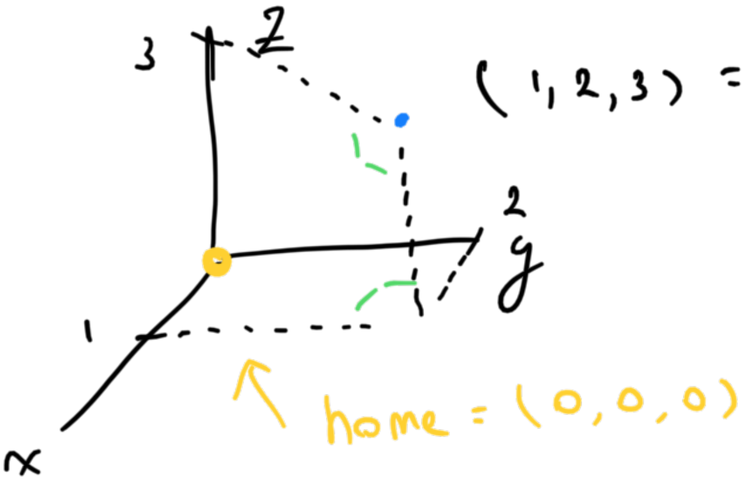
12.1 : 3D coordinates

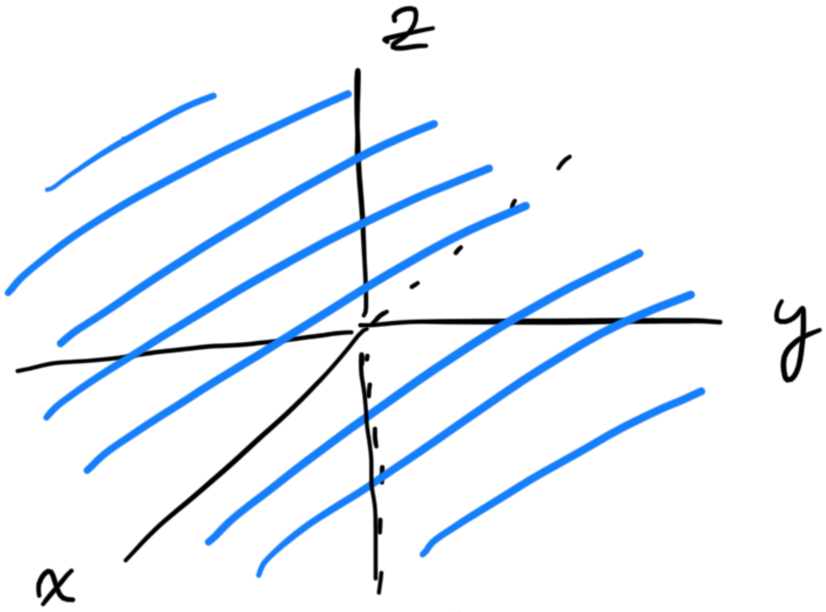


$(1,2,3) = P$

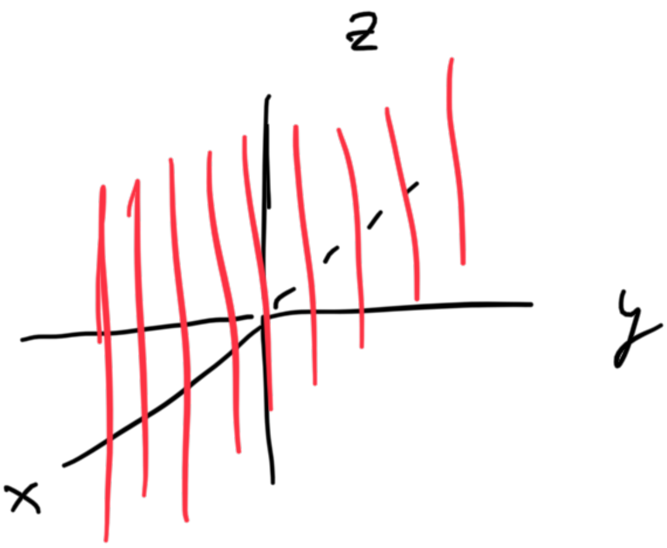
describes all points in 3-space uniquely.

Coordinate planes:

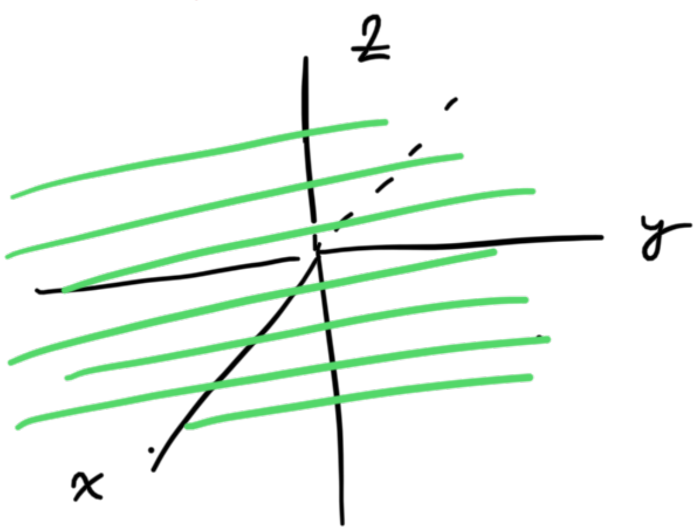
$x = 0$



$y = 0$

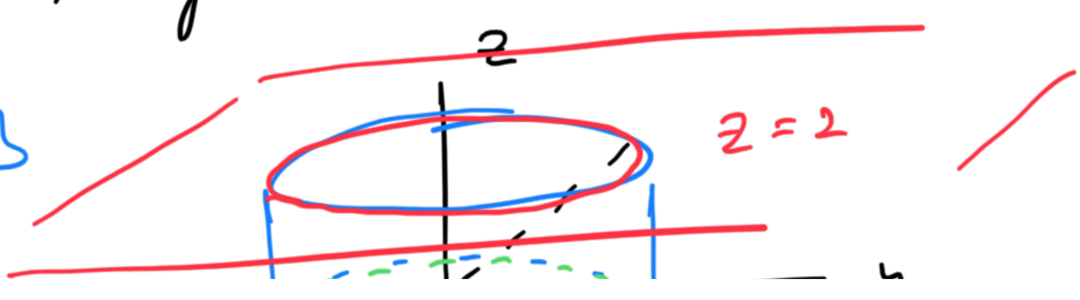


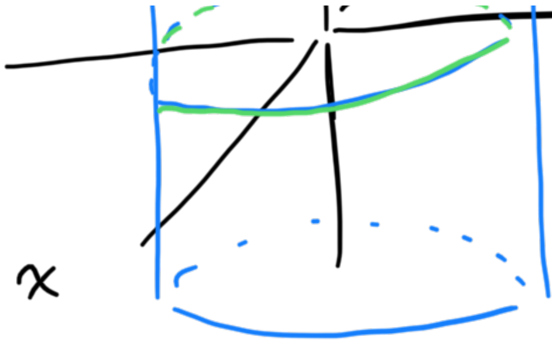
$z = 0$



What about $x^2 + y^2 = 9$ and $z = 2$

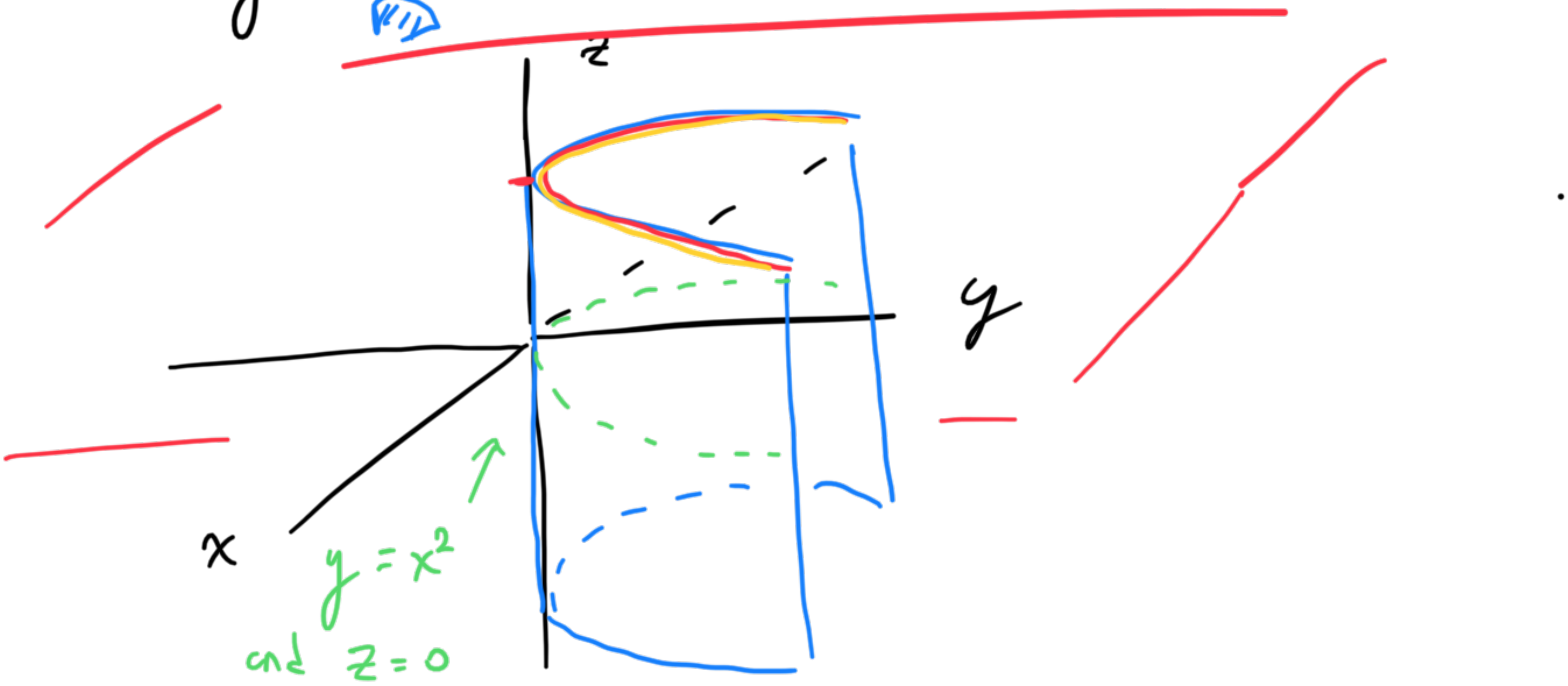
$x^2 + y^2 = 9$



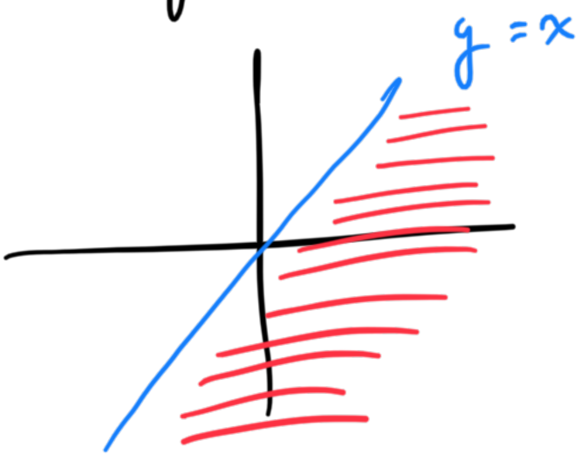


$x^2+y^2=0$ and $z=0$

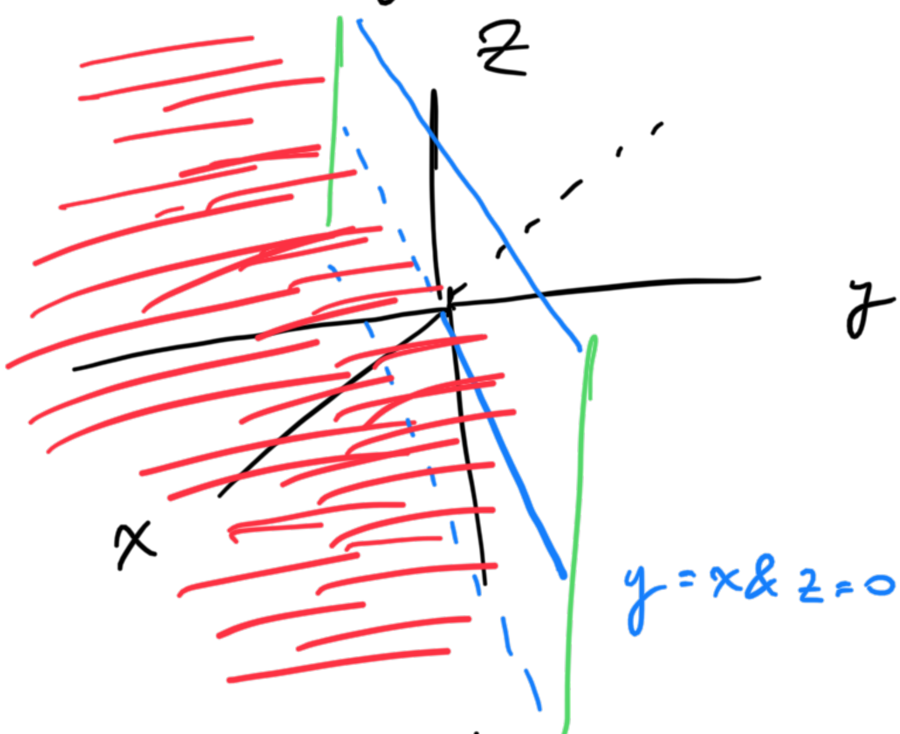
ex: $y = x^2$ and $z = 2$



ex: $y \leq x$ in 2D



$y \leq x$ in 3D ?



all that stuff

Distance in 3 space?

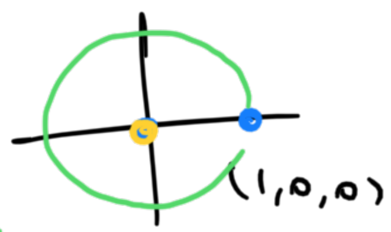
$Q = (x_2, y_2, z_2)$

$\Delta w = w_2 - w_1$

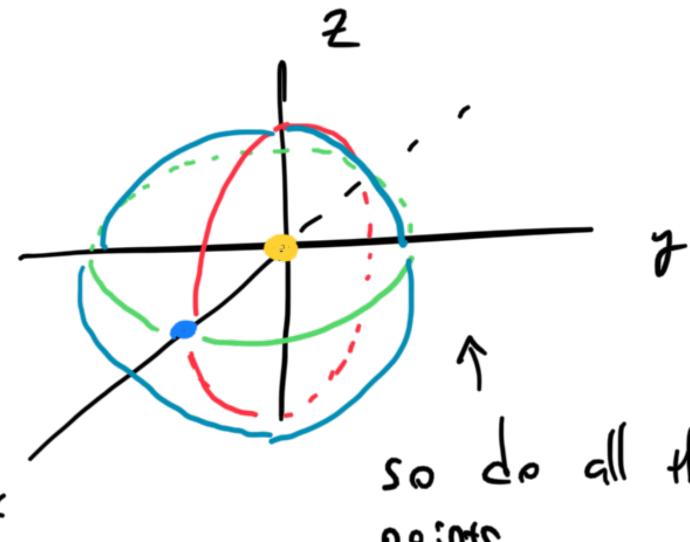
$d = \sqrt{\Delta x^2 + \Delta y + \Delta z}$

$P = (x_1, y_1, z_1)$

ex: $d((0,0,0),(1,0,0)) = \sqrt{\Delta^2+\Delta^2+\Delta^2} = \sqrt{1} = 1$



all same distance from (0,0,0)



so do all these points

Observation: The set of all points a fixed distance from a fixed point = sphere!

algebraically: $\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$

$\Leftrightarrow (x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2$

point (x_0, y_0, z_0) is center and r is the radius.

ex: Find the radius and center of

$x^2+y^2+z^2-4z=0$

$\frac{-4}{2} \xrightarrow{\square/2} -2 \xrightarrow{\square^2} 4$

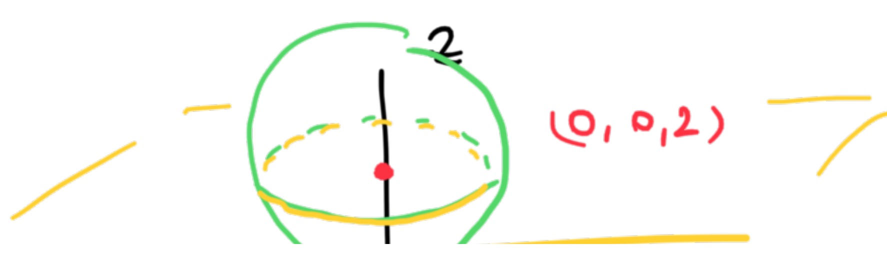
$x^2+y^2+z^2-4z=0$

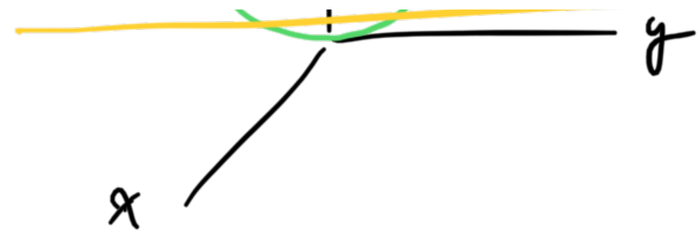
$x^2+y^2+z^2-4z+\underline{4}=\underline{4}$

$x^2+y^2+(z-2)^2=4$

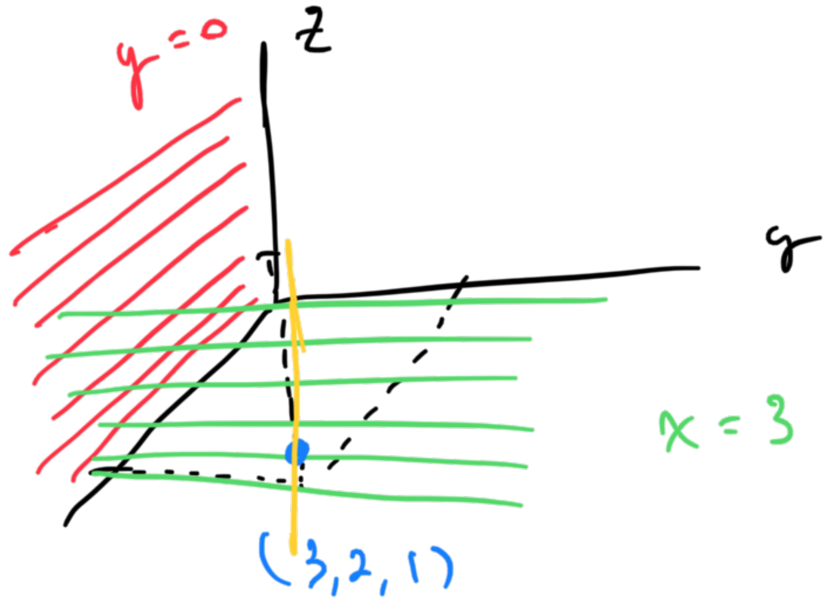
center = $(0,0,2)$

radius = 2





ex: Find plane through $(3, 2, 1)$ perpendicular to xz-plane.



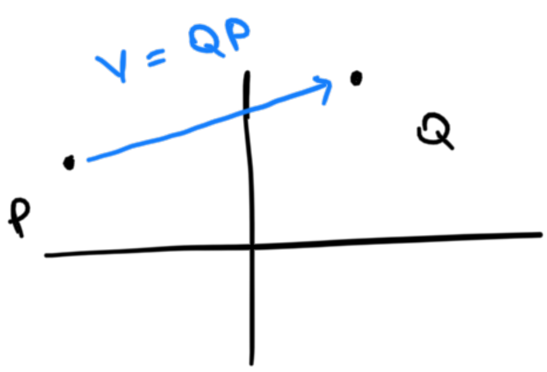
Line through $(3,2,1)$ parallel to z-axis?

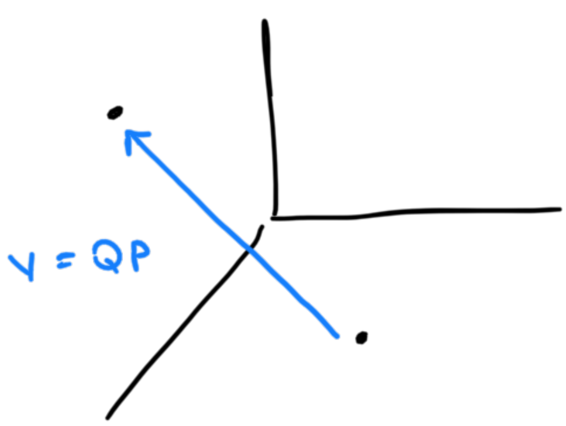
$x=3$ & $y=2$

12.2 Vectors

Vectors algebraically $v = (x, y) \in \mathbb{R}^2$ or $v = (x, y, z) \in \mathbb{R}^3$

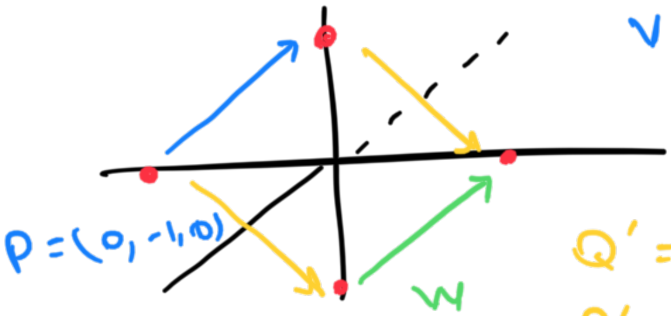
Represent direction and magnitude





Book uses < , > but I'm going to avoid that, just stylistic choice.

ex: $Q = (0,0,1)$



$v = QP = (0,0,1) - (0,-1,0)$
$= (0,1,1)$

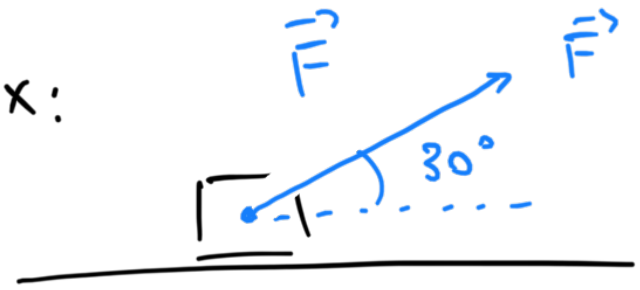
$Q' = Q$ moved over $(0,1,-1)$
$P' = P$ moved over $(0,1,-1)$

$w = Q'P' = (0,1,0) - (0,0,-1)$
$= (0,1,1)$

Every vector $v \in \mathbb{R}^3$ breaks up into components

$$v = (v_x, v_y, v_z)$$

ex:



$\vec{F}$ is 20 lb force then

$\vec{F} = (20\cos 30°, 20\sin 30°)$
$= (20 \cdot \frac{\sqrt{3}}{2}, 20 \cdot \frac{1}{2})$
$= (10\sqrt{3}, 10)$ lb

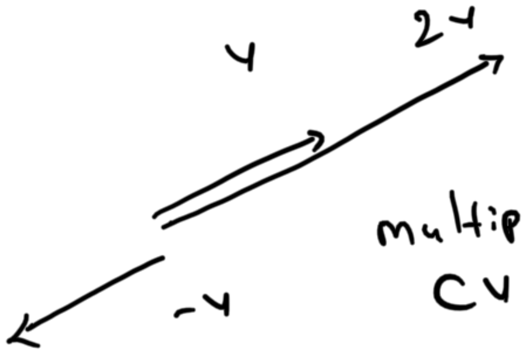
$F_x = 10\sqrt{3}$ $F_y = 10$ (lbs)

Each vector has a direction and magnitude (except 0)

in previous example, $|F| = 20$, or $|v| = \sqrt{2}$

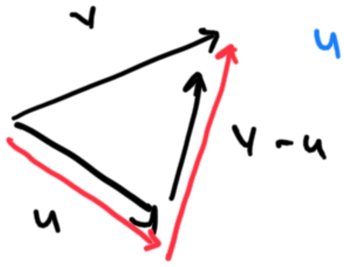
$$|v| = \sqrt{v_x^2 + v_y^2 + v_z^2} \leftarrow \text{magnitude}$$

Vector algebra:



multiply vectors
cv

get all nice things



$u + (v-u) = v$

v wrt, pg. 718

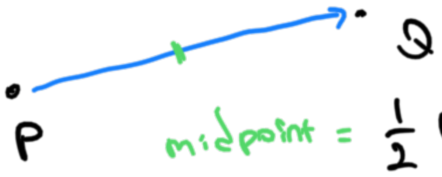
$u + v = v + u \qquad (u+v)+w = u+(v+w)$

$u + 0 = u \qquad u + (-u) = 0$

$0u = 0 \qquad 1u = u$

$a(bu) = (ab)u \qquad a(u+v) = au + v$

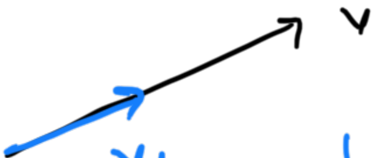
$(a+b)u = au + bu$



coordinate-wise

midpoint $= \frac{1}{2}(Q+P)$

direction:



$v/|v|$ = direction of v

makes sense for all $v \neq 0$.

