolic structures on N are equivalent. That is, up to isometry, there's only 1-way to put a hyperbolic structure on N. Consequently $\rho_*: \Gamma \longrightarrow \mathrm{Isom}(\mathbb{H}^3)$ has to be trivial. (Here we're using compact Riemannian manifolds are complete and holonomy must be injective in this case)

Algebraically, $H^1(\Gamma; \mathfrak{sl}(2,\mathbb{C})_{Ad\,\rho}) = 0$

for $\Gamma = \pi_1(N)$ and ρ the holonomy of N.



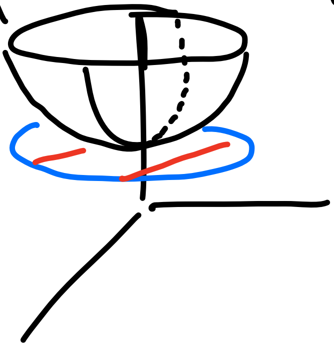
there's only 1 complete hyperbolic structure on us!

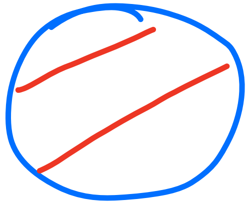
We're interested in Projective Structures on 3-manifolds

$G = PGL(4,\mathbb{R}) \quad X = \mathbb{R}P^3$

hyperboloid model

$x^2+y^2-z^2=-1$





Klein model of $\mathbb{H}^2$

projectivization of $\vec{v} \in \mathbb{R}^{3,1}$ with $\langle \vec{v}, \vec{v} \rangle < 0$

$x^3+y^2+z^2-w^2.$

$PSO(3,1) < PGL(4,\mathbb{R})$

Hyperbolic geometry sub-geometry to projective geometry

Mostow Rigidity is silent on whether we can projectively deform.

So naturally, Q: Given a hyperbolic 3-manifold, does its projective structure deform in the larger group $PGL(4,\mathbb{R})$?

State of the Art

Cooper-Long-Thistlethwaite: Surveyed a bunch (4,500) census manifolds and found at most 61 do.

Heusener-Porti: Infinitely many Dehn-surgeries on figure-eight knot complement do not.

Daly: 2000 on the figure-eight knot complement do not for example (2,3).

Ballas-Danciger-Lee-Marquis: Under a mild cohomological condition, once-cusped hyperbolic 3-manifolds admit infinitely many projectively rigid Dehn-surgeries

A bunch of arguments rely on a condition called 'infinitesimal projective rigidity rel cusp(s).' (IIP)

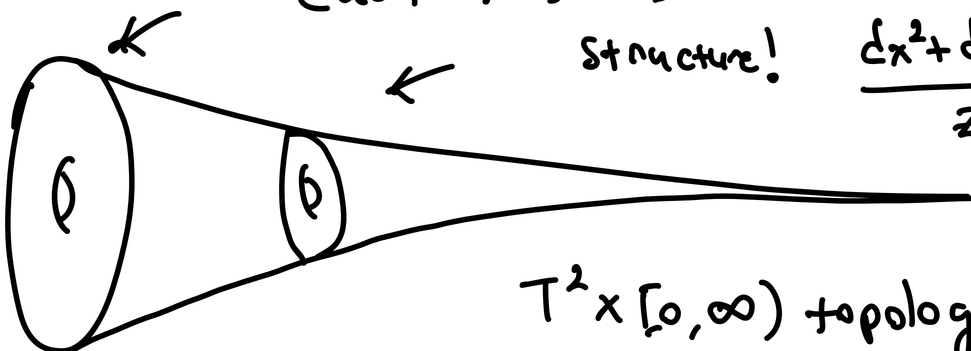
Let M be a once-cusped hyperbolic 3-manifold.

We say M is infinitesimally projectively rigid rel cusp iff

$$H^1(M; \mathfrak{sl}(4,\mathbb{R})) \xrightarrow{i^*} H^1(\partial M; \mathfrak{sl}(4,\mathbb{R}))$$

is injective.

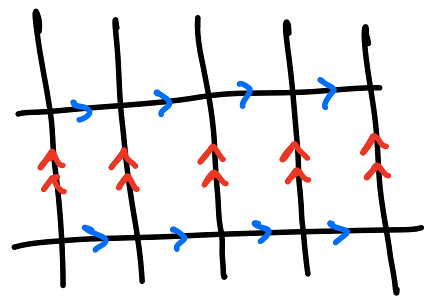
each of us has a euclidean structure! $\dfrac{dx^2+dy^2+dz^2}{z^2}$ with $z = c_i$.