Mention some notation: $\vec{F} = (F_x, F_y, F_z)$

$= F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$

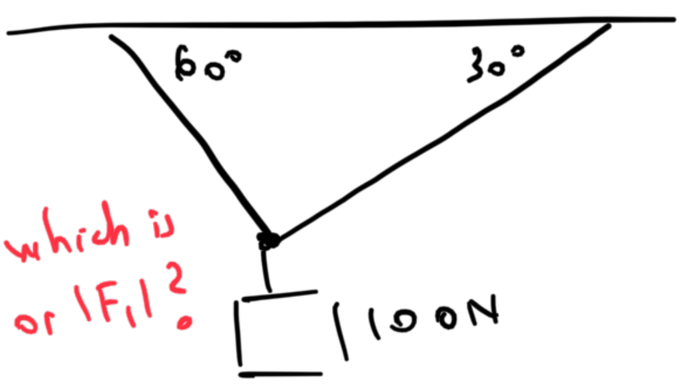
$\hat{i} = (1,0,0)$

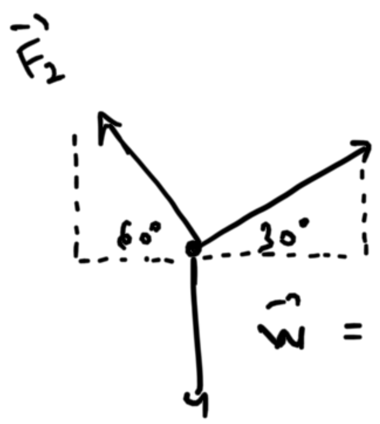
$\hat{j} = (0,1,0)$

$\hat{k} = (0,0,1)$

ex:

9 pg. 729





$\vec{w} = (0, -100)$

Intuitively which is bigger $|F_2|$ or $|F_1|$?

Find magnitudes

(*) $\vec{F_1} + \vec{F_2} + \vec{w} = \vec{0}$

$(F_{1x}, F_{1y}) + (F_{2x}, F_{2y}) = (0, +100)$

$F_{1x} + F_{2x} = 0 \qquad F_{1x} = |F_1| \cos 30°$

$F_{1y} + F_{2y} = +100 \qquad F_{1y} = |F_1| \sin 30°$

$F_{2x} = -|F_2| \cos 60°$

$F_{2y} = |F_2| \sin 60^\circ$

$|F_1|\left(\frac{\sqrt{3}}{2}\right) - |F_2|\left(\frac{1}{2}\right) = 0$, $|F_1| = \frac{1}{\sqrt{3}}|F_2|$

$|F_1|\left(\frac{1}{2}\right) + |F_2|\left(\frac{\sqrt{3}}{2}\right) = 100$

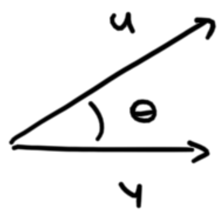
$$\left(\frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2}\right)|F_2| = 100$$

$$|F_2| = 100 / \text{mess}$$

$$|F_1| = \frac{1}{\sqrt{3}}|F_2|$$

12.3: Dot Product

Given two vectors $u, v \in \mathbb{R}^3$, how to find $\angle(u,v)$?



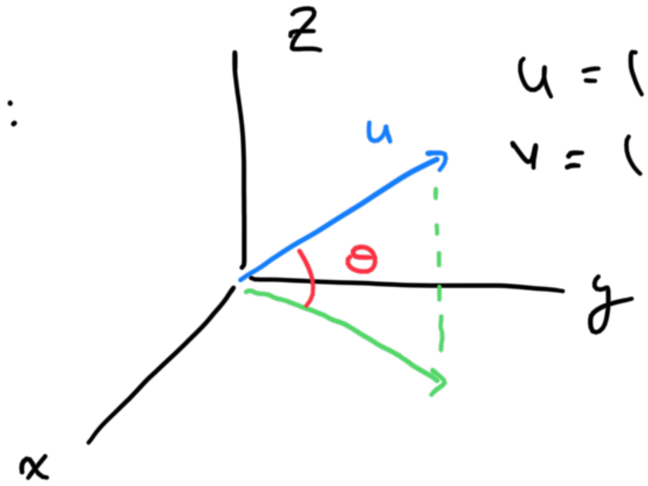
$\theta \in [0, \pi]$

theorem: $\theta = \arccos\left(\dfrac{\overbrace{u_1v_1 + u_2v_2 + u_3v_3}^{u \cdot v}}{|u||v|}\right)$

$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 \in \mathbb{R}$ not another vector.

Another way of saying this is $\cos\theta = \dfrac{u \cdot v}{|u||v|}$

ex:



$u = (1,1,1)$

$v = (1,1,0)$

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{1+1+0}{\sqrt{3}\sqrt{2}}$$

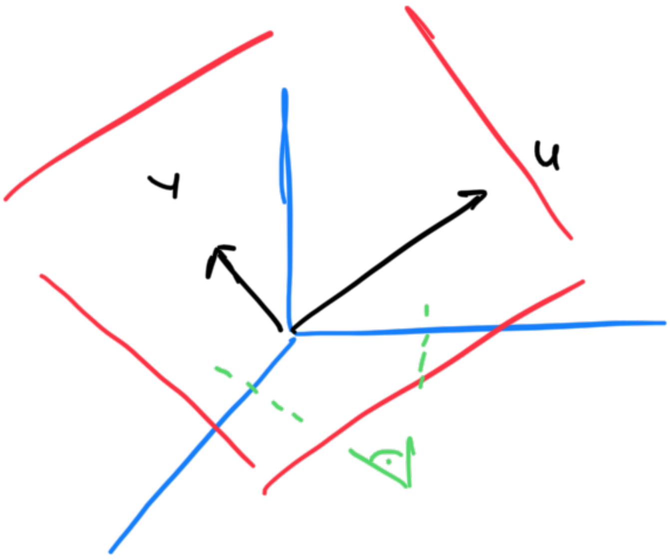
$$= \frac{\sqrt{2}}{\sqrt{3}}$$

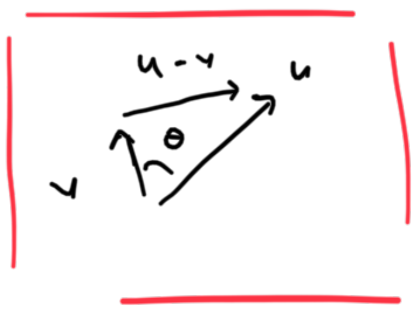
$\theta = \arccos(\sqrt{2}/\sqrt{3})$ $> 45^\circ$ or $< 45^\circ$ or $= 45^\circ$?

$\approx 35^\circ$

$\theta = \arctan\left(\frac{|w|}{|v|}\right)$ more v than w.

Why does $\cos\theta = \frac{u \cdot v}{|u||v|}$? Use some trig on this picture.





law of cosines
$|u-v|^2 = |u|^2 + |v|^2 - 2|u||v|\cos\theta$

dot product gives us a nice way to see if two vectors are perpendicular (orthogonal).

$$\cos\theta = \frac{u \cdot v}{|u||v|} = 0 \text{ iff } \underline{u \cdot v = 0}.$$

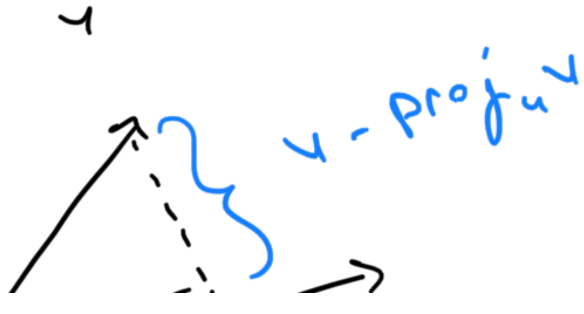
if want to see if $(1,1)$ and $(1,-1)$ $\perp$ just check with dot product!

Nice algebra : $u \cdot v = v \cdot u \qquad (cu) \cdot v = c(u \cdot v)$

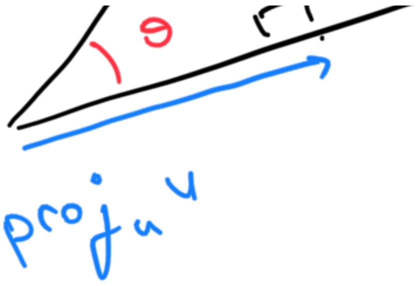
$u \cdot (v + w) = u \cdot v + u \cdot w$

$u \cdot u = |u|^2 \qquad \vec{0} \cdot u = 0$

ex:

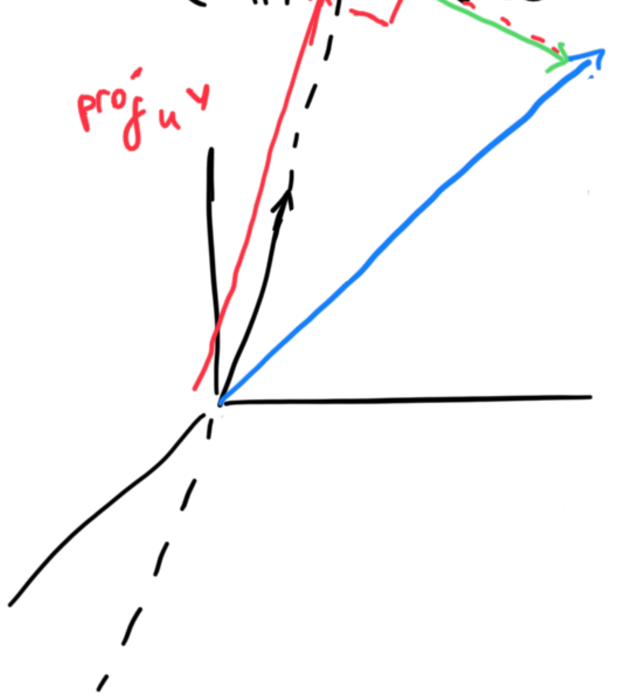


$$\text{proj}_u v = \underbrace{(|v|\cos\theta)}_{\in \mathbb{R}} \underbrace{(\text{direction } u)}_{\text{vector}}$$



$$u \quad = |v| \left(\frac{u\cdot v}{|u||v|}\right)\left(\frac{u}{|u|}\right)$$

$$= \underbrace{\left(\frac{u\cdot v}{|u|}\right)}_{\in \mathbb{R}} \underbrace{\left(\frac{u}{|u|}\right)}_{\text{vector}} = \underbrace{\left(\frac{u\cdot v}{|u|^2}\right)}_{\in \mathbb{R}} u$$



ex: $u = (1,1,2)$

$v = (3,3,3)$

$$\text{proj}_u v = \left(\frac{u\cdot v}{|u|}\right)\frac{u}{|u|} = \frac{(3+3+6)}{\sqrt{6}}\left(\frac{(1,1,2)}{\sqrt{6}}\right)$$

$$= \frac{12}{6} = 2(1,1,2) = (2,2,4)$$

Want orthogonal part, then $v - \text{proj}_u v = (3,3,3) - (2,2,4)$

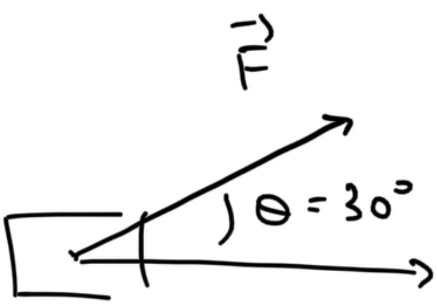
$= (1,1,-1)$

Sanity check $(1,1,-1) \perp u$?

$1+1-2 = 0$ ✓

Work: $W = \vec{F}\cdot\vec{D}$ where $\vec{F}$ force over displacement $\vec{D}$.

ex:



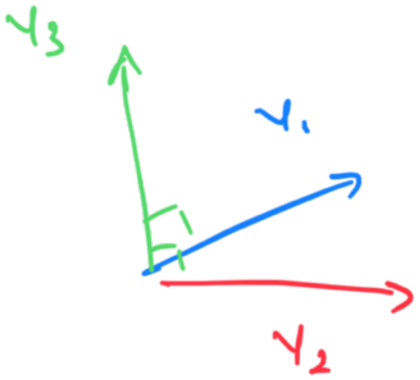
direction displacement

Say $|D| = \frac{4}{\sqrt{2}}$ m and $|\vec{F}| = 100$ N then $\vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos \theta = 100 \cdot \frac{4}{\sqrt{3}} \cdot \left(\frac{\sqrt{3}}{2}\right)$

$= 200 \text{ J}$ Joules $= N \cdot m$

12.4: Cross-Products

Given two vectors $v_1, v_2 \in \mathbb{R}^3$, they span a plane (typically). Then up to scale, there's a natural choice of a complementary vector. Just take one $\perp$ to the plane. Here's a cartoon.



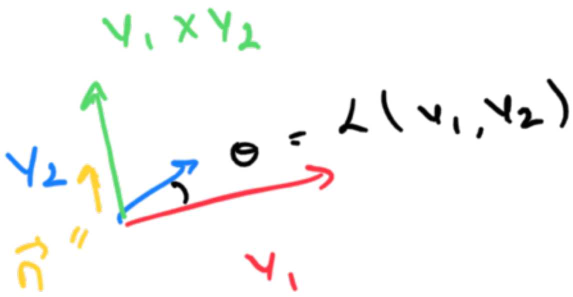
However, there's some ambiguity. There's a bunch of choices; let's agree on an orientation. Let's say that 3-ordered vectors are standardly oriented or right-hand rule oriented if you can rotate, scale, shear, etc to agree with $(\hat{\imath}, \hat{\jmath}, \hat{k})$. So the set (v_1, v_2, v_3) depicted above is not standardly oriented.
(Show them the hand thing)

We define the cross product of v_1, v_2 to be

$$v_1 \times v_2 = (|v_1||v_2| \sin\theta)\,\vec{n}$$ where $\vec{n}$ is

a unit vector so that (v_1, v_2, n) is standard orientation, and $\theta = \angle(v_1, v_2)$.

Here's picture:



Here are some properties: (pg. 435)

(i). $(ru) \times (sv) = (rs)\, u \times v$

(ii). $u \times (v + w) = u \times v + u \times w$

(iii). $v \times u = -(u \times v)$

(iv). $(v + w) \times u = v \times u + w \times u$

(v). $\vec{0} \times u = \vec{0}$

(vi). $u \times (v \times w) = (u \cdot w) v - (u \cdot v) w$

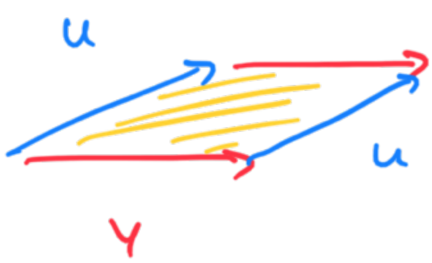
(vii). $u \times v = 0$ for $u, v \neq 0$ iff u, v are parallel.

Most fundamental ones to remember are

$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{i} = \hat{j}$$

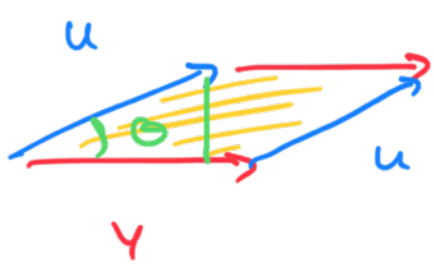
Easy to remember by picture

example: let $u, v \in \mathbb{R}^3$. Form parallelogram as pictured below. Then $|u \times v| = \text{Area}(u, v)$.



Area $= |u \times v|$

Just because

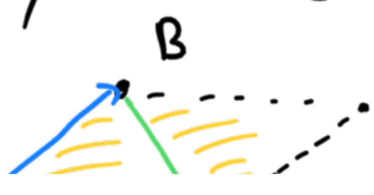


Area $= |\text{base}||\text{height}|$

$= |v| (|u| \sin \theta)$

$= |u \times v|$ bc $|n| = 1$

Similarly if have 3-generic points in $\mathbb{R}^3$, then



form parallelogram take the area

and 1/2 it to get area of triangle of ΔABC.

That is, $\text{Area } \Delta ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

How to actually calculate it? Before starting, let us recall determinant of a 2x2-matrix.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(actually the signed area of parallelogram formed by $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ in $\mathbb{R}^2$)

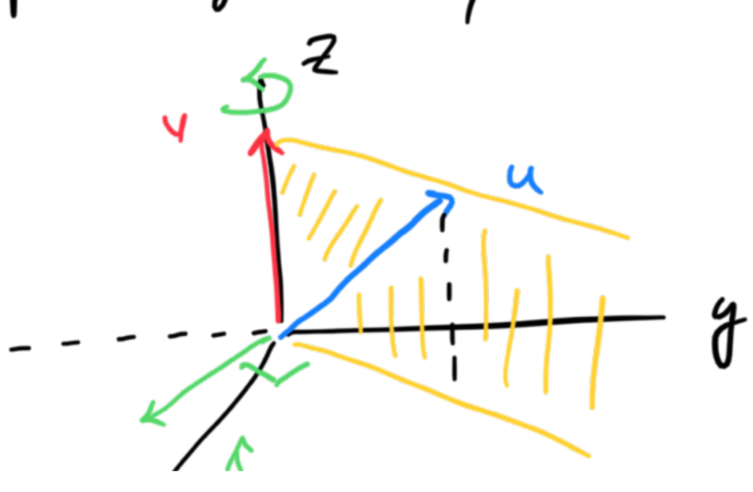
okay, so $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$

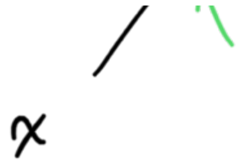
then
$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

(show then how to see this as minors of matrix)

ex: Find a unit vector perpendicular to the plane generated by $(1,1,1)$ and $(0,0,2)$





this intuitively will do, say $w = (1, -1, 0)$
(kinda)

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$$

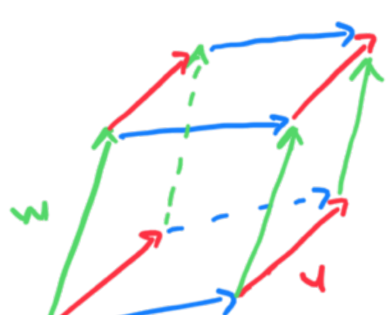
$$= (2, -2, 0)$$

then divide out by norm, $w = (2,-2,0)/\sqrt{8}$

However, note, $w \parallel$ to our original choice $(1,-1,0)$.

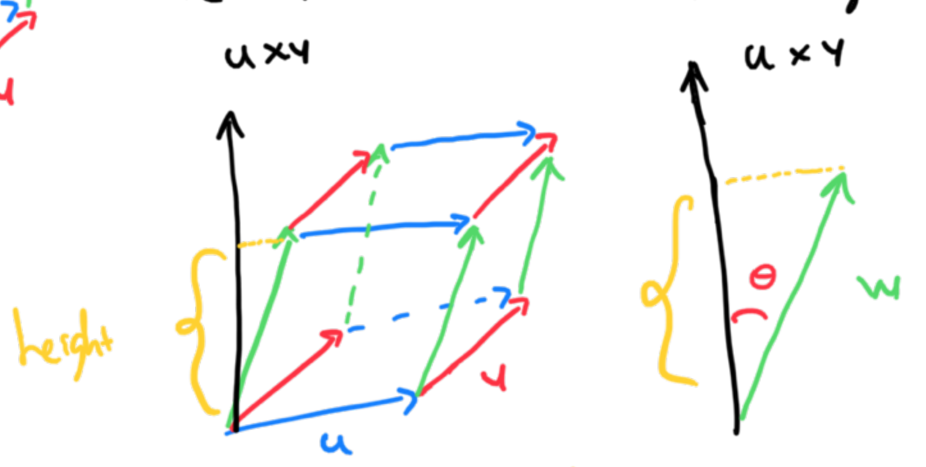
Also note right hand rule.

Note 1 vector determines a line segment
2 vectors determine a parallelogram (typically)
3 vectors determine a ... parallelpiped



What the volume of this.

Recall Volume = area · height



$h = |w||\cos\theta|$

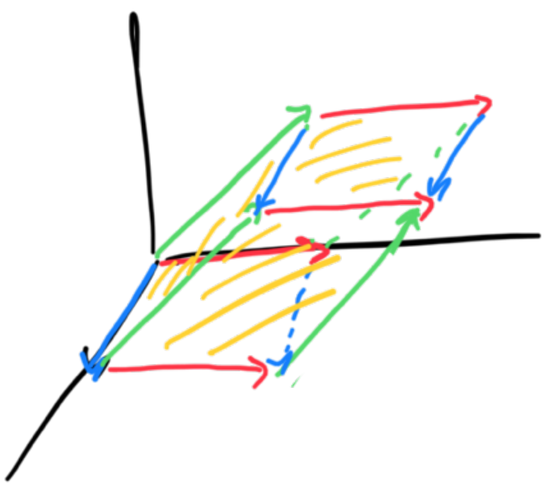
thus Volume $= (\text{Area of parallelogram } (u,v))(\text{height})$

$$= |u \times v||w||\cos\theta| = |(u \times v) \cdot w|$$

if you work it out...

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

ex; $u = (1,0,0)$ $v = (0,1,0)$ $w = (0,1,1)$



Volume $= 1$ (why?)

verify:

$$V = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0 + 0$$

$$= 1$$

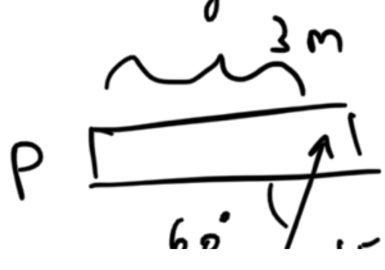
Torque: __Vector__ quantity that expresses turning (kinda)

Just think of opening door.

if push is it easier to open at point A or B?

$\vec{\tau} = \vec{r} \times \vec{F}$.

Calculate magnitude of torque generated by $\vec{F}$ at pivot in figure (example 5 pg. 738)



so $|\vec{\tau}| = |\vec{r}||\vec{F}||\sin\theta|$

$= 3|20||\sin 60°|$

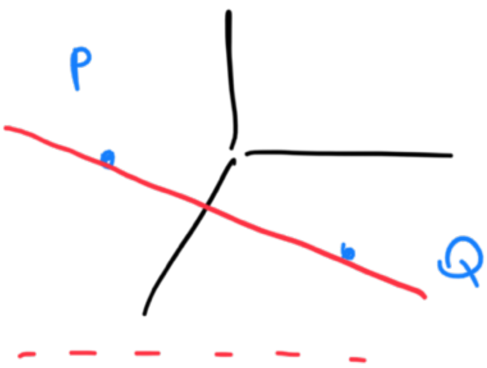
$/ \ |F| = 20\,N$ $= 60 \cdot \frac{\sqrt{3}}{2}\ Nm$

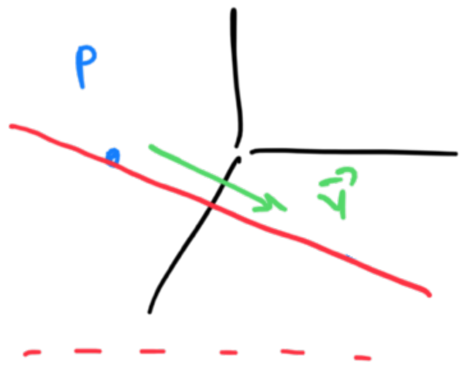
$= 30\sqrt{3}\ Nm.$

12.5: Lines and Planes

A line is uniquely determined by either

- 2 distinct points
- 1 point and a non-zero vector.





Parametric equation for a line. let v and P be as above. The line determined by P and v is algebraically given by $\vec{r}(t) = P + tv$ for $t \in \mathbb{R}$.

the idea being t is time along the line. $t = 0$ is starting point, namely P.

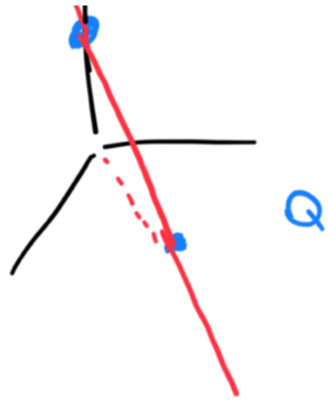
Another way of writing this is

$$\begin{aligned} x &= x_0 + t v_1 \\ y &= y_0 + t v_2 \\ z &= z_0 + t v_3 \end{aligned} \quad \sim \quad \vec{r}(t) = P + tv$$

with $P = (x_0, y_0, z_0)$
$v = (v_1, v_2, v_3)$.

ex: Parametrize line joining $P = (0, 0, 1)$
$Q = (1, 1, 0)$

in such a way that ...



in such a way that at $t = 0$,
located at the mid point of P and Q.

$$\text{midpoint}(P, Q) = \tfrac{1}{2} P + \tfrac{1}{2} Q = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right).$$

Vector defined by P, Q $= \vec{PQ} = (1,1,0) - (0,0,1) = (1,1,-1)$

equation: $\vec{r}(t) = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) + t(1,1,-1)$.

unit speed meaning $r(t) = P + t\vec{v}$ with $|\vec{v}| = 1$.

$$\vec{r}(t) = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) + t\left(\frac{(1,1,-1)}{\sqrt{3}}\right)$$

unit vector

Planes: Plane are determined by

(typically)

- 1 point, 2 vectors
- 3 points
- 2 points, 1 vector
- 1 point, 1 vector ← (less obvious)
- etc

Algebraically they are described by the set of all $(x,y,z) \in \mathbb{R}^3$

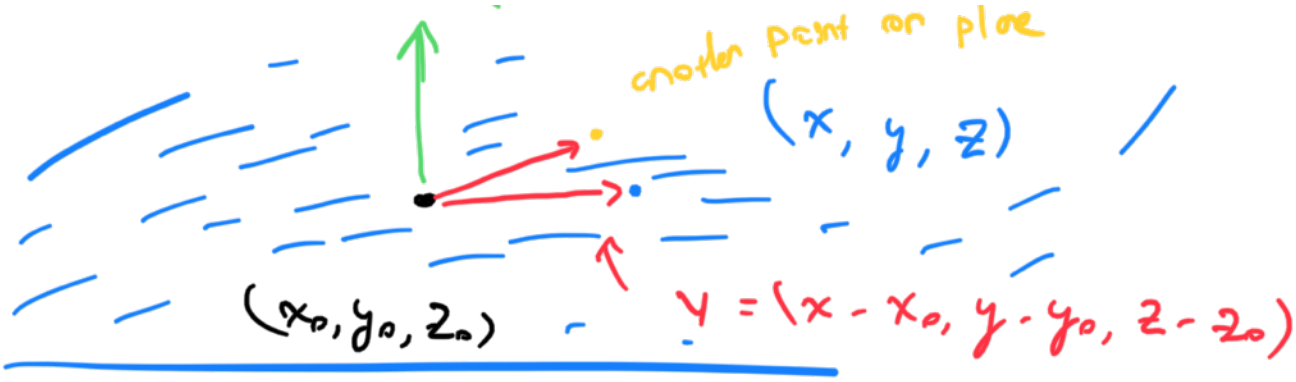
$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

for fixed $A, B, C \in \mathbb{R}$ and a point $P = (x_0, y_0, z_0)$.

Note equation is $\simeq$ to

$$(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

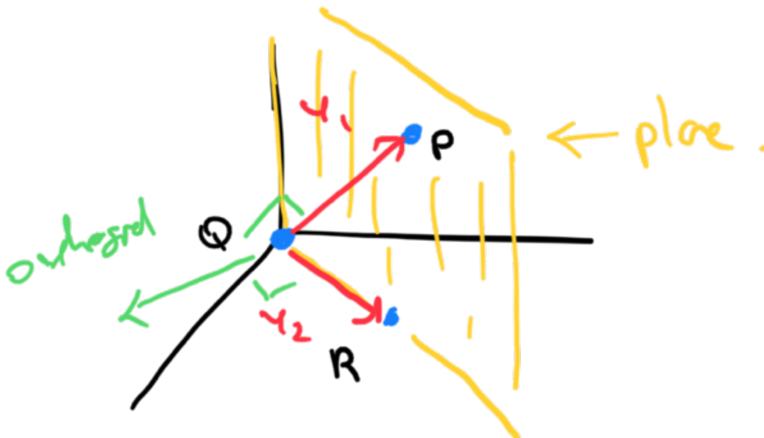
(A, B, C)



note vector $(A, B, C) \perp$ plane

ex: Find equation of plane containing points

$P = (1,1,1) \quad Q = (0,0,0) \quad R = (1,1,0)$



$v_1 = (1,1,1)$

$v_2 = (1,1,0)$

$w = v_2 \times v_1 = (1,-1,0)$

$= (A, B, C)$

equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

here chose point R, does it matter though?

$(1)(x-1) + (-1)(y-1) + 0(z-0) = 0$

$(x-1) - y + 1 = 0, \quad \underline{x - y = 0}$

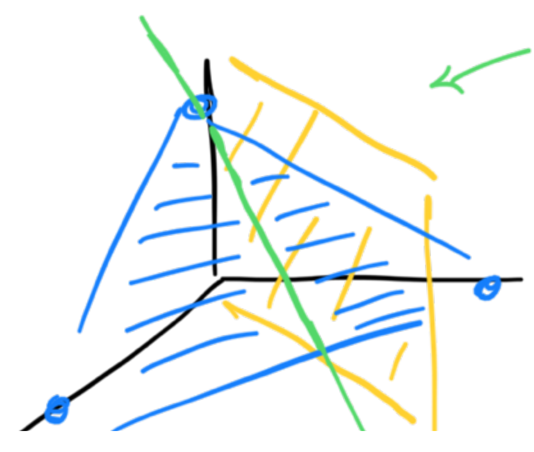
ex: let plane 1 be defined by $P_1: x - y = 0$

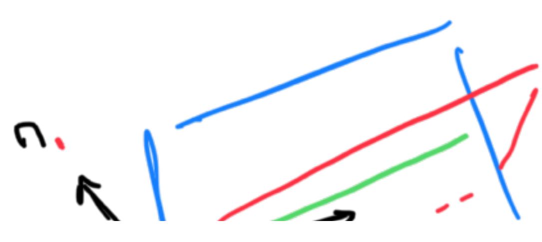
plane 2 be defined by $P_2: x + y + z = 1$

determine the equation of the line intersecting these two planes.

we already know it!

However, there's another way.

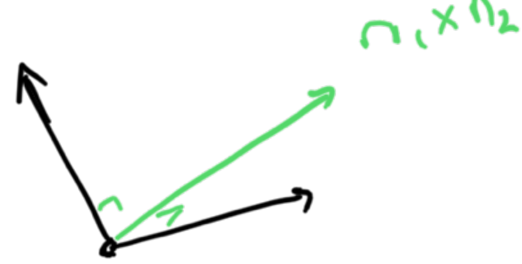




$n_1 = (1,-1,0)$

$n_2 = (1,1,1)$

$$n_1 \times n_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$



$$v = \hat{i}(-1) - \hat{j}(1) + \hat{k}(2) = (-1,-1,2) \sim (1,1,-2)$$

Quick sanity: $n_1 \cdot v = 0$ and $n_2 \cdot v = 0$

all need is a point on both P_1 & P_2.

$$x - y = 0$$
$$x + y + z = 1$$

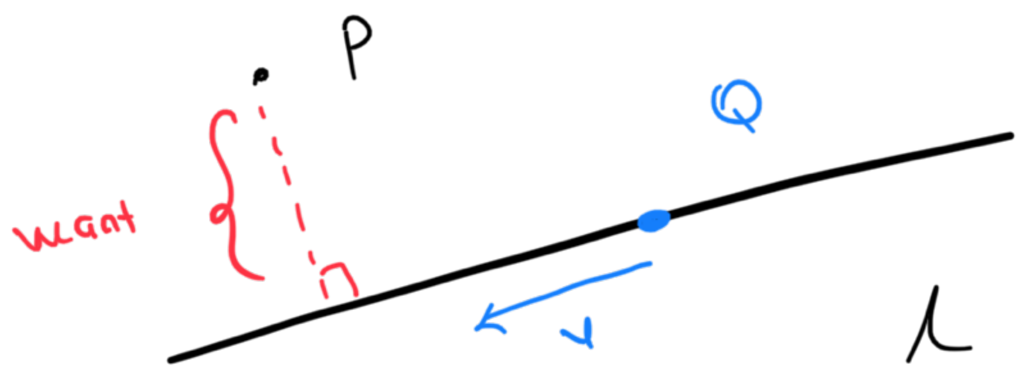
$P = (0,0,1)$ is good.

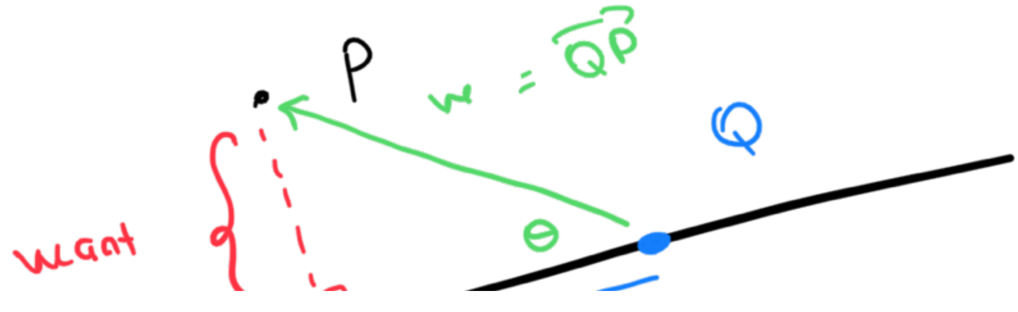
$\ell : (0,0,1) + t(1,1,-2)$. Good way to check?

ensure $\ell \subseteq P_1$ and $\ell \subseteq P_2$!

Distances:

ex: Point and a line



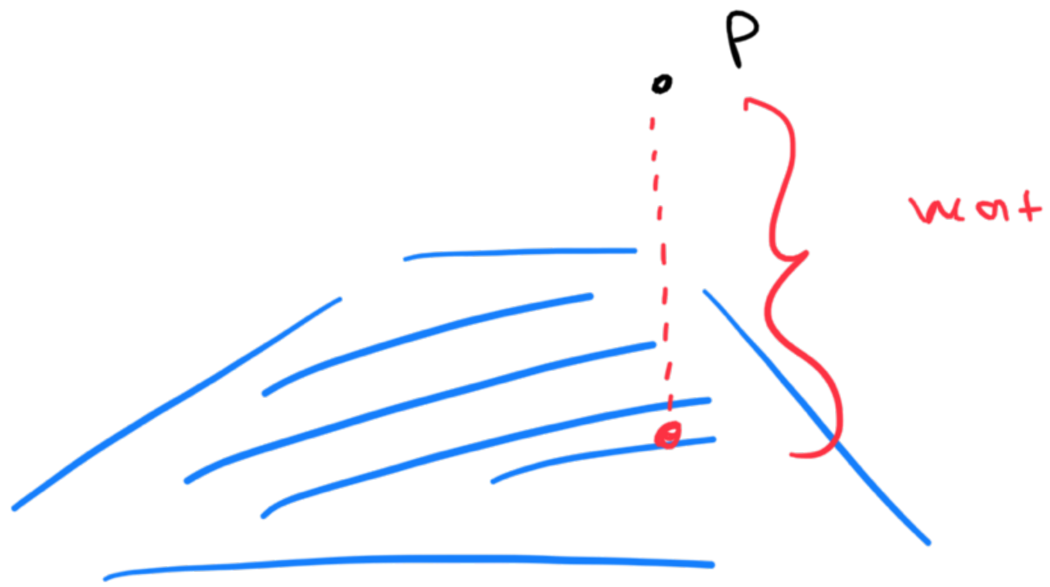


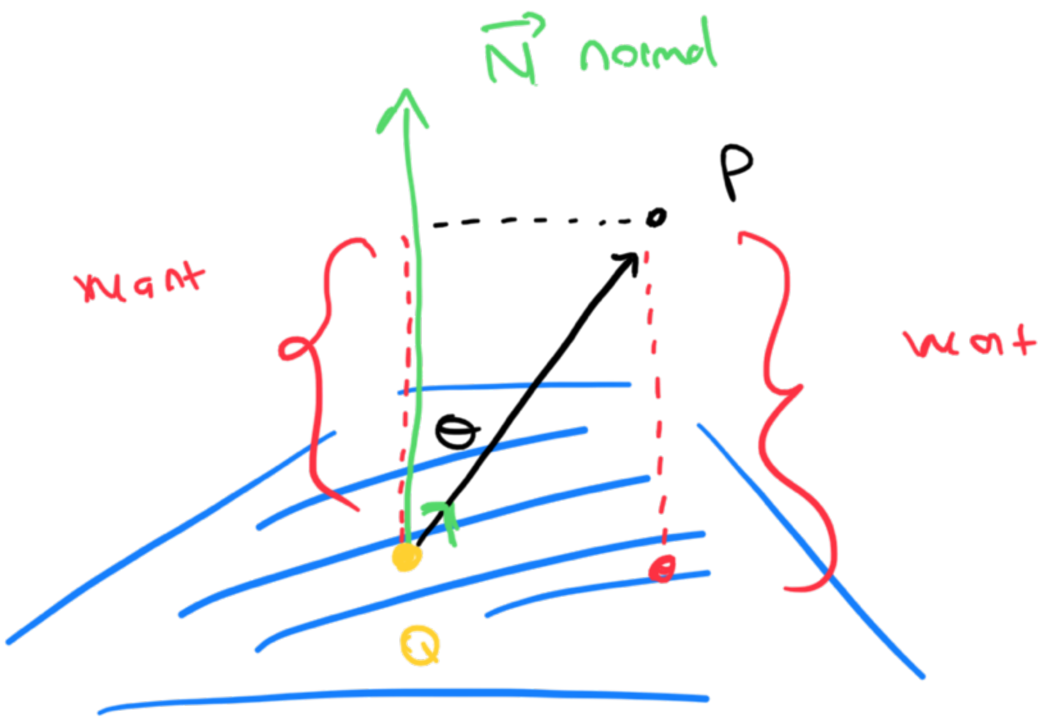
want $|\vec{QP}||\sin\theta| = \dfrac{|\vec{QP}||v||\sin\theta|}{|v|}$

$$\text{dist}(P,\ell) = |\vec{QP}\times\vec{v}|\,/\,|\vec{v}|$$

$\vec{v}$ parallel to ℓ and Q any point on it

ex: Distance between point and a plane?





distance is equal to $|\vec{QP}||\cos\theta|$

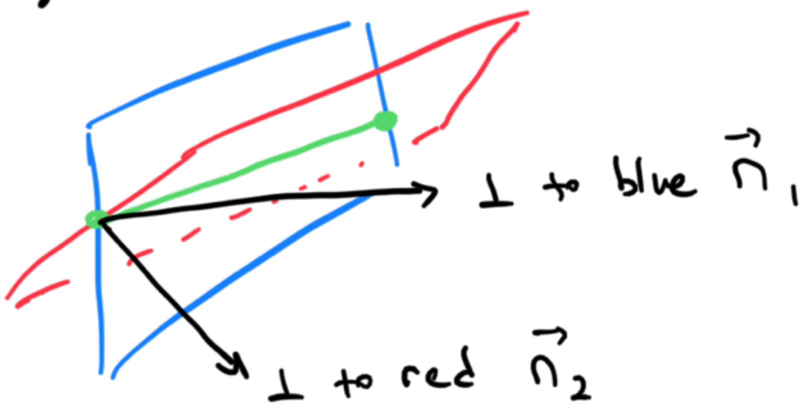
$$= \frac{|\vec{QP}|\,|\vec{N}|\,|\cos\theta|}{|\vec{N}|}$$

$$\text{dist}(P, \text{plane}) = |\vec{QP} \cdot \vec{N}| / |\vec{N}|$$

$\vec{N}$ normal to plane

Q any point on it

ex: Angle between two planes



$$\cos(\theta(\text{plane}_1, \text{plane}_2)) = \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

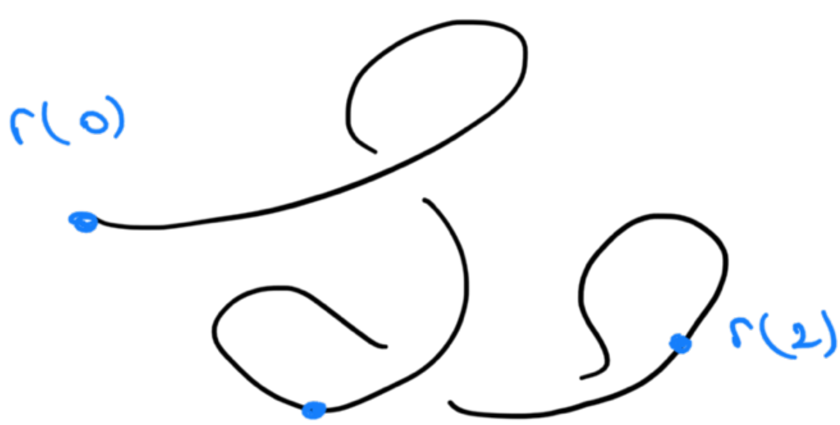
where n_1, n_2 normal to plane 1, plane 2 respectively

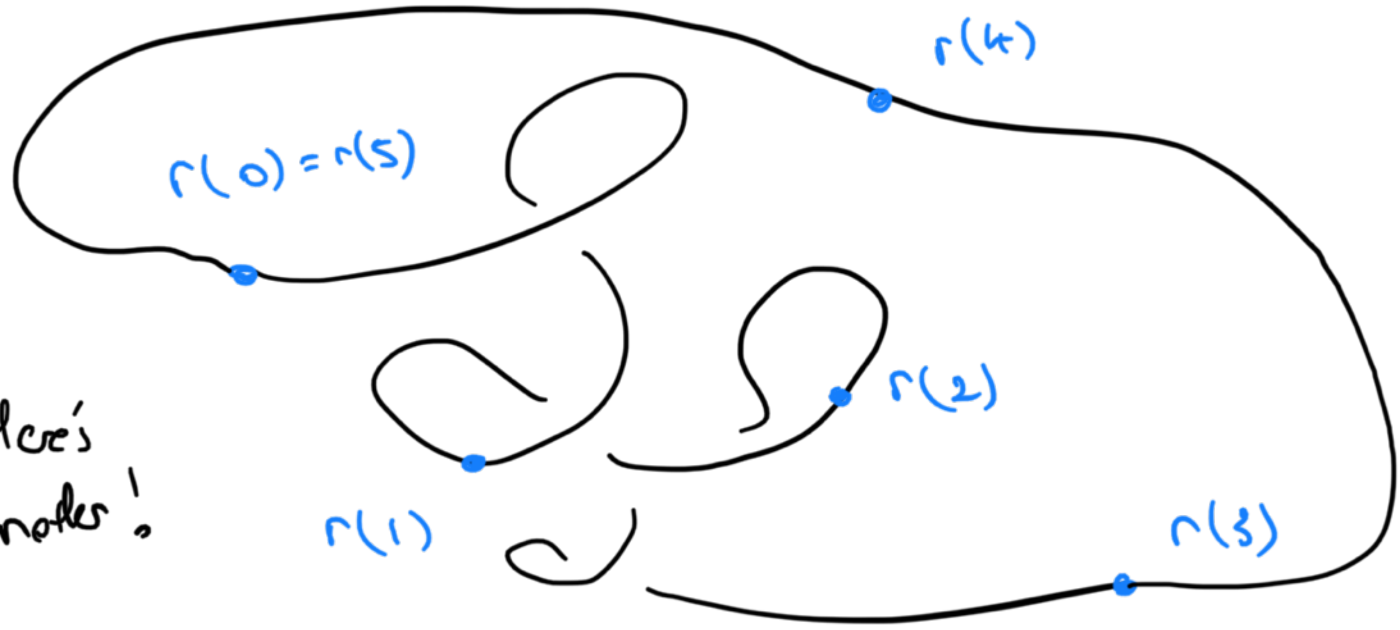
13.1 : Curves in space and tangents

A curve in space is a map

$\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^3$ typically written as

$\vec{r}(t) = (x(t), y(t), z(t))$. You imagine the t-parameter is time. Here's a curve.





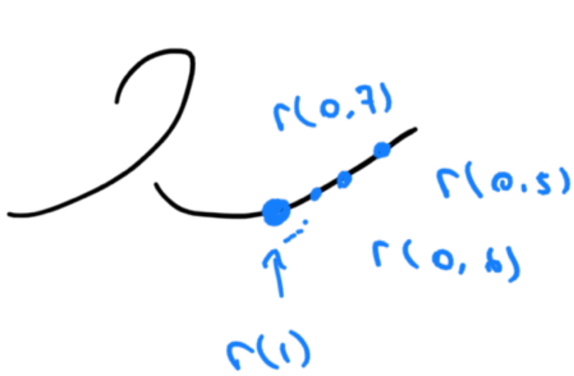
Here's another!

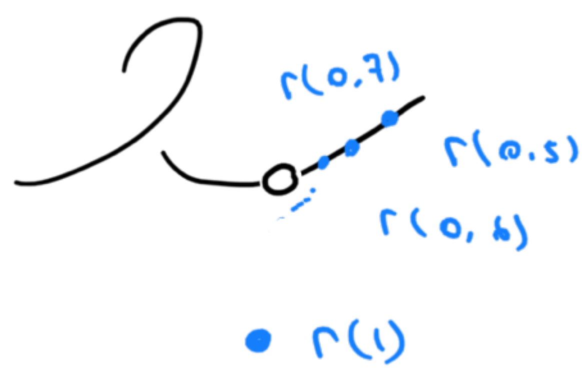
typically we want our curves to be smooth meaning differentiable. Want to avoid stuff like

but it happens.

Just like in 2-dimensional calculus we have limits and derivatives.

ex:





$\lim_{t \to 1} r(t)$ exists

$\lim_{t \to 1} r(t)$ exists but not

equal to $r(1)$.

continuous iff for each point $t \in$ domain $\lim_{s \to t} r(s) = r(t)$.

Above or right not cont.

theorem: if $r(t) = (x(t), y(t), z(t))$ then $r(t)$ continuous at s iff each $x(t), y(t), z(t)$ continuous at s.

ex: $r(t) = (t, t^2, t^3)$ good (also interesting curve)

ex: $r(t) = (1/t, \sin(t), t^2)$ not cont at $t = 0$, however, good everywhere else.

Derivatives: let $\vec{r}(t)$ be curve in $\mathbb{R}^3$.