$T^2 \times [0,\infty)$ topologically

Non-trivial deformations of M produce non-trivial deformations of ∂M.

$$p(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad p(\ell) = \begin{pmatrix} 1 & 2\sqrt{3}\, i \\ 0 & 1 \end{pmatrix}$$



Why do we care about inf. proj. rig. rel. cusp?

1. Appears in studying long-exact sequence of $(\partial M, M)$

$$0 \to H^0(\partial M) \to H^1(M,\partial M) \to \underbrace{H^1(M) \xrightarrow{i^*} H^1(\partial M)} \to H^2(M,\partial M) \to \dots$$

2. (DBL) if a hyperbolic 3-manifold is iprrc then $[\rho] \in \mathrm{Hom}(\Gamma; PGL(4,\mathbb{R}))/PGL(4,\mathbb{R})$ is a smooth point.

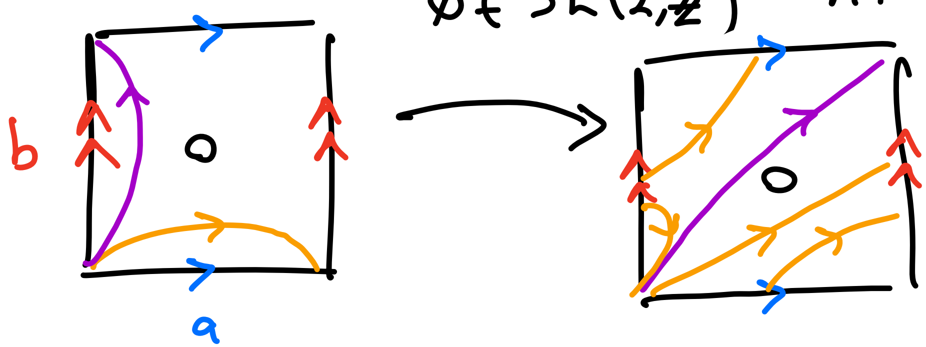
3. (DBL) there exists convex projective structures on the <u>double</u> of such manifolds.

4. (HP) Can inform you on deformation theory of Dehn-surgeries on manifolds.

Goal: Construct an explicit infinite family of once-cusped hyperbolic 3-manifolds.

Set off to look at hyperbolic once-punctured torus bundles. Topologically

$\phi \in SL(2,\mathbb{Z})$ with $|\mathrm{tr}\,\phi| > 2$, $\phi = i^{\varepsilon} R^{n_1} L^{m_1} \cdots R^{n_k} L^{m_k}$



$R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$i = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\varepsilon = 0, 1$

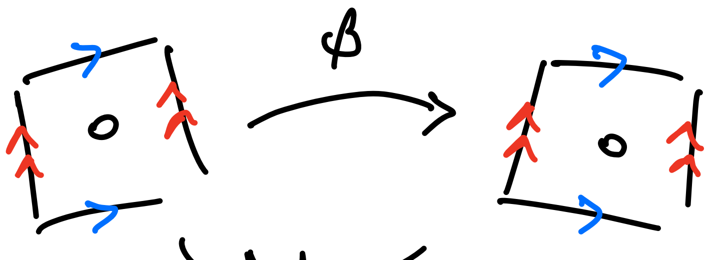
$\phi = RL = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow$ $(RL)(a) = R(ab) = aba$

$(RL)(b) = R(b) = ba$

$\Gamma = \langle a, b, x \mid xax^{-1} = \phi(a), xbx^{-1} = \phi(b) \rangle$

$F \hookrightarrow M \twoheadrightarrow S^1$ topologically

By (HP) inf. proj. rig. rel cusp $\simeq$ dim $H^1(\Gamma; \mathfrak{sl}(4,\mathbb{R})) = 3$.
(Because of Half-Lives Half-Dies theorem)



fibres F

$$\begin{array}{c} M \\ \downarrow \\ S^1 \end{array}$$

$F_2 \hookrightarrow \Gamma \twoheadrightarrow F_1$

So $\Gamma/F_2 \simeq F_1$ acts on
$H^1(F_2; \mathfrak{sl}(4,\mathbb{R}))$

Lyndon-Hochschild-Serre Spectral sequence

$$H^p(\Gamma/F_2; H^q(F_2; \mathfrak{sl}(4,\mathbb{R}))) \Rightarrow H^{p+q}(\Gamma; \mathfrak{sl}(4,\mathbb{R}))$$