We say $\vec{r}(t)$ is differentiable at t iff

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \text{ exists.}$$

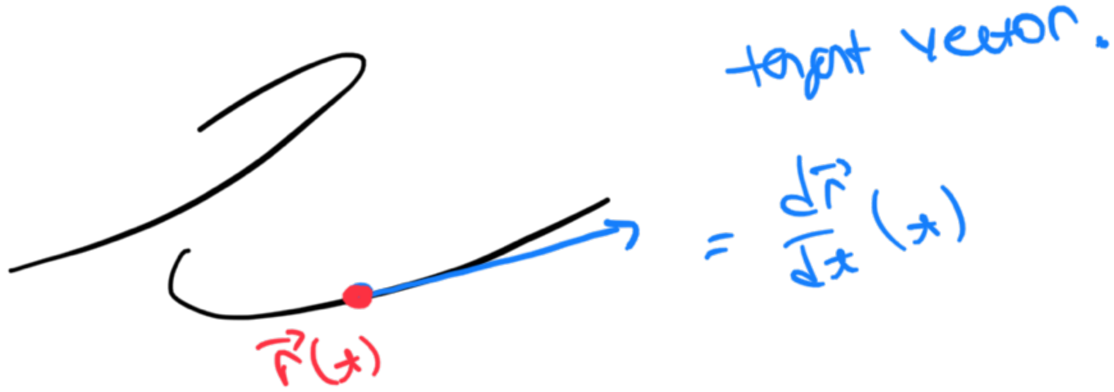
In this case we denote the limit by $\vec{r}'(t)$ or $\frac{d\vec{r}}{dt}(t)$.

theorem: $\vec{r}(t)$ is differentiable at t iff each $x(t)$, $y(t)$, $z(t)$ is too.

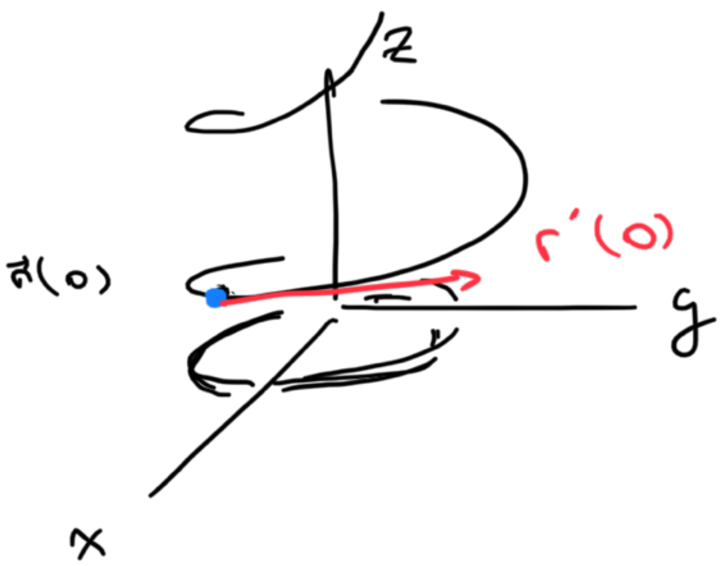
if so then $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

upshot: Just look at components.

Here's a picture:



ex: $\vec{r}(t) = (\cos t, \sin t, e^t)$



$\vec{r}\,'(t) = (-\sin t, \cos t, e^t)$

$\vec{r}\,'(0) = (0, 1, 1)$

Calculate the tangent line at $t = 0$.

Tangent line should just be what it was in 2D, but now in 3D.

$F(t) - F(a) \approx \frac{dF}{dt}(a)(t-a)$ now with vectors.

$L(t) = F(a) + \frac{dF}{dt}(a)(t-a)$

for linear.

$$L(t) = \vec{r}(0) + (0,1,1)(t-0)$$
$$= (1,0,1) + t(0,1,1)$$

Acceleration: $\frac{d^2\vec{r}}{dt^2}(t) = \frac{d}{dt}(-\sin t, \cos t, e^t)$

$$= (-\cos t, -\sin t, e^t)$$

Here some helpful rules on pg. 765

(i). $\frac{d}{dt}\vec{C} = 0$ where $\vec{C}$ constant

(ii). $\frac{d}{dt}(c\,\vec{u}(t)) = c\,\vec{u}'(t)$ c constant scaler

(iii). $\frac{d}{dt}(f(t)\,\vec{u}(t)) = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$
f(t) scalar function

(vi). $\frac{d}{dt}(\vec{u}(t)+\vec{v}(t)) = \vec{u}'(t) + \vec{v}'(t)$

(v). $\frac{d}{dt}(\vec{u}(t)-\vec{v}(t)) = \vec{u}'(t) - \vec{v}'(t)$

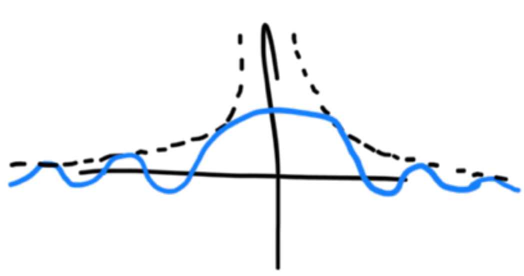
(vi). $\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}'(t)\cdot v(t) + \vec{u}(t)\cdot\vec{v}'(t)$

(vii). $\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'(t)\times v(t) + \vec{u}(t)\times\vec{v}'(t)$

(viii). $\frac{d}{dt}(\vec{u}(f(t)) = \vec{u}'(f(t))\,f'(t)$

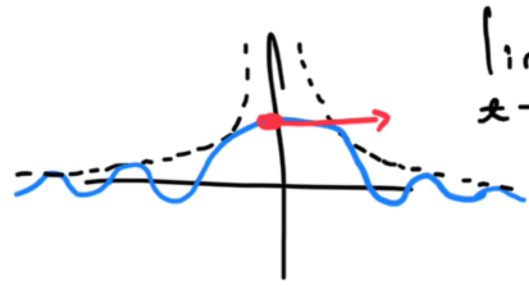
ex: $\vec{r}(t) = \left(t, \frac{\sin t}{t}, 1\right)$

Want $\lim_{t\to 0}\vec{r}(t)$ and $\lim_{t\to 0}\vec{r}'(t)$



$\lim_{t \to 0} \vec{r}(t) = (0, 1, 1)$ $\vec{r}'(t) = \left(1, \frac{t\cos t - \sin t}{t}, 0\right)$

$$\lim_{t \to 0} \vec{r}'(t) = (1, 0, 0)$$



$\lim_{t \to 0} \vec{r}'(t)$

13.2 : Integrals and projectile motion

Just like functions of a single variable, we say $\vec{r}(t)$ has an anti-derivative $\vec{R}(t)$ if

$\frac{d\vec{R}}{dt}(t) = \vec{r}(t)$. Or in integral notation,

$$\vec{R}(t) = \int \vec{r}(t)\, dt.$$

Note just like 1-variable, there's ambiguity.

$$\vec{S}(t) = \vec{R}(t) + \vec{C}$$ where $\vec{C}$ constant

is perfectly valid too.

(oops forgot
$$\int_a^b \frac{d\vec{F}}{dt}(t)\, dt = \vec{F}(b) - \vec{F}(a)$$
be sure to mention)

Motivated by physical quantities.

eg: $\vec{v}(t) = (-\sin t, \cos t, 1)$, velocity of particle. Want to find a particle's path whose velocity is $\vec{v}(t)$. Set up an integral.

$$\int \vec{v}(t)\, dt = \int -\sin t\, dt\ \hat{i} + \int \cos t\, dt\ \hat{j} + \int dt\ \hat{k} = (\cos t, \sin t, t) + \vec{C}.$$

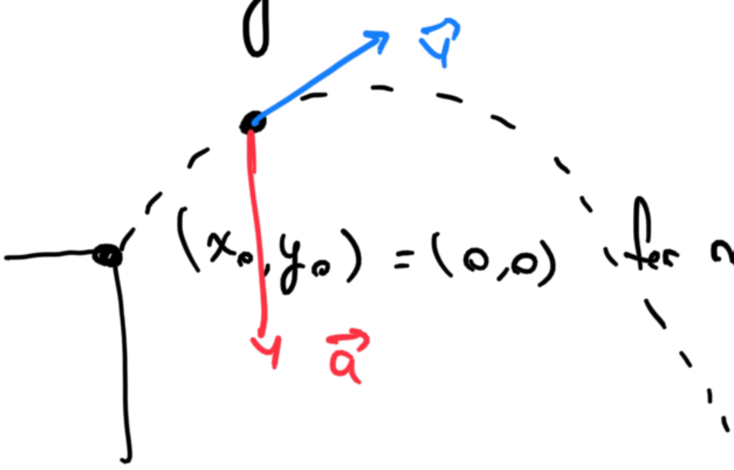
If our particle starts at point $(1, 0, 1)$ then solution is unique.

$$\vec{r}(t) = (\cos t, \sin t, t) + (C_1, C_2, C_3)$$

$\vec{r}(0) = (1, 0, 1) = (1, 0, 0) + (c_1, c_2, c_3)$

so $\vec{r}(t) = (\cos t, \sin t, t+1)$

ex: Say a ball thrown with initial velocity $\vec{v} = (v_x, v_y)$ at position $(x_0, y_0) = \vec{r}_0$. Calculate its trajectory assuming no forces other than gravity.



$(x_0, y_0) = (0,0)$ for now, then shift later.

$$\frac{d^2\vec{r}}{dt^2}(t) = \vec{a} = (0, -g) \qquad g = \text{gravitational constant} \approx 10 \text{ m/s}$$

$$\int_0^s \frac{d^2\vec{r}}{dt^2}(t) = \int_0^s 0 \, dt \, \hat{i} + \int_0^s -g \, dt \, \hat{j}$$

$$\vec{v}(s) - \vec{v}(0) = (0, -gs) \quad (\text{replace with } t)$$

$$\vec{v}(t) = \vec{v}(0) + (0, -gt) = (v_x, v_y - gt)$$

$$\int_0^s \vec{v}(t) \, dt = \int_0^s \frac{d\vec{r}}{dt}(t) \, dt = \vec{r}(s) - \vec{r}(0)$$

$$= \int_0^s v_x \, dt \, \hat{i} + \int_0^s (v_y - gt) \, dt \, \hat{j}$$

$$= \left(v_x s, \; v_y s - \tfrac{1}{2} g s^2\right)$$

So $\vec{r}(t) = (x_0 + v_x t,\ y_0 + v_y t - \frac{1}{2} g t^2)$ $(\ast)$

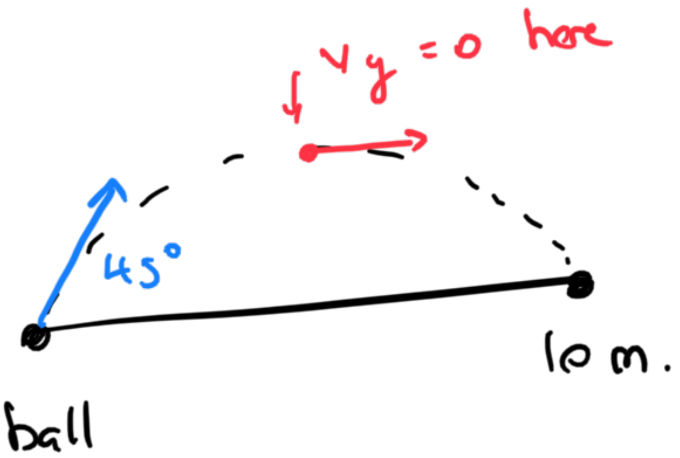
ex 13.2.27: Hit ball at $\theta = 45°$ and it lands level 10 m away.

(i). What was balls initial speed?

(ii). Find 2 angles that make range 6m.

why?

Speed is $|\vec{v}(t)|$. We want $|\vec{v}_0|$.



We can solve for half time by $v_y = 0$

$\vec{r}(t) = (v_x t,\ v_y t - \frac{1}{2} g t^2)$

$\vec{v}(t) = (v_x,\ v_y - g t)$

so $v_y - g t = 0$ thus

$$t^* = v_y / g.$$

this means total flight time is $2t^*$, thus range

$$R = v_x (2t^*) = 2 v_x v_y / g \quad (\ast)$$

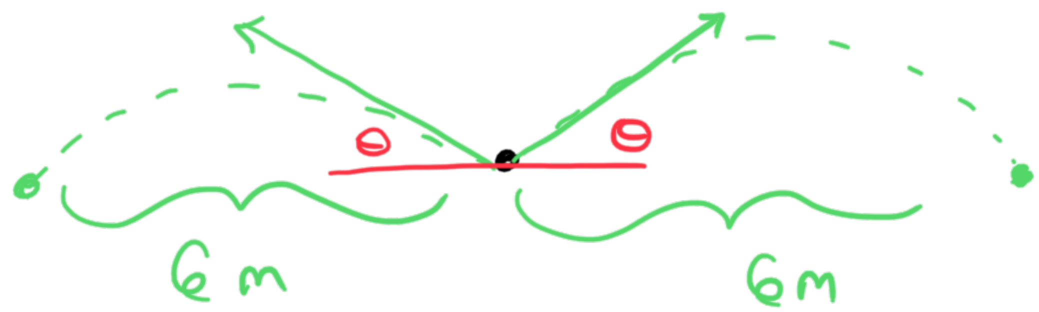
$$= 2 v_0^2 \cos\theta \sin\theta / g = \frac{1}{g} v_0^2 \sin 2\theta$$

$10\text{ m} = \frac{1}{g} v_0^2 \sin(2 \cdot 45°)$ so $|v_0| = \cancel{\sqrt{20}}\ 10$ m/s

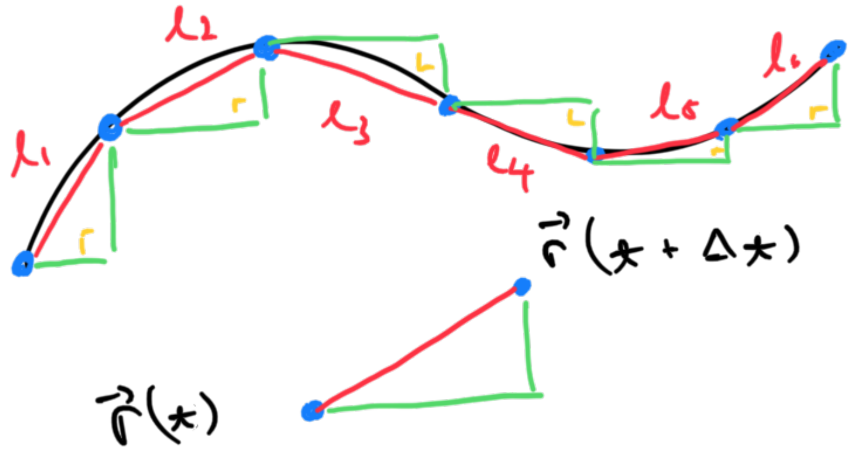
If however want range to be 6m at initial speed of $\sqrt{20}$ m/s, then

6

$$6m = \frac{1}{2} v_0^2 \sin(2\theta) \qquad \frac{\cdots}{20} = \sin(2\theta)$$



13.3 : Arc length



$$|\vec{r}(t+\Delta) - \vec{r}(t)| \approx \text{red length}$$

$$\approx \left| \frac{d\vec{r}}{dt}(t) \right| |\Delta t| = |\vec{v}(t)| \, |\Delta t|$$

So we define arc-length of a curve $\vec{r}(t)$ from $[a, b]$ to be

$$L = \int_a^b |\vec{v}(t)| \, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

ex: $\vec{r}(t) = (\sin t, 2\sin t, \sin t)$

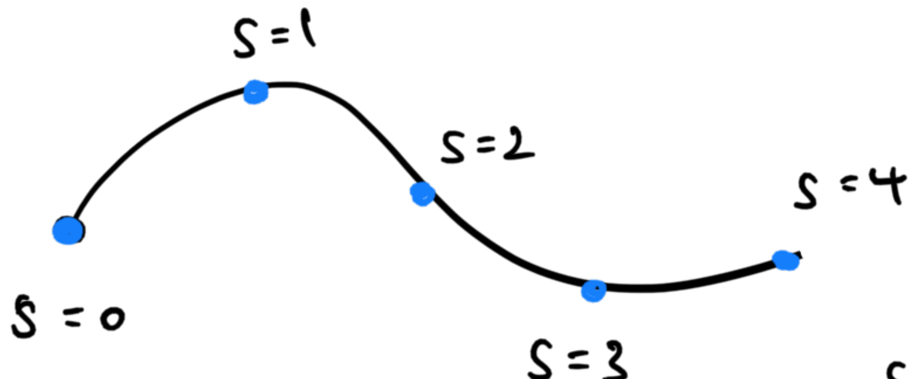
What length of curve from $t=0$ to $t=\pi/2$

$$L = \int_0^{\pi/2} \sqrt{(x')^2+(y')^2+(z')^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{\cos^2 t + 4\cos^2 t + \cos^2 t}\, dt$$

$$= \int_0^{\pi/2} \sqrt{6}\, |\cos t|\, dt = \sqrt{6}$$

Weird way of parametrizing, but okay. A 'natural' way to parametrize a smooth curve is by arclength itself!



where $s(t) = \int_0^t |v(\tau)|\, d\tau$

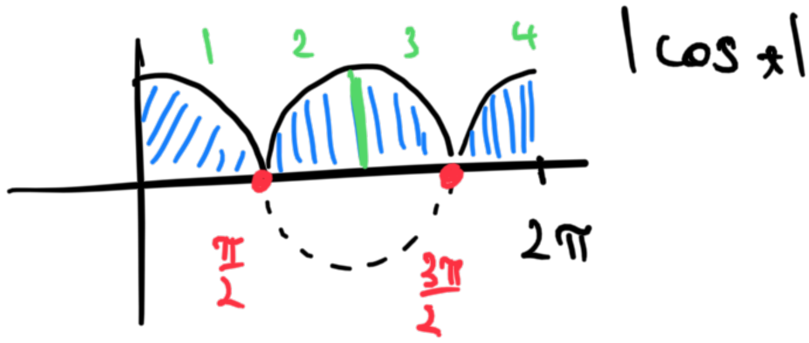
s is a measure of how far you've gone.

ex: $\vec{r}(t) = (\sin t, 2\sin t, \sin t)$ then after t-seconds we've traveled

$$s(t) = \int_0^t \sqrt{\cos^2\tau + 4\cos^2\tau + \cos^2\tau}\, d\tau$$

$$= \sqrt{6} \int_0^t |\cos\tau|\, d\tau$$

So for example after 2π, it's gone



So $s(2\pi) = \sqrt{6}\cdot 4$ and note that
$\vec{r}(2\pi) = \vec{r}(0)$.

Unit tangent vectors: let $\vec{r}(t)$ be some path. Unit tangent vector records <u>direction</u> of velocity vector (so long as $\vec{v}(t) \neq 0$).

denoted
$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}.$$

One property is that $\vec{T}(s) = \dfrac{d\vec{r}}{ds}(s)$ (good exercise).

ex: $\vec{r}(t) = (C\cos t, C\sin t, 1)$ $\quad C$ constant > 0

$\vec{v}(t) = (-C\sin t, C\cos t, 0)$

$|\vec{v}(t)| = |C| = C$

so $T(t) = (-\sin t, \cos t, 0)$ indeed

unit norm.

Also note, $s(t) = \int_0^t |\gamma'(t)|\,dt = Ct$ for $t \geq 0$.

$s = Ct$ so $t = s/c$.

so $\vec{r}(s) = (C\cos(s/c), C\sin(s/c), 1)$

$\frac{d\vec{r}}{dt}(s) = (-\sin(\underset{t}{\overset{''}{s/c}}), \cos(\underset{t}{\overset{''}{s/c}}), 0)$

$= \vec{T}(s)$!

14.1: Functions of several variables

Before: $t \in \mathbb{R} \longmapsto (x(t), y(t), z(t)) \in \mathbb{R}^3$

Now: $(x,y,z) \in \mathbb{R}^3$ or $(x,y) \in \mathbb{R}^2 \longmapsto f(x,y,z) \in \mathbb{R}$ or $f(x,y) \in \mathbb{R}$

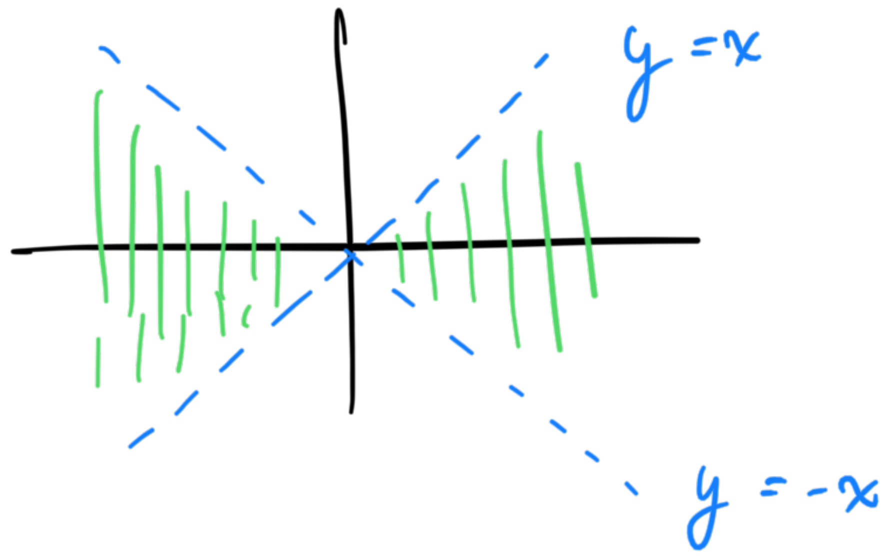
Real-valued functions. Can think of them as measurements of some variables, e.g. temperature along a plate of food or elevation as a function of position.

When you have a multivariable function typically defined by $f(x,y,z)$ or $f(x,y)$. The set of all points (x,y,z) for which f makes sense is called the domain.

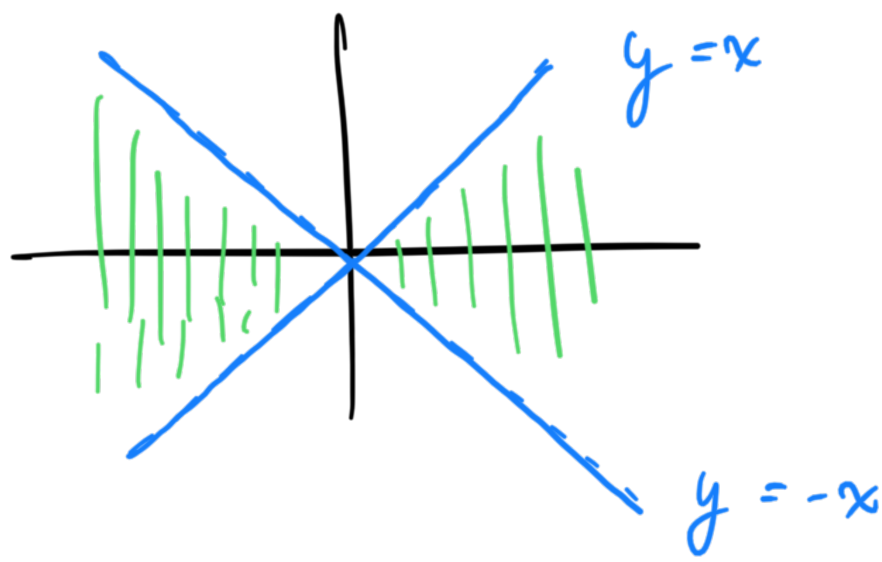
ex: Describe the domain of map

$$f(x,y) = \log(x^2 - y^2)$$

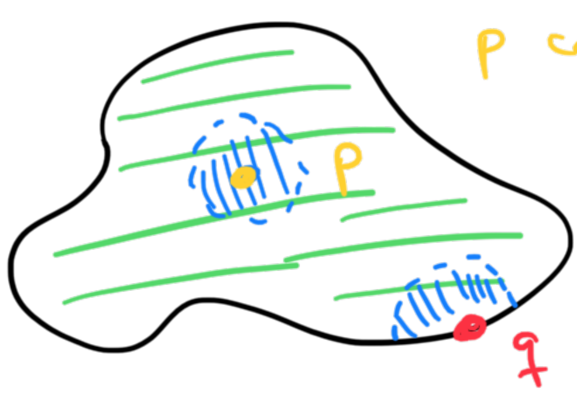
Need $x^2 - y^2 > 0$ (not equal)



What if $f(x,y) = \sqrt{x^2 - y^2}$? then



domain:

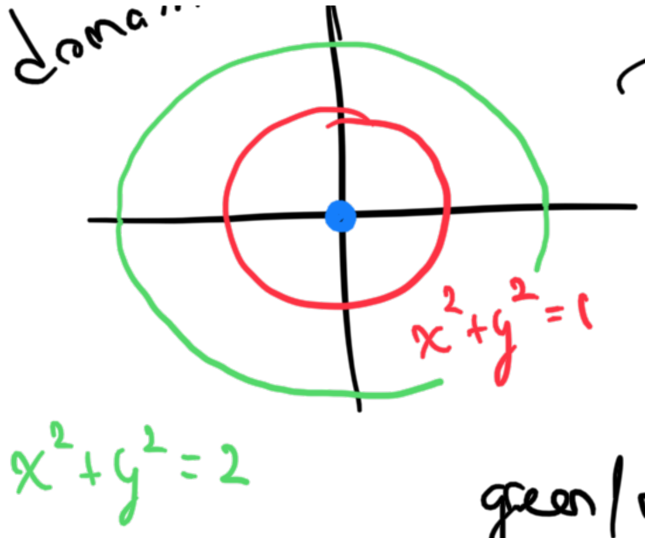


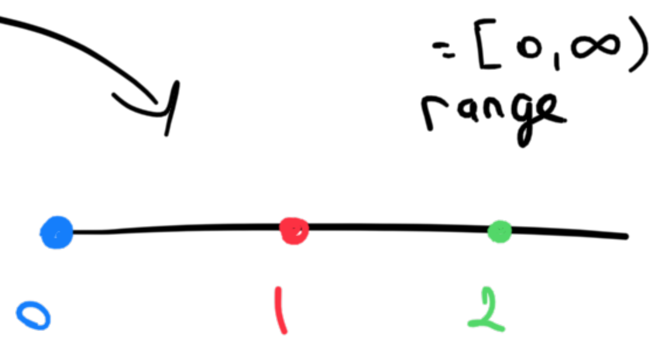
p called an interior point

q called a boundary point

ex: $f(x,y) = x^2 + y^2$.

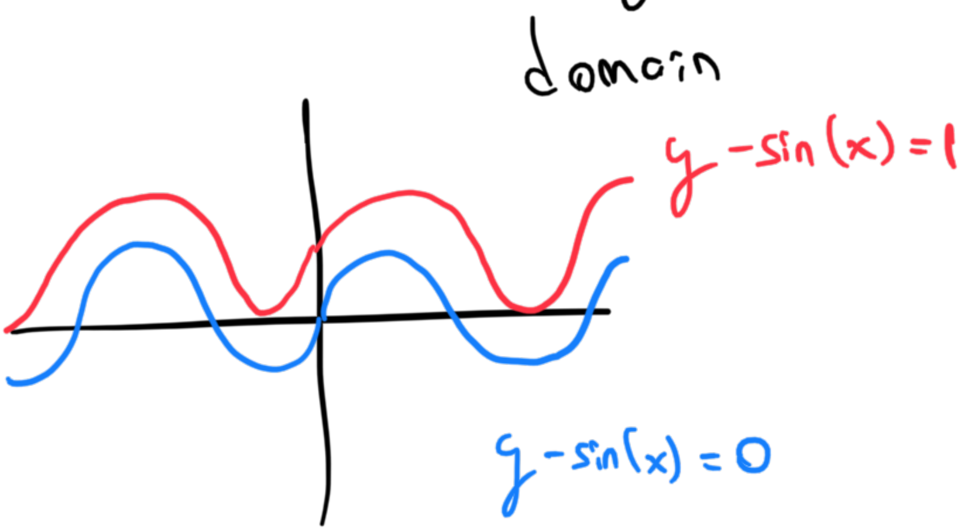
f

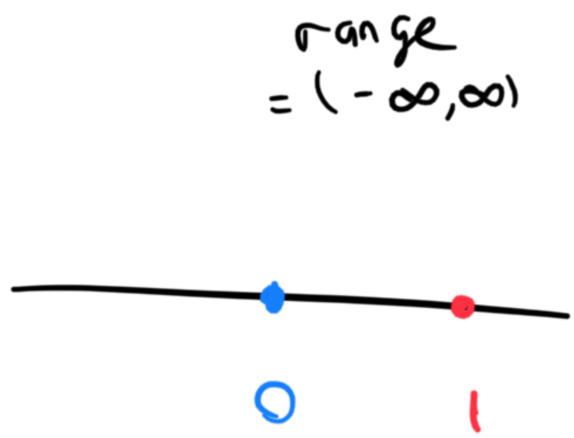




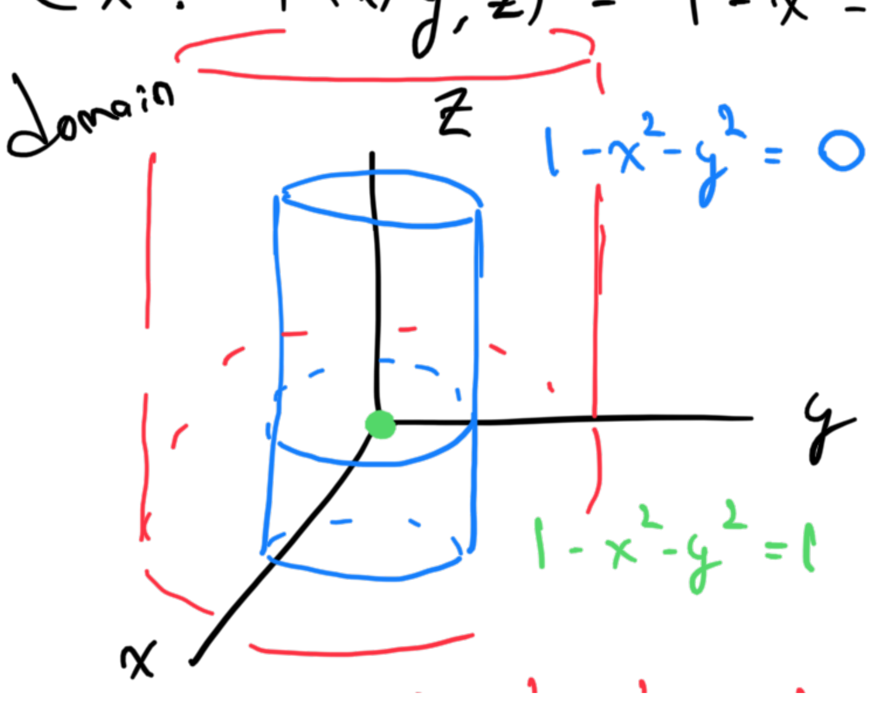
green/red/blue in domain are called 'level curves'. These are curves where the function has a constant value. Can define some notion of a 'level-surface'.

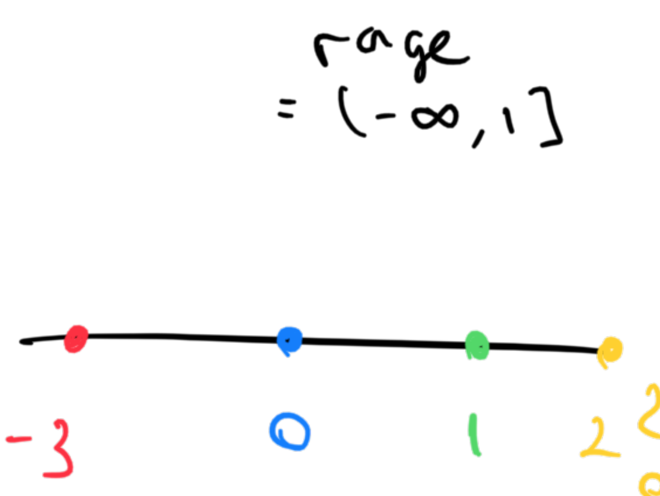
ex: $f(x,y) = y - \sin(x)$





ex: $f(x,y,z) = 1 - x^2 - y^2$

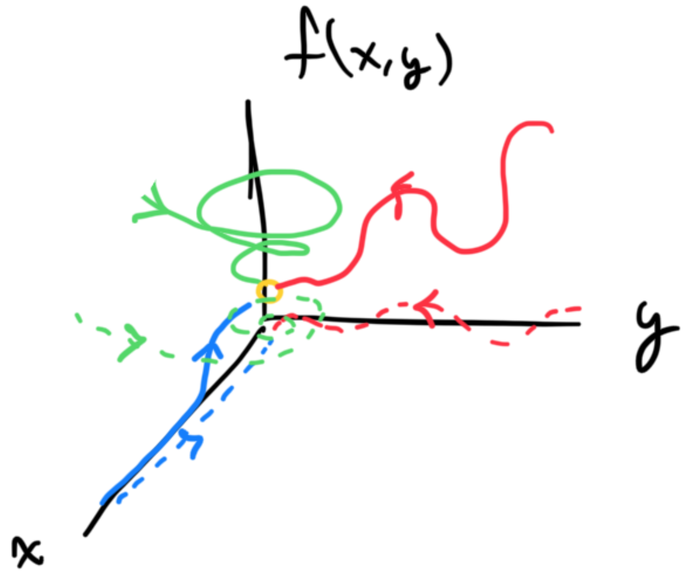




$1 \cdot x^2 \cdot y^2 = -3$

14.2: limits and continuity

limits in 2-variable are <u>much</u> more complicated than usual. Reason why is because you have a bunch of directions to approach now.



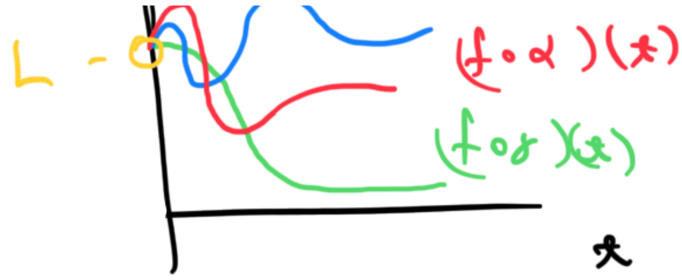
Let's not formally define limits, but instead use rules to figure out what limits are equal to.

However here's a piece of intuition.

Let $f(x,y)$ be a function. Consider a point (a,b). We say $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ iff

for every smooth path $\gamma(t)$ in $\mathbb{R}^2$ with $\gamma(0) = (a,b)$

we have $$\lim_{t\to 0} (f\circ\gamma)(t) = L.$$



Note in particular if we find two paths that disagree, then limit does not exist! that is if find an α and β so that

$$\lim_{t\to 0} (f\circ\alpha)(t) \neq \lim_{t\to 0} (f\circ\beta)(t)$$ then limit

does <u>not</u> exist!

Bunch of rules on pg. 812 look exactly like ones you familiar with but now in 2-variables.

e.g. if $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$ & $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = M$

then $$\lim_{(x,y)\to(a,b)} f(x,y)+g(x,y) = L+M$$

$$\lim_{(x,y)\to(a,b)} f(x,y)\cdot g(x,y) = L\cdot M \quad \text{etc.}$$

We say a function is continuous if for every point (a,b) in the domain we have

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

ex: $\displaystyle \lim_{(x,y)\to(1,1)} \frac{x^4 - y^4}{x^2 - y^2} = \lim_{(x,y)\to(1,1)} \frac{(x^2 - y^2)(x^2+y^2)}{(x^2-y^2)}$

$\displaystyle = \lim_{(x,y)\to(1,1)} x^2+y^2 = 1+1=2$

ex: 14.2.19

define $f(x,y) = \dfrac{\sqrt{2x-y}-2}{2x-y-4}$ for $2x-y-4 \neq 0$

$= 7$ if $2x-y-4=0$

is $f(x,y)$ continuous at $(2,0)$?

$\displaystyle \lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$

$\displaystyle = \lim_{(x,y)\to(2,0)} \frac{(\sqrt{2x-y}-2)}{(\sqrt{2x-y}-2)(\sqrt{2x-y}+2)} = \lim_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2}$

$= \dfrac{1}{2+2} = 1/4 \neq 7$ so no, function not continuous at $(2,0)$

ex: Just because a function has a limit along every line, doesn't mean it has a limit!

$f(x,y) = \dfrac{2x^2y}{x^4+y^2}$ pg. 815

note along y-axis $\alpha(t) = (0,t)$ for $t \neq 0$

$(f\circ\alpha)(t) = \dfrac{0}{t^2} = 0$ for $t \neq 0$

$\longrightarrow 0$

However for $\beta(t) = (t, mt)$

$$(f \circ B)(t) = \frac{2(t^2)(mt)}{t^4 + (mt)^2} = \frac{2m t^3}{m^2 t^2 + t^4}$$

$$= \frac{2m t}{m^2 + t^2} \quad \text{for } t \neq 0$$

so $\lim_{t \to 0} (f \circ B)(t) = 0$ true for any m.

However limit is different for $C(t) = (t, kt^2)$
check is good exercise.

ex: $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y}$, $u = x+y$

$\lim_{u \to 0} \frac{\sin(u)}{u} = 1$ can do replacement if function is a

composition $(f \circ g)(x,y)$ where $f(u)$ continues at $\lim g(x,y) \to L$

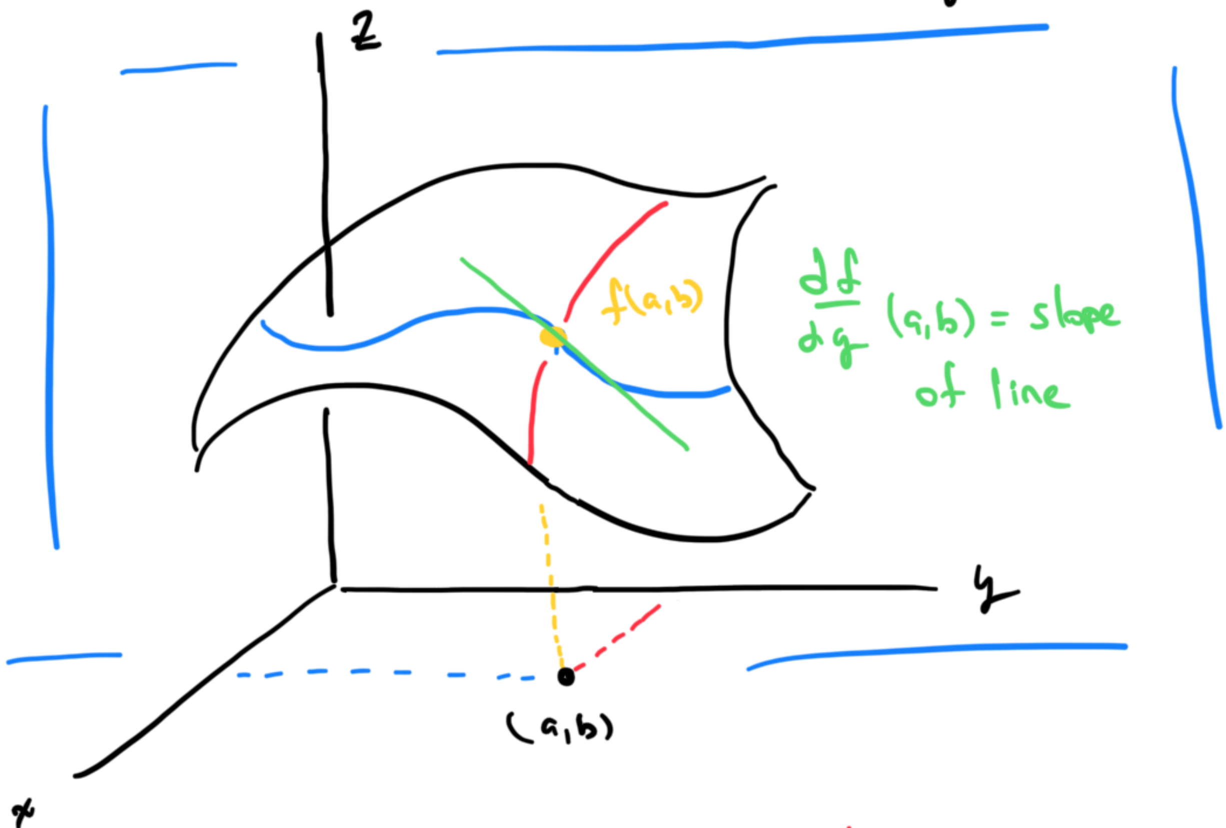
14.3: Partial derivatives

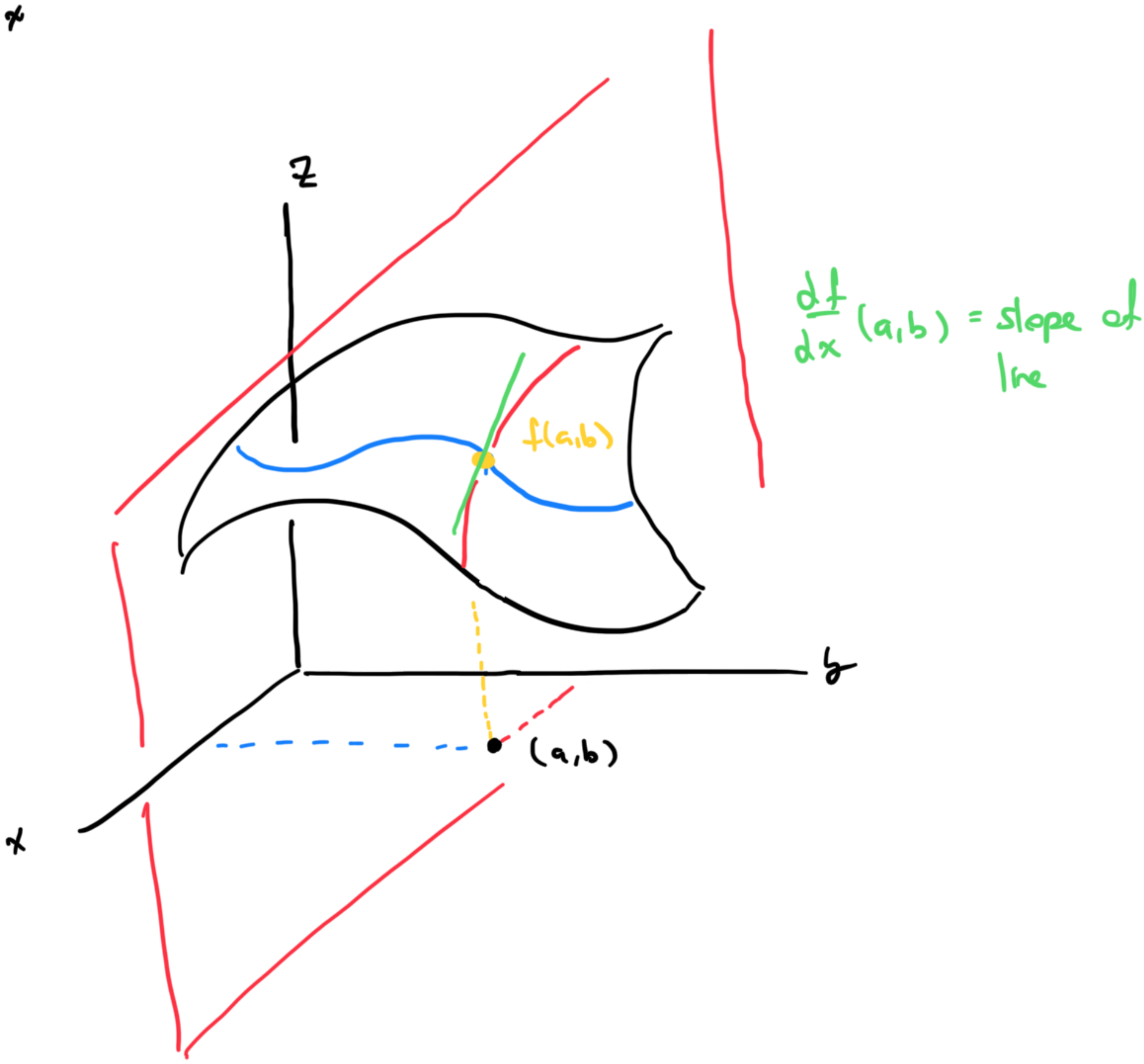
$z = f(x,y)$ then we can define partial derivatives which measure how a function changes as one variable changes and the other stays fixed.

$$\frac{\partial f}{\partial x}(a,b) := \lim \frac{f(a+\Delta x, b) - f(a,b)}{}$$

$$\Delta x \to 0 \qquad\qquad \Delta x$$

$$\frac{df}{dy}(a,b) := \lim_{\Delta y \to 0} \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y}$$





ex: $f(x,y) = x\cos(xy)$

$$\frac{df}{dy} = x\,\frac{d}{dy}\cos(xy) = x\left(-\sin(xy)\,\frac{d}{dy}(xy)\right)$$

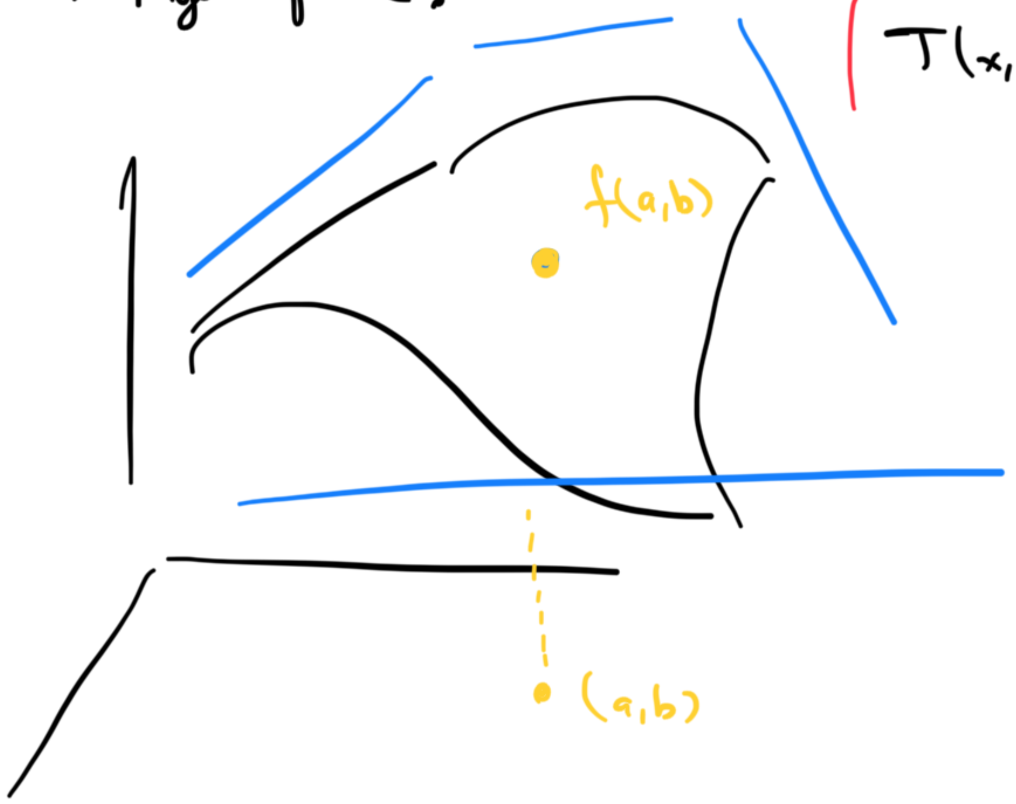
$$= -x\sin(xy)(x) = -x^2\sin(xy)$$

$$\frac{df}{dx} = \frac{d}{dx}\left(x\cos(xy)\right) = \left(\frac{d}{dx}x\right)\cos(xy) + x\frac{d}{dx}\left(\cos(xy)\right)$$

$$= 1\cos(xy) + x\left(-\sin(xy)\frac{d}{dx}(xy)\right)$$

$$= \cos(xy) - x\sin(xy)(y) = \cos(xy) - xy\sin(xy)$$

Tangent plane:



$$\boxed{\begin{aligned} T(x,y) = {} & f(a,b) \\ & + (x-a)\frac{df}{dx}(a,b) \\ & + (y-b)\frac{df}{dy}(a,b) \end{aligned}}$$

ex: $f(x,y) = xe^{x\sin(y)}$

Calculate tangent plane to $f(x,y)$ at point $(x,y) = (0,\pi)$

$$\frac{df}{dx} = \left(\frac{d}{dx}x\right)e^{x\sin(y)} + x\frac{d}{dx}\left(e^{x\sin(y)}\right)$$

$$= e^{x\sin(y)} + xe^{x\sin(y)}\frac{d}{dx}(x\sin(y))$$

$$= e^{x\sin(y)} + xe^{x\sin(y)}\sin(y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x e^{x\sin(y)}\right) = x \frac{\partial}{\partial y}\left(e^{x\sin(y)}\right)$$
$$= x e^{x\sin(y)} \frac{\partial}{\partial y}\left(x\sin(y)\right)$$
$$= x e^{x\sin(y)} x\cos(y) = x^2 e^{x\sin(y)} \cos(y)$$

$\frac{\partial f}{\partial x}(0,\pi) = e^0 + 0 = 1$ $\quad \frac{\partial f}{\partial y}(0,\pi) = 0$

So tangent plane is and $f(0,\pi) = 0$

$$T(x,y) = f(0,\pi) + \frac{\partial f}{\partial x}(0,\pi)(x-0) + \frac{\partial f}{\partial y}(0,\pi)(y-0)$$
$$= 0 + x + 0 = x$$

ex: Is there a function $f(x,y)$ for which

$$\frac{\partial f}{\partial x}(x,y) = x^2 y \text{ and } \frac{\partial f}{\partial y}(x,y) = xy^2 \text{ ?}$$

Mixed partials (typically) agree.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(x^2 y) = x^2 \text{ and}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(xy^2) = y^2$$

So no!