Upshot:

$$H^0(\Gamma/F_2; H^1(F_2; \mathfrak{sl}(4,\mathbb{R}))) \cong H^1(\Gamma; \mathfrak{sl}(4,\mathbb{R}))$$

$$H^1(\Gamma/F_2; H^1(F_2; \mathfrak{sl}(4,\mathbb{R}))) \cong H^2(\Gamma; \mathfrak{sl}(4,\mathbb{R}))$$

So in context of hyperbolic once-punctured torus bundles,

inf. proj. rig. rel cusp $\cong$ Calculating

$$H^*(\Gamma/F_2; H^1(F_2; \mathfrak{sl}(4,\mathbb{R})))$$

$\Gamma/F_2 \cong \langle \bar{x} \rangle$ and given a $f: F_2 \longrightarrow \mathfrak{sl}(4,\mathbb{R})$

$$(x f)(g) = (x \cdot f)(x^{-1} g x) \quad \text{for all } g \in F_2$$

$H^0(\Gamma; M) = M^\Gamma$ for any Γ-module M.

Interested in Γ/F_2 - action on $H^1(F_2; \mathfrak{sl}(4,\mathbb{R}))$

$$H^1(F_2; \mathfrak{sl}(4,\mathbb{R})) \simeq \mathfrak{sl}(4,\mathbb{R}) \times \mathfrak{sl}(4,\mathbb{R}) \Big/ \{((1-a)X, (1-b)X), X \in \mathfrak{sl}(4,\mathbb{R})\}$$

$$(xf)(a) = x \cdot f(x^{-1}ax) = x f(w_1)$$

$$(xf)(b) = x \cdot f(x^{-1}bx) = x f(w_2)$$

express $f(w_1)$ and $f(w_2)$ in terms of $f(a)$, $f(b)$

using crossed homomorphism properties.

ex: $x a x^{-1} = aba$

$x b x^{-1} = ba$

$$(f(a), f(b)) \mapsto ((1+ab)f(a) + a f(b),\ b f(a) + f(b))$$

Interested in invariants of Γ/F_2 - action so

look at characteristic polynomial.

One way to do this is

$$SL(4,\mathbb{R}) \hookrightarrow SL(4,\mathbb{R}) \times SL(4,\mathbb{R}) \twoheadrightarrow H^1(F_2; SL(4,\mathbb{R}))$$

$$\overset{\cong}{B^1(F_2; SL(4,\mathbb{R}))} \qquad \overset{\cong}{Z^1(F_2; SL(4,\mathbb{R}))}$$

x-action here is easy, its just $Ad(x)$

this is the difficult part...

1. Holonomy of Γ in $SL(2,\mathbb{C})$

$$p(a) = \begin{pmatrix} \frac{1}{2}(1-\sqrt{3}i) & -i/\sqrt{3} \\ \frac{1}{2}(3-\sqrt{3}i) & 1 \end{pmatrix} \quad p(b) = \begin{pmatrix} 1+\sqrt{3}i & \frac{1}{2}(-1+i/\sqrt{3}) \\ -\frac{1}{2}(3+\sqrt{3}i) & 1/2(1-\sqrt{3}i) \end{pmatrix}$$

$$p(x) = \begin{pmatrix} -1 & -i/\sqrt{3} \\ 0 & -1 \end{pmatrix}$$

Calculate explicitly via 'trace relations'

$$tr(g) = tr(g^{-1}) \qquad tr(gh) = tr(g)tr(h) - tr(gh^{-1})$$

$$tr(\partial M) = 2 \quad (\text{parabolic boundary})$$

2. Holonomy in $SO(3,1) \cong SL(2,\mathbb{C})$

$$\rho(a) = \begin{pmatrix} 0 & \sqrt{3} & -1/2\sqrt{3} & 5/2\sqrt{3} \\ 0 & 1 & -3/2 & 3/2 \\ -2/\sqrt{3} & 1 & -2/3 & 4/3 \\ -1/\sqrt{3} & 2 & -4/3 & 8/3 \end{pmatrix}$$

$$\rho(b) = \begin{pmatrix} -3/2 & -\sqrt{3}/2 & -5/2\sqrt{3} & 7/2\sqrt{3} \\ 3\sqrt{3}/2 & -1/2 & 5/2 & -7/2 \\ -1/\sqrt{3} & 0 & 5/6 & -1/6 \\ -5/\sqrt{3} & 0 & -17/6 & 23/6 \end{pmatrix}$$

$$\rho(x) = \begin{pmatrix} 1 & 0 & -1/\sqrt{3} & -1/\sqrt{3} \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{3} & 0 & 5/6 & -1/6 \\ -1/\sqrt{3} & 0 & 1/6 & 7/6 \end{pmatrix}$$