What if $\frac{\partial f}{\partial x} = 2xy + y^2 + x^2$ and $\frac{\partial f}{\partial y} = x^2 + 2xy$

then quick check $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

also notationally $\quad f_{xy} = f_{yx}$

How to find an f?

$$\frac{\partial f}{\partial y} = x^2 + 2xy \quad \text{so} \quad f(x,y) = x^2 \int dy + 2x \int y \, dy$$
$$= x^2 y + xy^2 + \underline{h(x)}$$

$$\frac{\partial f}{\partial x} = 2xy + y^2 + h'(x) = 2xy + y^2 + x^2$$

get $h'(x) = x^2$ so $h(x) = \frac{1}{3}x^3 + C$

and $f(x,y) = x^2 y + xy^2 + \frac{1}{3}x^3 + C$

14.4 : Chain rule

Easiest explained by examples. However for now

let $z = f(x,y)$ and x be a function of t
y be a function of t

then $$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

ex: $z = x^2 - y^2$ and $x(t) = \cos(t)$
$y(t) = \sin(t)$

then $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$\frac{\partial z}{\partial x} = 2x$ $\quad \frac{\partial z}{\partial y} = -2y$ $\quad \frac{dx}{dt} = -\sin t$ $\quad \frac{dy}{dt} = \cos t$

$= 2\cos t$ $\quad = -2\sin t$ so

$$\frac{dz}{dt} = (2\cos t)(-\sin t) + (-2\sin t)(\cos t)$$

$$= -2\sin(2t)$$

In fact, $z(t) = \cos^2 t - \sin^2 t = \cos(2t)$

so $\dfrac{dz}{dt} = -\sin(2t)(2)$

ex: $f(x,y,z) = \sin x \cos y \tan z$ and want $\dfrac{df}{dt}(0)$ where $x(t) = t$ $\quad y(t) = -t$ and $z(t) = \tan^{-1}(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\frac{\partial f}{\partial x} = \cos x \cos y \tan z = (\cos^2 t)\tan(\tan^{-1}(t))$$
$$= t\cos^2 t$$

$$\frac{\partial f}{\partial y} = -\sin x \sin y \tan z = -(\sin^2 t)\, t$$

$$\frac{\partial f}{\partial z} = \sin x \cos y \sec^2 z = \sin x \cos y (1+\tan^2 z)$$
$$= \sin t \cos t (1+t^2)$$

$$\frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -\sin t \qquad \frac{dz}{dt} = \frac{1}{1+t^2}$$

So $\dfrac{df}{dt}(0) = (0)(1) + (0)(0) + (0)(1) = 0$

Can see this simply by

$$f(t) = (\sin t \cos t)\, t = \frac{1}{2}\sin(2t)\, t$$

$$\frac{df}{dt} = \left(\frac{1}{2}\cos(2t)\cdot 2\right)\cdot t + \left(\frac{1}{2}\sin(2t)\right)\cdot 1$$

$$\frac{df}{dt}(0) = 0$$

What if have more than one variable, i.e. a surface?

ex: $w = f(x,y,z)$ where $x(u,v), y(u,v), z(u,v)$?

Pretty much the same thing.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$ and

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

ex: $w = x^2 + y^2 + z^2$ $\quad x = \sin u \cos v$

Want $\frac{\partial w}{\partial u}$. $\quad y = \sin u \sin v$

$\quad z = \cos u$

$$\frac{\partial w}{\partial x} = 2x \quad \frac{\partial w}{\partial y} = 2y \quad \frac{\partial w}{\partial z} = 2z$$

$$w_x = 2\sin u \cos v \quad w_y = 2\sin u \sin v \quad w_z = 2\cos u$$

$$\frac{\partial x}{\partial u} = \cos u \cos v \quad \frac{\partial y}{\partial u} = \cos u \sin v \quad \frac{\partial z}{\partial u} = -\sin u$$

$$\begin{aligned} w_u &= (2\sin u \cos v)(\cos u \cos v) \\ &\quad + (2\sin u \sin v)(\cos u \sin v) \\ &\quad + (2\cos u)(-\sin u) \\ &= (2\sin u \cos u)\underline{\cos^2 v} + (2\sin u \cos u)\underline{\sin^2 v} \\ &\quad - 2\cos u \sin u = 0 \end{aligned}$$

let's try w_v. We already have some terms. Just need $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}$

$$\frac{dx}{dv} = -\sin u \sin v \quad \frac{dy}{dv} = \sin u \cos v \quad \frac{dz}{dv} = 0$$

so
$$\frac{dw}{dv} = (2\sin u \cos v)(-\sin u \sin v) + (2\sin u \sin v)(\sin u \cos v) + 0 = 0$$

So $w_v = 0$

In fact...
$$w(u,v) = (\sin u \cos v)^2 + (\sin u \sin v)^2 + (\cos u)^2$$
$$= (\sin^2 u)(\cos^2 v + \sin^2 v) + (\cos^2 u)$$
$$= 1$$

So $\frac{dw}{du} = 0$ and $\frac{dw}{dv} = 0$!

implicit differentiation:

Say have z implicitly defined in terms of x, y, e.g. $x^2 + y^2 - z^2 = \cancel{1}\,3$

Want to calculate $\frac{dz}{dx}$ at a point, e.g. $(2, 0, 1)$

You do so by differentiating.

$$\frac{d}{dx}(x^2 + y^2 - z^2) = \frac{d}{dx}(3)$$
$$2x + 0 - 2z\frac{dz}{dx} = 0 \quad \text{so}$$
$$\frac{dz}{dx} = \frac{x}{z} \quad \text{thus} \quad \frac{dz}{dx}(2,0) = \frac{2}{1} = 2$$

ex: $\sin(xy^2)+e^z = 1$ calculate $\frac{dy}{dx}$ and $\frac{dy}{dz}$

at $(\pi, 1, 0)$.

$$\frac{d}{dx}\left(\sin(xy^2)+e^z\right) = \frac{d}{dx}(1)$$

$$\cos(xy^2)\frac{d}{dx}(xy^2) + 0 = 0$$

$$\cos(xy^2)\left(1 + x \cdot 2y\frac{dy}{dx}\right) = 0 \quad \text{plug in } (\pi, 1, 0)$$

$$\cos(\pi)\left(1 + 2\pi\frac{dy}{dx}\right) = 0 \quad \text{so} \quad \frac{dy}{dx}(\pi, 0) = -1/2\pi$$

$$\frac{d}{dz}\left(\sin(xy^2)+e^z\right) = \frac{d}{dz}(1)$$

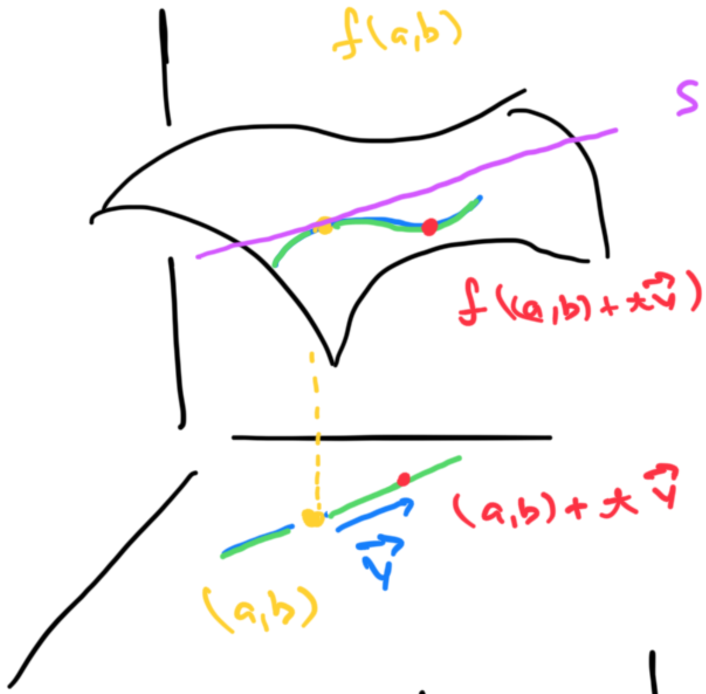
$$\cos(xy^2)\frac{d}{dz}(xy^2) + e^z = 0$$

$$\cos(xy^2)\,x\left(2y\frac{dy}{dz}\right) + e^z = 0 \quad \text{at } (\pi, 1, 0),$$

$$\cos(\pi)(\pi\, 2)\frac{dy}{dz}(\pi, 0) + 1 = 0 \quad \text{so}$$

$$\frac{dy}{dz}(\pi, 0) = +1/2\pi$$

14.5: Directional derivatives and gradient



slope $= \dfrac{df}{d\vec{v}}(a,b)$

$= \left(\dfrac{df}{ds}\right)_{(\vec{v},(a,b))}$

$$\text{slope} = \lim_{t \to 0} \frac{f((a,b)+t\vec{v}) - f(a,b)}{t}$$

= scalar quantity measure how fast f is changing along the vector $\vec{v}$.

Warning: Book only defines this for unit vectors. This is absurd, I have no idea why.

So when they ask something like "find the derivative of the function f at (a,b) in the direction of $\vec{v}$" they mean make $\vec{v}$ a unit vector first!

ex: find the derivative of $f(x,y) = xy^2$ at the point $(1,1)$ along the vector $\vec{v} = (3,4)$. (then do the same for the direction).

1. $f((1,1)+t(3,4)) - f(1,1)$

$$\lim_{t \to 0} \frac{\quad}{t}$$

$$= \lim_{t\to 0} \frac{f(1+3t,\ 1+4t) - f(1,1)}{t} = \lim_{t \to 0} \frac{(1+3t)(1+4t)^2 - 1}{t}$$

$$= \lim_{t\to 0} \frac{(1+3t)(1+8t+16t^2) - 1}{t}$$

$$= \lim_{t \to 0} \frac{11t + 40t^2 + 48t^3}{t} = \lim_{t\to 0} \frac{11t}{t} + \text{stuff} \cdot t = 11$$

if the book asks for direction, $u = \left(\frac{3}{5}, \frac{4}{5}\right)$

$$\lim_{t\to 0} \frac{f\left((1,1) + t\left(\frac{3}{5}, \frac{4}{5}\right)\right) - f(1,1)}{t} = \frac{11}{5} \quad \text{why?}$$

$$= (D_u f)(1,1)$$

In general note for a vector $\vec{v}$,

$$(D_{\vec{v}} f)(a,b) = \frac{df}{d\vec{v}}(a,b) = \lim_{t\to 0} \frac{f((a,b) + t(v_x, v_y)) - f(a,b)}{t}$$

$\gamma(t) = (a,b) + t(v_x, v_y)$ then

$(D_{\vec{v}} f)(a,b) = \left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma)(t)$ which we may use chain-rule,

$$= \frac{df}{dx}(\gamma(0)) \frac{dx}{dt}(0) + \frac{df}{dy}(\gamma(0)) \frac{dy}{dt}(0)$$

$$= \left(\frac{df}{dx}(a,b),\ \frac{df}{dy}(a,b)\right) \cdot (v_x, v_y)$$

so $(D_{\vec{v}} f)(a,b) = \left(\frac{df}{dx}(a,b),\ \frac{df}{dy}(a,b)\right) \cdot \vec{v}$

we define $(\nabla f)(a,b) =$ called

the gradient of f //

ex: Find the directional derivative of $f(x,y) = x\sin(y)$ at $(1,\pi)$ along the vector $\vec{v} = (1,2)$.

$$\vec{u} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$(D_{\vec{u}} f)(1,\pi) = \nabla f(1,\pi) \cdot \vec{u}$$

$$(\nabla f)(1,\pi) = (\sin(y), x\cos(y))\Big|_{(1,\pi)}$$
$$= (0, -1)$$

so $(D_{\vec{u}} f)(1,\pi) = (0,-1) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{-2}{\sqrt{5}} < 0$

The formula $(D_{\vec{v}} f)(a,b) = \nabla f(a,b) \cdot \vec{v}$ is very telling insofar as it gives us that

$$(D_{\vec{v}} f)(a,b) = \nabla f(a,b) \cdot \vec{v} = |\nabla f(a,b)||\vec{v}| \cos\theta$$

so $-|\nabla f(a,b)||\vec{v}| \leq (D_{\vec{v}} f)(a,b) \leq |\nabla f(a,b)||\vec{v}|$

bc $-1 \leq \cos\theta \leq 1$. This quantity is maximized if

$\cos\theta = 1$, i.e. $\theta = 0$, so $\vec{v}$ and $\nabla f(a,b)$ point same direction.

$\cos\theta = -1$ i.e. $\theta = \pi$, so $\vec{v}$ and $\nabla f(a,b)$ point opposite direction.

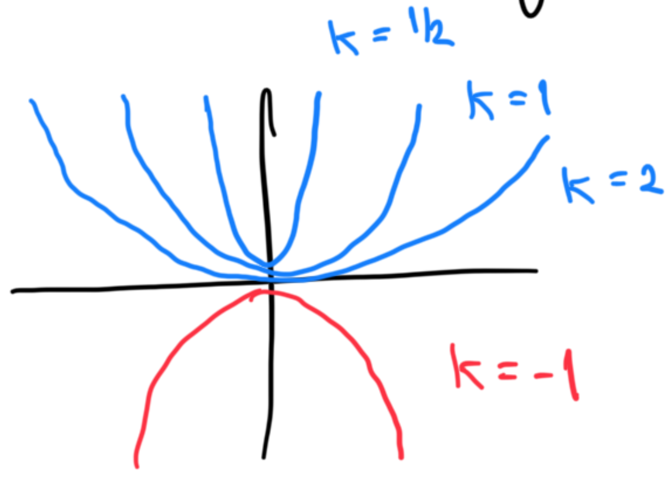
thus f increases most in direction of ∇f
and f decreases most in direction of $-\nabla f$.

Similarly, $(D_{\vec{v}} f)(a,b) = \nabla f(a,b) \cdot \vec{v} = 0$
iff $\vec{v} \perp \nabla f(a,b)$, thus f is
not changing if $\vec{v} \perp \nabla f(a,b)$!
so f is constant along direction $\perp$ to ∇f!

ex: $f(x,y) = \dfrac{x^2}{y}$ on domain $y \neq 0$.

Draw level curves. $\dfrac{x^2}{y} = k$ for constant k

means that $y = k^{-1} x^2$

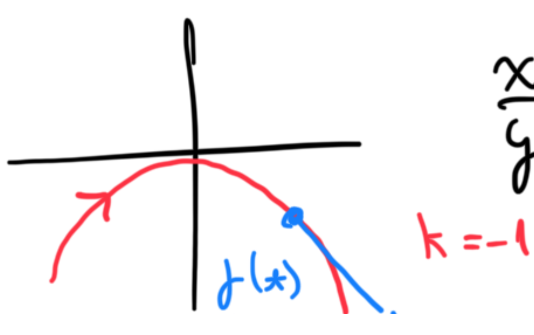


Okay, let's calculate the gradient at a point (a,b).

$$(\nabla f)(a,b) = \left(\frac{df}{dx}(a,b), \frac{df}{dy}(a,b) \right)$$

$$= \left(\frac{2x}{y}, -\frac{x^2}{y^2} \right)\Big|_{(a,b)} = \left(\frac{2a}{b}, -\frac{a^2}{b^2} \right)$$

let's choose a level curve. Say $k = -1$.



$\dfrac{x^2}{y} = -1$. Now you can parametrize by $\gamma(t) = (t, -t^2)$

so $\gamma'(t) = (1, -2t)$

$(t, -t^2)$ $\quad \gamma'(t) = (1, -2t)$. $\quad$ so $\gamma'(t) = (1, -2t)$.

So $\vec{u}(t) = \dfrac{(1, -2t)}{\sqrt{1+4t^2}}$

So note if take $(D_{\vec{u}(t)} f)(\gamma(t))$

$$= (\nabla f)(\gamma(t)) \cdot \vec{u}(t)$$

$$= \left(2\frac{t}{-t^2}, -\frac{t^2}{t^4}\right) \cdot (1, -2t) / \sqrt{1+4t^2}$$

$$= \left(-\frac{2}{t}, -\frac{1}{t^2}\right) \cdot \frac{(1, -2t)}{\sqrt{1+4t^2}} = \frac{-\frac{2}{t} + \frac{2}{t}}{\sqrt{1+4t^2}} = 0 \quad \text{for } t \neq 0$$

that means f not changing in the direction of $\gamma'(t)$! (But this is not surprising at all, it's because $\gamma(t)$ is a level curve!)

this principle holds in general.

— the gradient is orthogonal to level curves!

Some properties:

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$

$$\nabla(cf) = c\nabla f \quad c \text{ constant}$$

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - (\nabla g)f}{g^2}$$

ex: $f(x, y, z) = x^2 + y^2 - z^2$.

at the point $(1,2,3)$ find the direction

for which f is increasing the most.

$$\nabla f = (2x, 2y, -2z) \text{ so}$$

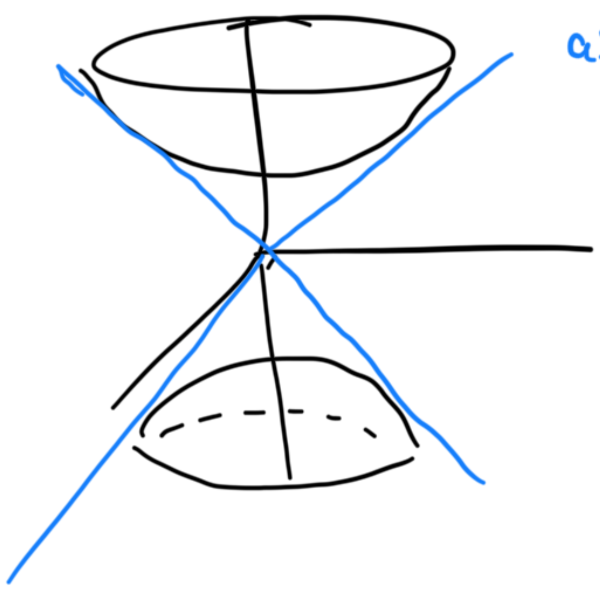
$(\nabla f)(1,2,3) = (2, 4, -6)$, direction of greatest increase

is $\dfrac{(2,4,-6)}{|(2,4,-6)|} = \dfrac{(1,2,-3)}{\sqrt{14}}$

Calculate a vector normal to the surface

$x^2+y^2-z^2=-1$. $(\nabla f)(x,y,z) = (2x, 2y, -2z)$

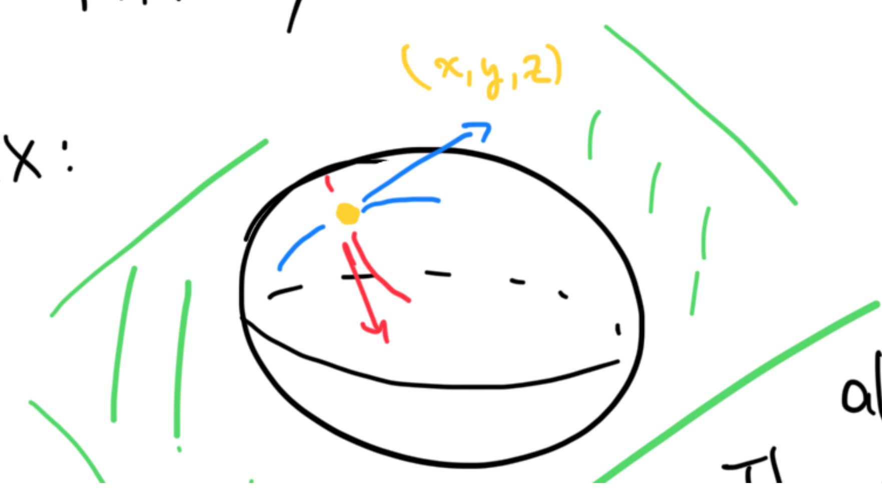
asymptote cone $x^2+y^2-z^2=0$



14.6: Tangent planes and differentials

intuitively have an idea of this

ex:



consider all paths through point (x,y,z) on your surface S and take

all possible tangent lines.

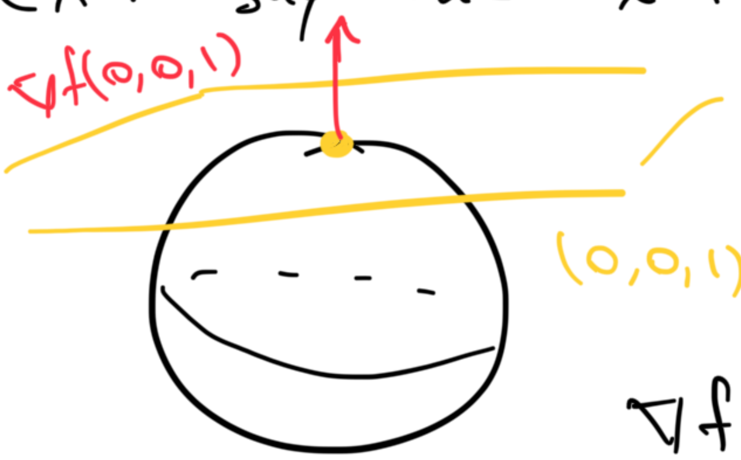
This is your tangent plane.

As we know a plane is also specified by a point and normal vector. So we have the following.

Let S be surface defined by $f(x,y,z) = C$.
Let (a,b,c) be a point on S. The tangent plane is going to be all lines through (a,b,c) that are $\perp$ to $\nabla f(a,b,c)$ because $\nabla f(a,b,c)$ is $\perp$ to level curves. So get equation

$$\nabla f(a,b,c) \cdot (x-a, y-b, z-c) = 0 \quad (*)$$

ex: Say have $x^2+y^2+z^2=1$



so intuitively plane is $z=1$.

Check. $f(x,y,z) = x^2+y^2+z^2$

$$\nabla f = (2x, 2y, 2z)$$

So $\nabla f(0,0,1) = (0,0,2)$ tangent plane is

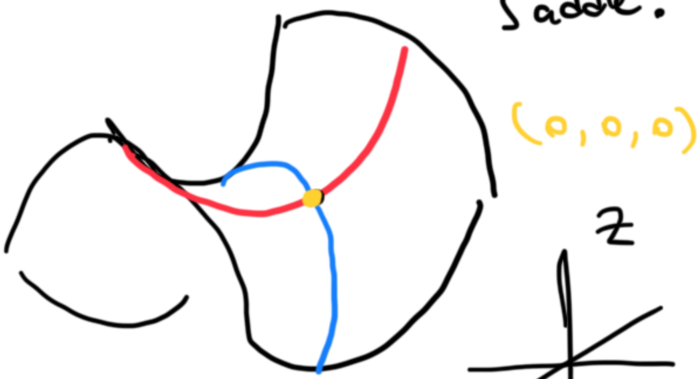
$$\nabla f)(0,0,1) \cdot (x-0, y-0, z-1) = 0$$

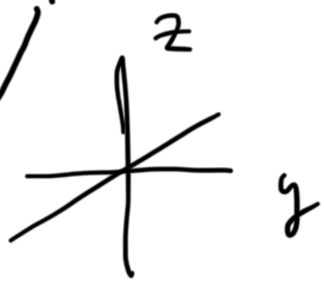
$$(0,0,2) \cdot (x, y, z-1) = 0$$

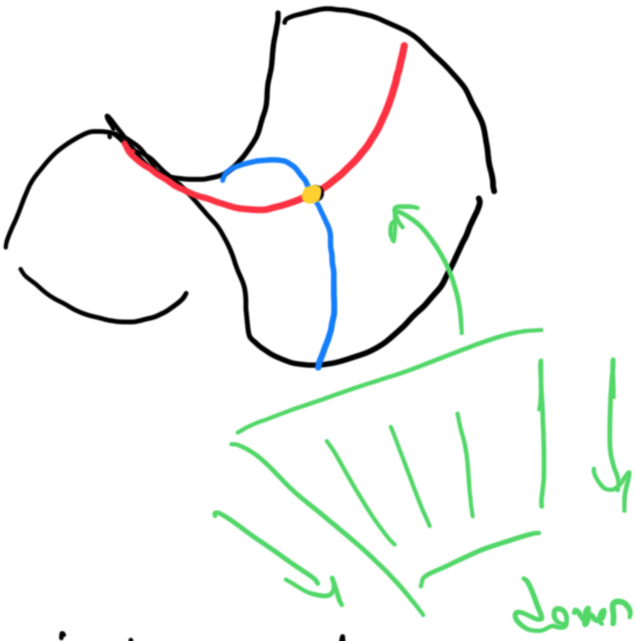
$$2(z-1) = 0 \sim z = 1 \quad !$$

ex: $f(x,y) = x^2 - y^2 = z$

saddle. want tangent plane at $(0,1,-1)$







$$g(x,y,z) = f(x,y) - z$$
$$= (x^2 - y^2) - z$$

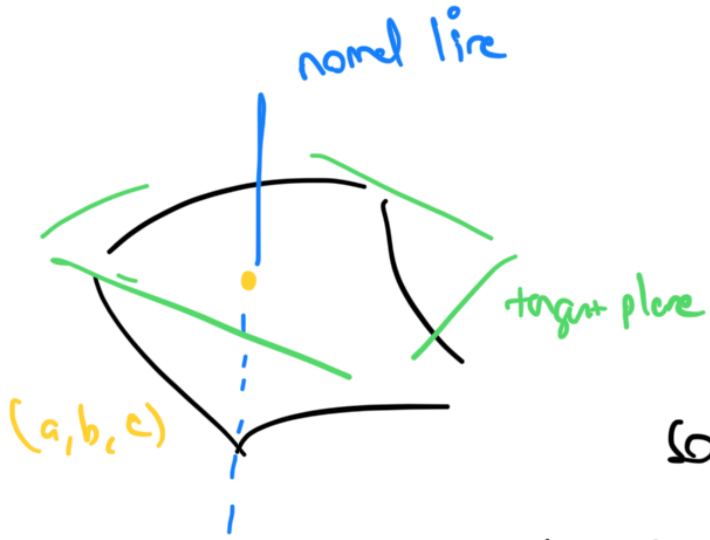
$\nabla g)(x,y,z) = (2x, -2y, -1)$ just proceed as normal.

$$\nabla g)(0,1,-1) = (0,-2,-1) = \vec{n}$$

So $\vec{n} \cdot (x-0, y-1, z-(-1)) = 0$

$$-2(y-1) - 1(z+1) = 1 - z - 2y = 0$$

Want normal line through a point at a surface?



if surface is $f(x,y,z) = c$

then $\nabla f(a,b,c)$ will be $\perp$ to level curve at (a,b,c).

so normal line is

$$L(t) = (a,b,c) + t\,\nabla f(a,b,c)$$

ex: Calculate normal line at point $(0,1,-1)$ as above.

$g(x,y,z) = x^2 - y^2 - z$ so $\nabla g = (2x, -2y, -1)$

thus $L(t) = (0,1,-1) + t(0,-2,-1)$
$= (0, 1-2t, -1-t)$

Why care about linearizing? Use it to approximate.

Recall that $\frac{df}{d\vec{v}}(a,b) = \nabla f(a,b) \cdot \vec{v}$

so we should have that $df(a,b)$ is related to $\nabla f(a,b) \cdot \vec{v}$ and $d\vec{v}$

Choose a small $t \approx 0$. Then we have,

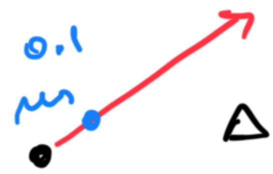
$$f((a,b) + t\vec{v}) \approx f(a,b) + t\,\nabla f(a,b) \cdot \vec{v} \quad (*)$$

first order taylor approximation.

so that is to say $\Delta(\vec{v}) \approx t\,\nabla f(a,b) \cdot \vec{v}$
(text uses df)

ex: $f(x,y) = x^2\cos(y)$ at $(1, \pi/2)$. Want to estimate the change of f if change the point $(1, \pi/2)$ 0.1 cm in the direction of $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

$\vec{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$



$(0, \pi/2)$

$\Delta f \approx (0.1)\,\nabla f(1, \pi/2) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\nabla f(x,y) = (2x\cos y, -x^2\sin(y))$

$\nabla f(1, \pi/2) = (0, -1)$ so $\Delta f \approx (0.1)(0,1) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\Delta f \approx -\frac{\sqrt{2}}{20} \approx -0.0707\ldots$ so that means

$f(\text{point}) - f(0,0) \approx -0.07$ and in fact,

$f((1,\frac{\pi}{2}) + (0.1)\vec{u}) \approx -0.08$ and $f(0,0) = 0$

So good estimate.

Another way is the tangent plane, though it's secretly the same method pretty much.

Recall

$$L(x,y) = f(a,b) + \frac{df}{dx}(a,b)(x-a) + \frac{df}{dy}(a,b)(y-b)$$

ex:

Say you're measuring height of some mountain and at coordinate $(a,b) = (10m, 20m)$ you estimate the elevation is changing -2 in the x-direction and $+3$ in the y-direction. (m/m ?)

if at point $(10m, 20m)$ you measure the height to be $100m$, approximate the height at $(10.1m, 20.3m)$

$$L(x,y) = f(10,20) + \frac{df}{dx}(10,20)(x-10) + \frac{df}{dy}(10,20)(y-20)$$

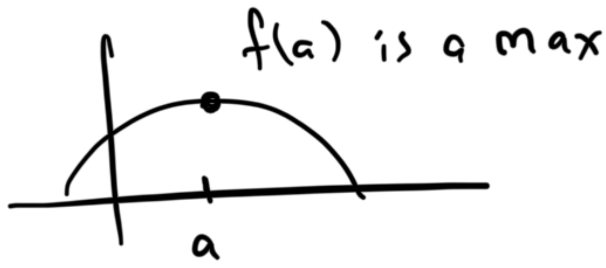
so $L(x,y) = 100 + (-2)(0.1) + (+3)(0.3)$

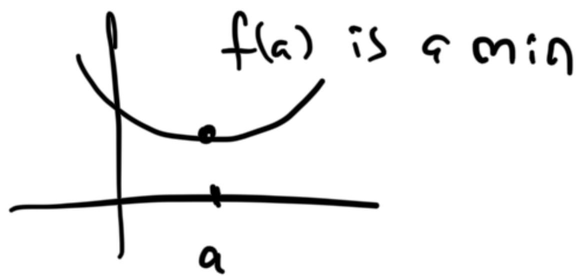
$= 100 - 0.2 + 0.9 = 100.7m$

So we estimate height to be $100.7m$ at $(10.1m, 20.3m)$

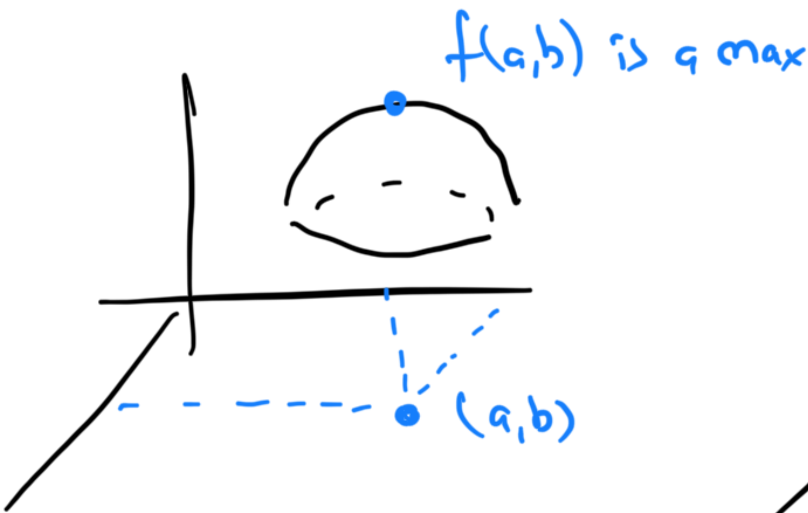
14.7: Extreme-values and saddle points

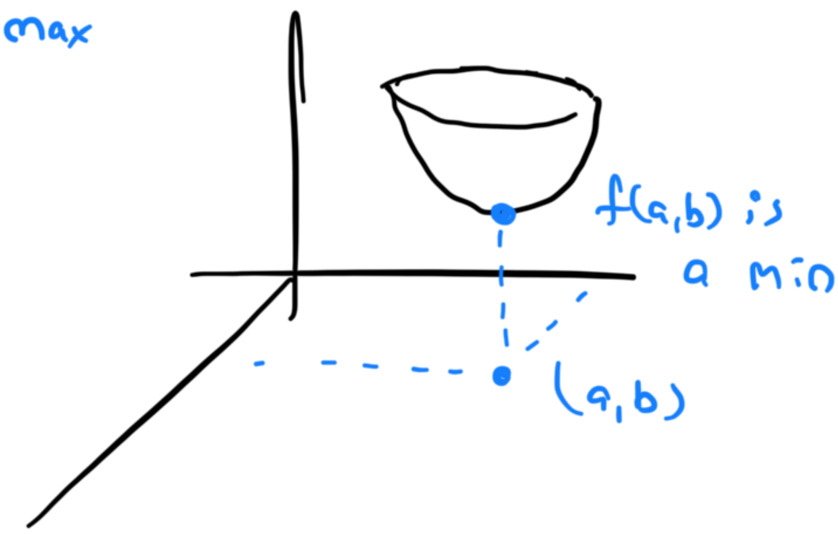
intuitively we have an idea from 1D-calc.



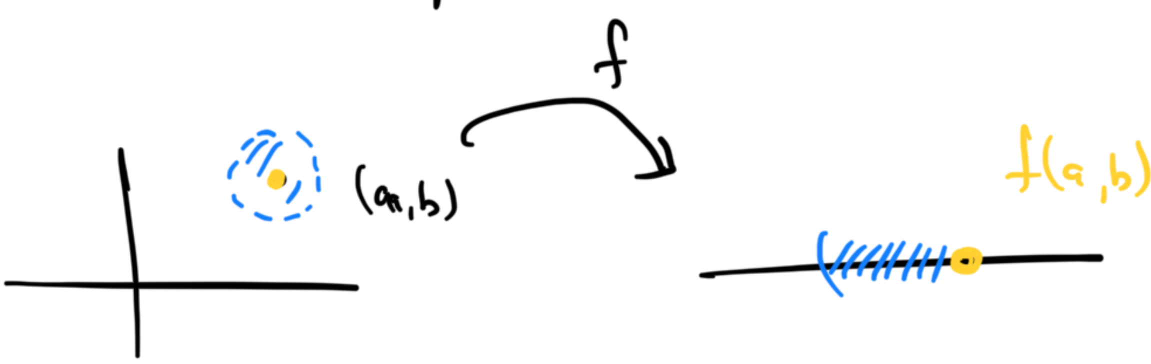


in 2D things are similar, but more subtle.





def: A point (a,b) is a <u>local</u> maximum if find a small ball about (a,b) so that for all points (x,y) in the ball, $f(x,y) \leq f(a,b)$.



$f(a,b)$ is the local max-value.

local min - similar picture / definition. (pg 860)

Recall from Calc 1 if $x = a$ is a max/min

then $f'(a) = 0$

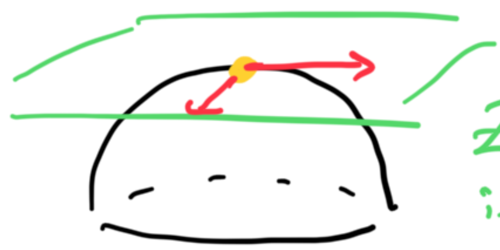
Similar statement holds in 2D.

<u>theorem:</u> if (a,b) is a local max/min,

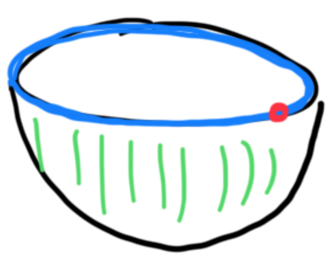
at an interior point of domain, then

$$\frac{df}{dx}(a,b) = \frac{df}{dy}(a,b) = 0$$

picture to have in mind is



interior point stuff is because



e.g: $f(x,y) = x^2 + y^2$ on disk

$x^2 + y^2 \leq 1$.

boundary is local maxes, however

$\frac{df}{dx}(a,b), \frac{df}{dy}(a,b)$ not necessarily zero.

However at point $(a,b) = (0,0)$, it is true that

$\frac{df}{dx}(\ldots) = \frac{df}{dy}(\ldots) = 0$. Also it's a

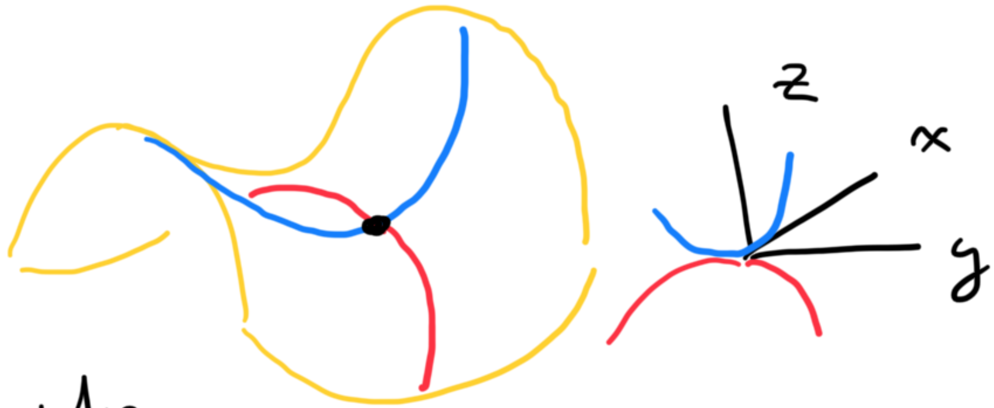
$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Also it's a

local min.

Warning! Just like in 1D, $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$ does not mean (a,b) is local min/max!

eg: $f(x,y) = x^2 - y^2$. Certainly $f_x(0,0) = f_y(0,0) = 0$

However,



So there are points where $f(x,y) > f(0,0)$ and points where $f(x,y) < f(0,0)$.

def: A critical point for a function $f(x,y)$ is a point (a,b) in the <u>interior</u> of the domain of $f(x,y)$ where either

(i). $f_x(a,b) = f_y(a,b) = 0$ or

(ii). either $f_x(a,b)$ or $f_y(a,b)$ does not exist.

Can think of these as 'potential' max/min.

def: A critical point (a,b) is called a saddle point if there are points in every ball about (a,b) that evaluate to values both bigger and less than $f(a,b)$.

How to tell if a critical point is a local max or local min?

theorem: 2nd derivative test for $f(x,y)$.

let (a,b) be a point where f_x and f_y are continuous around it. Assume also that $f_x(a,b) = f_y(a,b) = 0$ so (a,b) is a critical point.

(i). if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ (at (a,b)), then (a,b) is a local max

(ii). if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ then (a,b) is a local min

(iii). if $f_{xx}f_{yy} - f_{xy}^2 < 0$ then (a,b) is a saddle point

(iv). if $f_{xx}f_{yy} - f_{xy}^2 = 0$ then test is inconclusive.

the quantity $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ is called the Hessian or discriminant of f at (a,b).

$= f_{xx}f_{yy} - f_{xy}^2$

ex: $f(x,y) = x^2 + \ldots^2$

$f(x,y) = x^2 + xy - y^2$

$f_x(x,y) = 2x + y \quad f_y(x,y) = x - 2y$

$$\begin{array}{l} 2x + y = 0 \\ x - 2y = 0 \end{array} \Rightarrow y = 0 \text{ and } x = 0 \text{ so } (a,b) = (0,0).$$

$f_{xx}(x,y) = 2 \quad f_{xy}(x,y) = 1 \quad f_{yy}(x,y) = -2$

so $f_{xx} f_{yy} - f_{xy}^2 = -4 - 1 = -5 < 0$

so $(0,0)$ is a <u>saddle point</u>.

ex: $f(x,y) = 3x^3 + 2xy + y^2$

$f_x = 9x^2 + 2y \quad f_y = 2x + 2y$

So setting both to zero, get $x = -y$

and then $9(-y)^2 + 2y = 0 = y(9y + 2) = 0$

So $y = 0$ <u>or</u> $y = -2/9$

so $(a,b) = (0,0)$ <u>or</u> $(a,b) = (2/9, -2/9)$.

Now $f_{xx} = 18x \quad f_{xy} = 2 \quad f_{yy} = 2$

so $f_{xx} f_{yy} - f_{xy}^2 = 36x - 4 = 4(9x - 1)$

$H(0,0) = -4 < 0$ so $(0,0)$ is <u>saddle</u>

$H(2/9, -2/9) = 4(2-1) > 0$ so either max/min.

note $f_{xx}(2/9, -2/9) = 18(2/9) > 0$

So <u>local min</u>

ex: $f(x,y) = 3x^2y - y^3$ Classify all critical points.

$f_x(x,y) = 6xy$ $\quad f_y(x,y) = 3x^2 - 3y^2$.

So critical points mean $6xy = 0$ and $3x^2 - 3y^2 = 0$

so either $x = 0$ & $-3y^2 = 0 \Rightarrow x = y = 0$

<u>or</u> $y = 0$ & $3x^2 = 0 \Rightarrow x = y = 0$

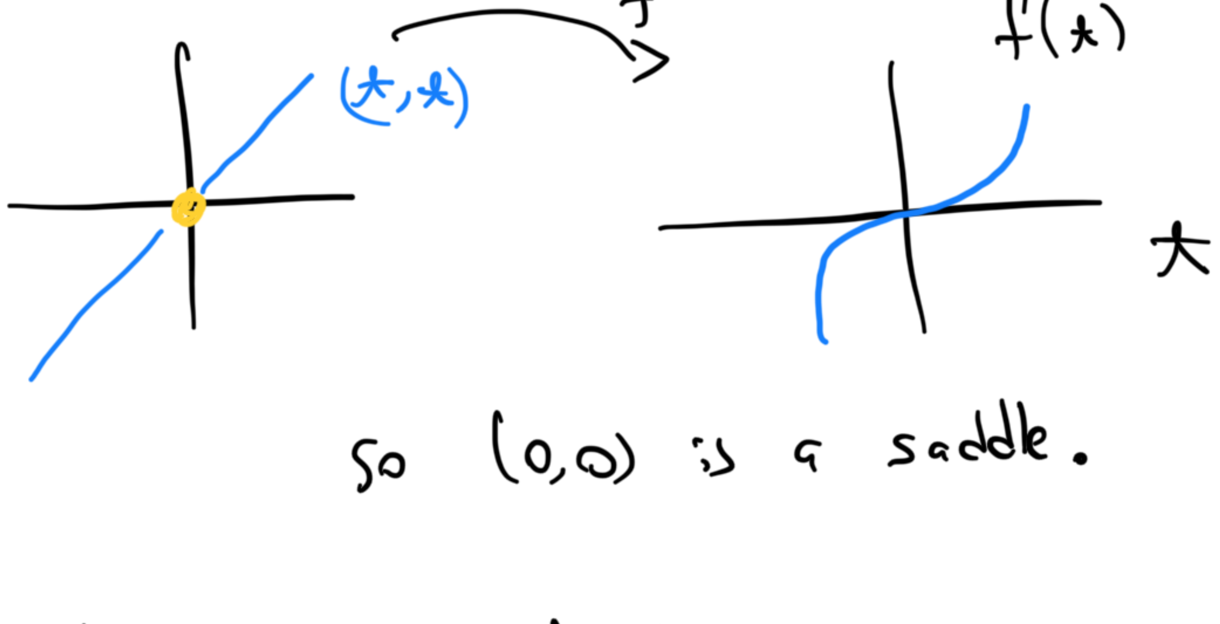
So only one option $x = y = 0$. $(a,b) = (0,0)$

Note $f_{xx}(x,y) = 6y$, $f_{yy}(x,y) = -6y$ and $f_{xy}(x,y) = 6x$. Thus

$H(0,0) = 0$. inconclusive. However, note that $f(x,x) = 3x^3 - x^3 = 2x^3$.

So along $\gamma(t) = (t,t)$, there are points so that $f(0,0) = 0 < f(t,t)$ if $t > 0$

<u>and</u> $f(0,0) = 0 > f(t,t)$ if $t < 0$.



So $(0,0)$ is a saddle.

One last example is boundary problems.

ex: let $f(x,y) = x^2 - y^2$ on unit disk $x^2+y^2 \le 1$.

Calculate local max/min and saddles.

$f_x = 2x, \quad f_y = -2y \quad = 0$ iff $x = y = 0$.

$f_{xx} = 2 \quad f_{yy} = -2 \quad f_{xy} = 0$ so

$H(x,y) = -4 < 0$ so $(0,0)$ is a saddle.

What about boundary? Parameterize by

$\gamma(t) = (\cos t, \sin t)$. Then

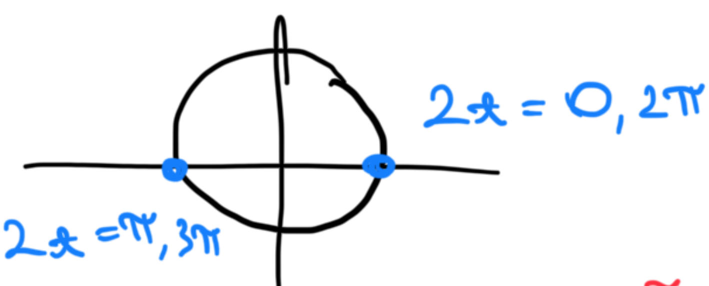
$\gamma(t)$ in white $= (\cos t, \sin t)$

then along $\gamma(t)$ we have

$(f \circ \gamma)(t) = \cos^2 t - \sin^2 t$. Reduce problem to 1-dimension.

$= \cos(2t)$. Minimize/maximize by

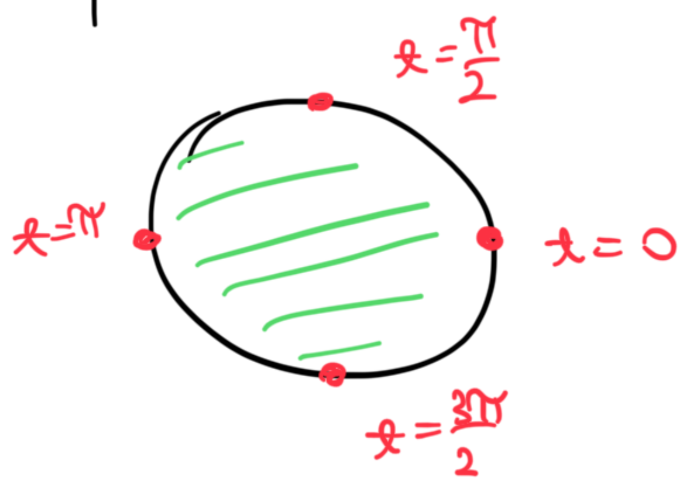
derivative or picture.