See how this acts on $\mathfrak{sl}(4,\mathbb{R})$ via adjoint representation. $\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{so}(3,1) \oplus V$

$\uparrow$ $\mathfrak{so}(3,1)^{\perp}$

killing form

In case of RL,

$\lambda_{\Gamma/F_2}(t) =$

$$(-1+t)^5(1-5t+t)^2(1-15t+27t^2-42t^3+27t^4-15t^5+t^6)$$

symmetric polynomial.

thm (Daly): let M be a hyperbolic once-punctured torus bundle. If the characteristic polynomial of the Γ/F_2-action on $H^1(F_2; sl(4,\mathbb{R}))$ is equal to the twisted Alexander polynomial of the representation $Ad\circ\rho : \Gamma \longrightarrow Aut(sl(4,\mathbb{R}))$ and $\alpha : \Gamma \longrightarrow \Gamma/F_2 \cong \mathbb{Z}$.

thm (Daly): let M be a hyperbolic once-punctured torus bundle. If the twisted Alexander polynomial of $Ad\,\rho : \Gamma \to Aut(\mathfrak{sl}(4,\mathbb{R}))$ and $\alpha : \Gamma \to \Gamma/F_2 \cong \mathbb{Z}$ has 1 with multiplicity 5, then M is inf. proj. rig. rel. cusp.

Proof: Look at long-exact seque of Γ/F_2-modules associated to $(\partial F, F)$ where F is once-punctured torus fiber.

$$\begin{array}{ccccc} K & \hookrightarrow & H^1(F) & \xrightarrow{res} & H^1(\partial F) \leftarrow \text{dim } 5 \\ \big|\,\Gamma/F_2 & & \big|\,\Gamma/F_2 & & \big|\,\Gamma/F_2 \leftarrow \text{acts unipotently} \\ K & \hookrightarrow & H^1(F) & \underset{res_e}{\longrightarrow} & H^1(\partial F) \quad \text{with } \dim H^1(\partial F)^{\Gamma/F_2} = 3 \end{array}$$

A bunch of this can be rephrased to v in $Sl(4,\mathbb{R}) = SO(3,1) \oplus V$ because $SO(3,1)$ part well understood due to Thurston.

Action of Γ/F_2 - acting on $H^1(F_2; Sl(4,\mathbb{R}))$.

Purely group-theoretic in terms of LHS spectral sequence, however, has a nice interpretation in this context.

Let G be your favorite Lie group with $SL(2,\mathbb{C}) \hookrightarrow G$.

$\phi \in \mathrm{Out}(F_2) \cong GL(2,\mathbb{Z})$ acts on $\mathrm{Hom}(F_2, G) \cong G \times G$.

takes $\rho: F_2 \longrightarrow G$ to $(\phi\rho): F_2 \longrightarrow G$

via $(\phi\rho)(a) = \rho(\phi^{-1}(a))$.

Dynamics of this studied by Goldman, Gelander - Minsky, Forni - Goldman - Lawton - Silva Santos.

Our case interested in just $\phi \in \mathrm{Out}(F_2)$ acting on $\mathrm{Hom}(F_2, G) \simeq G \times G$. Each rep $\rho \simeq$ choice of $\rho(a)$, $\rho(b)$. For M once-punctured torus bundle with monodromy ϕ, get ρ by restricting to $F_2 < \Gamma \xhookrightarrow{\rho} SL(2, \mathbb{C})$.

If we post compose with conjugation by x, then ρ is a fixed point of this $\mathbb{Z}$-action in $\mathrm{Hom}(F_2, G)$.

The derivative of this map is Γ/F_2-action on $Z^1(F_2; \mathfrak{g})$.

thm (Daly): The Γ/F_2-action defined by LHS spectral sequence on $H^1(F_2; \mathfrak{g})$ is the differential of the ϕ-action on the character variety of $\mathrm{Hom}(F_2, G)/G$ at the fixed point $\rho : F_2 \longrightarrow G$ arising from the holonomy of the once-punctured torus fibre.

Thank You Folks!