$(f \circ \gamma)'(t) = -2\sin(2t)$

so $2t = 0, \pi, 2\pi, 3\pi$ so

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$



Now just check

$f(\gamma(0)) = f(1,0) = 1^2 - 0^2 = 1$

$f(\gamma(\pi/2)) = f(0,1) = 0^2 - 1^2 = -1$

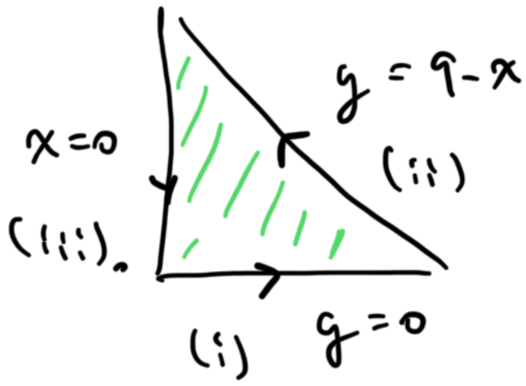
$f(\gamma(\pi)) = f(-1,0) = 1^2 - 0^2 = 1$

$f(\gamma(\frac{3\pi}{2})) = f(0,-1) = 0^2 - 1^2 = -1$

so global max is +1 and global min is −1

ex: Just going to loosely explain ex 6 on pg. 865.

$f(x,y) = 2 + 2x + 4y - x^2 - y^2$ on triangle



(////) you can do, just f_x, f_y, etc.

the other things let's explain.

(i). $y = 0$. $\gamma(t) = (t, 0)$ from $t = 0$ to $t = 9$

so $(f \circ \gamma)(t) = 2 + 2t - t^2$. $(f \circ \gamma)'(t) = 2 - 2t = 0$

iff $t = 1$ $\gamma(1) = (1, 0)$

add to list and $t = 0$

$t = 1$. $(0,0)$, $(0,9)$

(ii). $y = 9 - x$ $\gamma(t) = (9 - t, t)$ from $t = 0$ to $t = 9$.

$(f \circ \gamma)(t) = 2 + 2(9 - t) + 4(t) - (9 - t)^2 - t^2$

$= -61 + 20t - 2t^2$

$(f \circ \gamma)'(t) = 20 - 4t$ so $t = 5$, $\gamma(5) = (4, 5)$

and end points $\gamma(0) = (9, 0)$ and $\gamma(9) = (0, 9)$

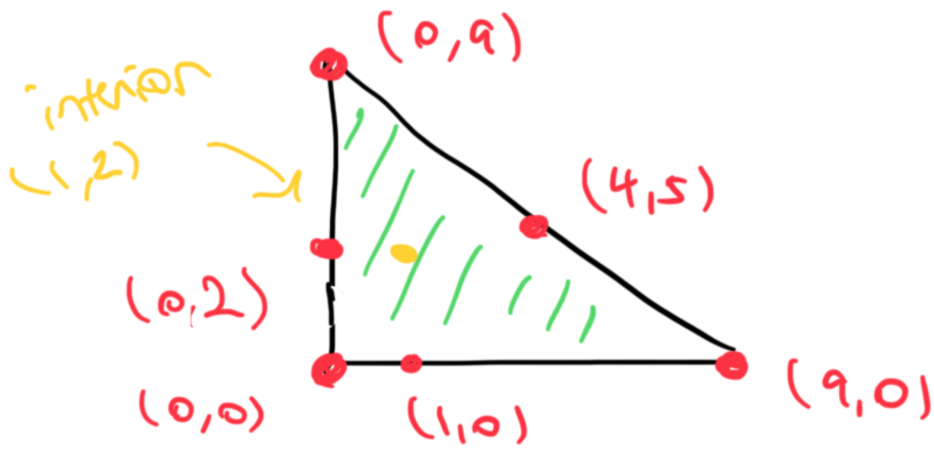
(iii) $x = 0$ $\gamma(t) = (0, 9 - t)$ from $t = 0$ to $t = 9$

$$(f\circ g)(x) = 2 + 4(9-x) - (9-x)^2$$
$$= -43 + 14x - x^2$$

So $(f\circ g)'(x) = 14 - 2x = 0$ at $x =$

So $g(7) = (0, 2)$

So list of points to on boundary are



14.8 : Lagrange Multipliers

Constrained optimization problems.

ex: Calculate the point on the plane

$2x + 3y + z = 5$ closest to the origin.

could think of this as minimizing the function

$$f(x,y,z) = \sqrt{x^2+y^2+z^2} = \text{dist}((x,y,z),(0,0,0)).$$

minimizing ↗ is same as minimizing the square though, so

instead lets take $f(x,y,z) = x^2+y^2+z^2$.

However, we want our point to lie in the plane

$z = 5 - 2x - 3y$. So minimize

$$f(x,y) = x^2 + y^2 + (5-2x-3y)^2$$

$$f_x = 2x + 2(5-2x-3y)(-2) = 2(-10+5x+6y) = 0$$

$$f_y = 2y + 2(5-2x-3y)(-3) = -30 + 12x + 20y = 0$$

so $\begin{aligned} 5x+6y &= 10 \\ 12x+20y &= 30 \end{aligned}$ $\quad (x,y) = (5/7, 15/14)$

Then you could check that $f_{xx}f_{yy} - f_{xy} > 0$

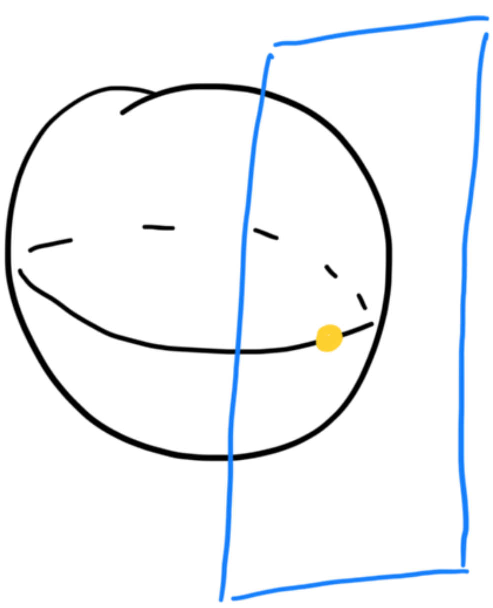
and $f_{xx} = 10 > 0$ so it's a local min,

so the point is $\left(5/7, 15/14, z = 5-2x-3y = 5/14\right)$

closest point

ex: Another way is to imagine level surfaces of

$$f(x,y,z) = x^2+y^2+z^2.$$



at point of contact, the two surfaces will have the same tangent plane (and therefore parallel normals).

this is expressible algebraically as

$\nabla f = \lambda \nabla g$ where f is function of (x,y,z) that defines sphere of radius a and $g(x,y,z)$ equation that defines the plane.

$f(x,y,z) = x^2+y^2+z^2-a^2$ = sphere of radius a

$g(x,y,z) = 2x+3y+z-5$ = plane.

$\nabla f = (2x, 2y, 2z)$ $\quad$ $2x = \lambda 2$

$\nabla g = (2, 3, 1)$ $\quad$ $2y = \lambda 3$ then solve.

$2z = \lambda 1$

$x = \lambda$

$y = \frac{3}{2}\lambda$ and we know $2(\lambda)+3(\frac{3}{2}\lambda)+\frac{1}{2}\lambda-5=0$

$z = \frac{1}{2}\lambda$

$\lambda\left(\frac{4+9+1}{2}\right) = 5$ so $\lambda = 5/7$

thus $(x,y,z) = \left(\frac{5}{7}, \frac{15}{14}, \frac{5}{14}\right)$

theorem of Lagrange Multipliers

Suppose $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq 0$ when $g(x,y,z)=0$.

To find the local max/min values of f subject to constraint $g(x,y,z)=0$ find the values of (x,y,z) that satisfy <u>both</u>

$\nabla f = \lambda \nabla g$ and $g(x,y,z)=0$

(...) Same thing for functions of 2-variables.

(if any) Same thing for [illegible]

ex 14.8.23: Find max / min to $f(x,y,z) = x - 2y + 5z$

on the sphere $x^2 + y^2 + z^2 = 30$.

Constraint $g(x,y,z) = x^2+y^2+z^2-30$

function $f(x,y,z) = x - 2y + 5z$

$\nabla f = (1, -2, 5) \qquad \nabla g = (2x, 2y, 2z)$

$$\begin{aligned} 1 &= \lambda 2x \\ -2 &= \lambda 2y \\ 5 &= \lambda 2z \end{aligned} \quad \text{or} \quad \begin{aligned} \lambda &= 2x \\ -2\lambda &= 2y \\ 5\lambda &= 2z \end{aligned} \qquad \begin{aligned} x &= \lambda/2 \\ y &= -\lambda \\ z &= 5/2\,\lambda \end{aligned}$$

& plug into g solve λ.

$$\left(\frac{\lambda}{2}\right)^2 + (-\lambda)^2 + \left(\frac{5}{2}\lambda\right)^2 - 30 = 0$$

$$\lambda^2\left(\frac{1}{4} + 1 + \frac{25}{4}\right) = 30$$

$$\lambda^2\left(\frac{30}{4}\right) = 30 \quad \text{so} \quad \lambda^2 = 4,\ \lambda = \pm 2.$$

thus $(a,b,c) = (1, -2, 5)$ or $(a,b,c) = (-1, 2, -5)$

$f(1,-2,5) = 1 - 2(-2) + 25 = 30$ max & $f(-1,-2,-5) = -30$ min

ex: 14.8.14

temperature on plate is $T(x,y) = 4x^2 - 4xy + y^2$.

ant on circle of radius 5 about origin walks

[illegible]

[illegible]. find highest / lowest temperatures.

function: $T(x,y) = 4x^2 - 4xy + y^2$

Constraint: $g(x,y) = x^2 + y^2 - 25$

$\nabla T = \lambda \nabla g \qquad \nabla T = (8x - 4y, -4x + 2y)$

$\nabla g = (2x, 2y)$

So $(8x - 4y) = \lambda 2x$,

$(-4x + 2y) = \lambda 2y$

$2y - 2y\lambda = 4x$

$2y(1-\lambda) = 4x$

$\frac{1}{2} y(1-\lambda) = x$

$4x - 2y = \lambda x$

$2y(1-\lambda) - 2y = \lambda\left(\frac{1}{2} y(1-\lambda)\right)$

$-4y\lambda = \lambda y(1-\lambda)$

$0 = \lambda\left(y(1-\lambda) + 4y\right) = \lambda(5y - y\lambda)$

$0 = y\lambda(5-\lambda)$

$y = 0 \Rightarrow x^2 + y^2 = x^2 = 25$ so $(-5, 0)$ & $(+5, 0)$

$\lambda = 0 \Rightarrow \nabla T = \vec{0} \quad 8x - 4y = 0$

$-4x + 2y = 0$ $\Rightarrow y = 2x$

so $x^2 + 4^2 x^2 = 25 \Rightarrow x = \pm\sqrt{5}$

$(-\sqrt{5}, -2\sqrt{5})$ or $(+\sqrt{5}, 2\sqrt{5})$

$\lambda = 5 \Rightarrow y(-4) = 2x$ so $4y^2 + y^2 = 25$ so

$x = -2y$ $\qquad y = \pm\sqrt{5}$

$(2\sqrt{5}, -\sqrt{5})$ or $(-2\sqrt{5}, \sqrt{5})$

$f(\pm 5, 0) = 100 \leftarrow$ doesn't count bc $\nabla g \neq \vec{0}$

$f(\sqrt{5}, 2\sqrt{5}) = 0 \leftarrow$ min

$f(2\sqrt{5}, \sqrt{5}) = 125 \leftarrow$ max ex: 14.5.7 (two constraints)

find maximum of $f(x,y,z) = x^2 + 2y - z^2$

subject to $2x - y = 0$ and $y + z = 0$

here $\nabla f = \lambda \nabla g + u \nabla h$

where f = above and $g = 2x - y$ $h = y + z$

$\nabla f = (2x, 2, -2z)$

$\nabla g = (2, -1, 0)$

$\nabla h = (0, 1, 1)$

$(2x, 2, -2z) = (2\lambda, -\lambda, 0) + (0, u, u)$

$2x = 2\lambda$

$2 = -\lambda + u \Rightarrow u = 2 + \lambda$

$-2z = u$

$x = \lambda$

$u = 2 + \lambda = -2z$

$z = -1 - \frac{\lambda}{2}$

$2x - y = 0$

so $2\lambda + z = 0$ bc $z = -y$

$2\lambda - 1 - \frac{\lambda}{2} = 0$

$\frac{3\lambda}{2} = 1$ so $\lambda = \frac{2}{3}$

and $u = 8/3$

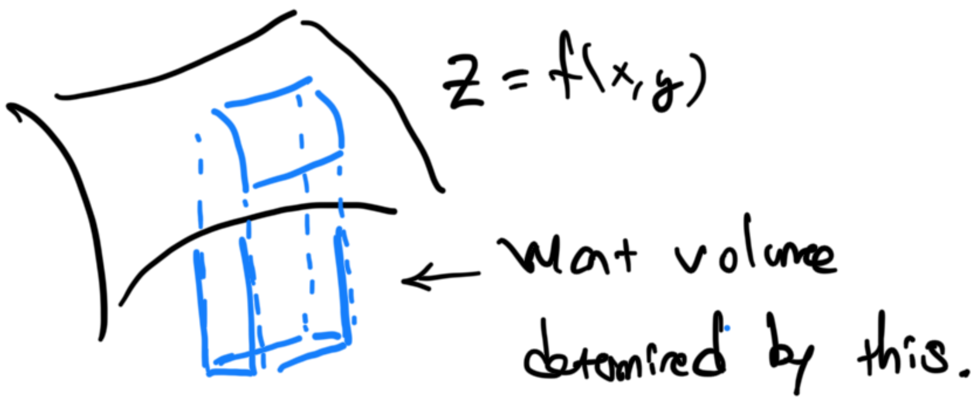
so $x = 2/3$

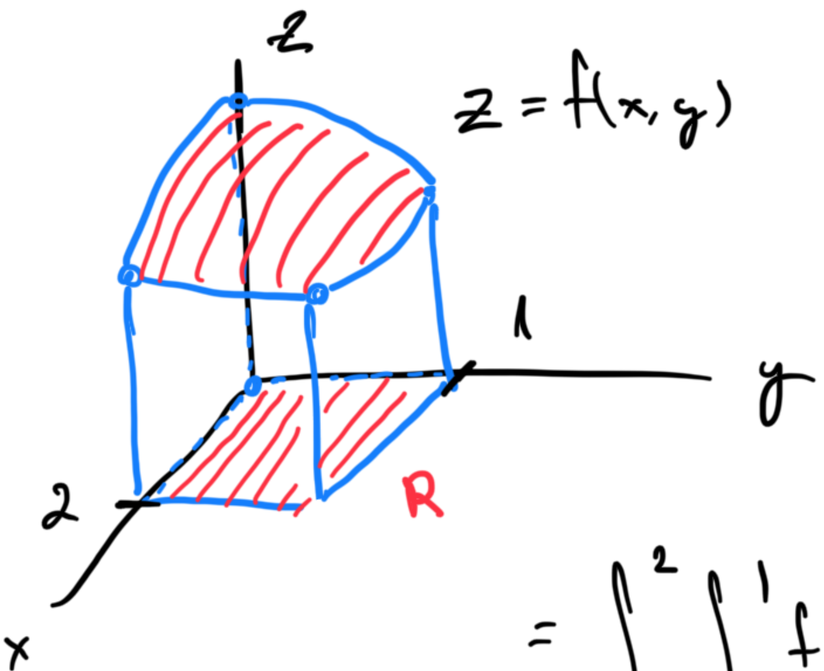
$z = -1 - 1/3 = -4/3$

$y = 4/3$

15.1: Double integrals over rectangles.

let $z = f(x,y)$ be some function.





$$\text{Volume} = \iint_R f(x,y)\, dA$$

$$= \int_0^1 \int_0^2 f(x,y)\, dx\, dy$$

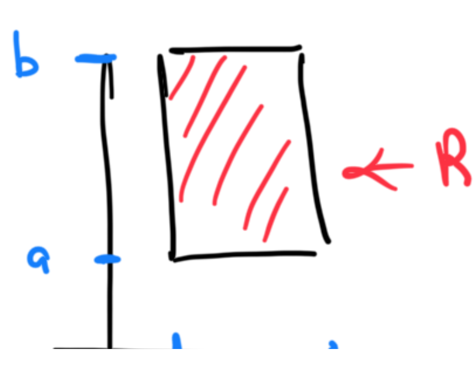
$$= \int_0^2 \int_0^1 f(x,y)\, dy\, dx$$

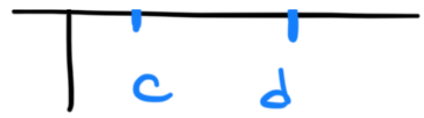
true in general: Fubini's theorem:

let $f(x,y)$ be defined over rectangular region R, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dx\, dy$$

$$= \int_c^d \int_a^b f(x,y)\, dy\, dx$$





ex: $\displaystyle\iint_R 4-x-y^2$ on region $R = \begin{matrix} 0 \le x \le 4 \\ 0 \le y \le 2 \end{matrix}$

$$= \int_0^4 \int_0^2 4-x-y^2 \, dy \, dx = \int_0^4 4y - xy - \frac{1}{3}y^3 \Big|_0^2 \, dx$$

$$= \int_0^4 8 - 2x - \frac{8}{3} \, dx = \frac{16}{3}\int_0^4 dx - 2\int_0^4 x \, dx$$

$$= \frac{16}{3}(x)\Big|_0^4 - 2\left(\frac{1}{2}x^2\right)\Big|_0^4 = \frac{16}{3}\cdot 4 - 2\cdot 8$$

$$= 16/3$$

$\underline{\underline{\text{or}}}$ $$\int_0^2\int_0^4 4-x-y^2 \, dx \, dy = \int_0^2 4x - \frac{1}{2}x^2 - y^2 x \Big|_0^4 \, dy$$

$$= \int_0^2 16 - 8 - 4y^2 \, dy = 8\int_0^2 dy - 4\int_0^2 y^2 \, dy$$

$$= 8(y)\Big|_0^2 - \frac{4}{3}y^3\Big|_0^2 = 16 - \frac{32}{3} = 16/3$$

ex: $\displaystyle\int_{-1}^{1}\int_{-\pi}^{\pi} \frac{\sin(xy^2)}{1+y^2} \, dy \, dx$??

However... $\displaystyle\int_{-\pi}^{\pi}\int_{-1}^{+1} \frac{\sin(xy^2)}{1+y^2} \, dx \, dy$

$$= \int_{-\pi}^{\pi} \frac{-\cos(xy^2)}{y^2(1+y^2)} \Big|_{x=-1}^{x=1} dy = \int_{-\pi}^{\pi} \frac{-\cos(y^2)+\cos(-y^2)}{y^2(1+y^2)} dy$$

$$= \int_{-\pi}^{\pi} \frac{0}{y^2(1+y^2)} dy = 0$$

(Sometimes reversing order of integration helps!)

15.2. Integration over general regions

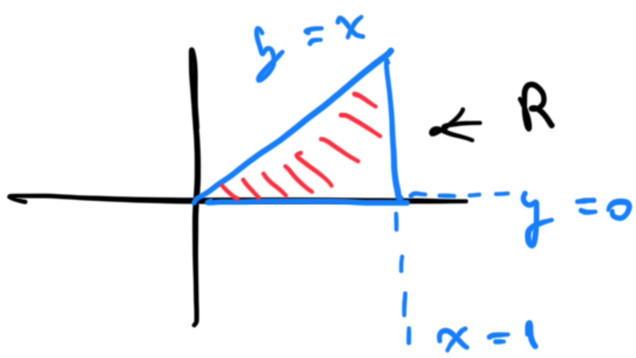
We know how to integrate over rectangles.

What about general regions?

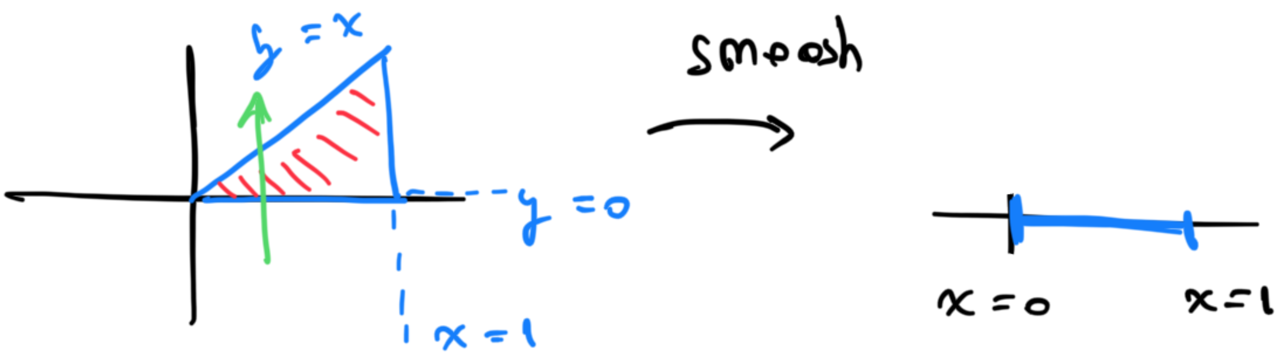
e.g.

$$\iint_R f(x,y)\, dA = ?$$

ex: $\iint_R x+y^2 \, dA$ over triangle defined by $y = x$, $y = 0$, $x = 1$.



(i).



enter at $y=0$

exit at $y=x$

$$\int_{\text{enter } y=0}^{\text{exit } y=x} f(x,y)\,dy \quad \text{then} \quad \int_{x=0}^{x=1}\int_{y=0}^{y=x} x+y^2\,dy\,dx$$

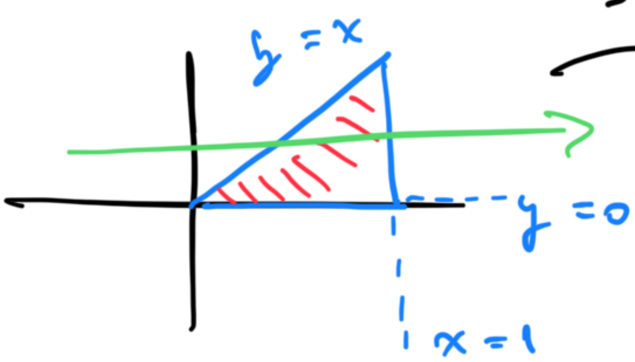
$$= \int_0^1\int_0^x x+y^2\,dy\,dx = \int_0^1 xy+\tfrac{1}{3}y^3\Big|_0^x\,dx$$

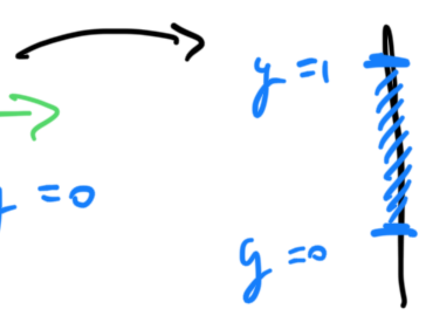
$$= \int_0^1 x^2+\tfrac{1}{3}x^3\,dx = \tfrac{1}{3}x^3+\tfrac{1}{12}x^4\Big|_0^1$$

$$= \tfrac{1}{3}+\tfrac{1}{12} = 5/12$$

<u>or</u>

smoosh



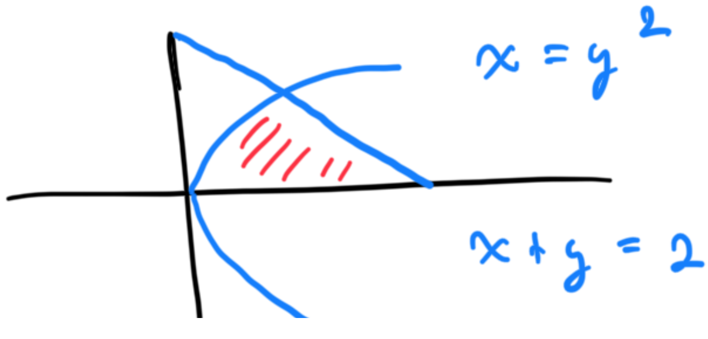


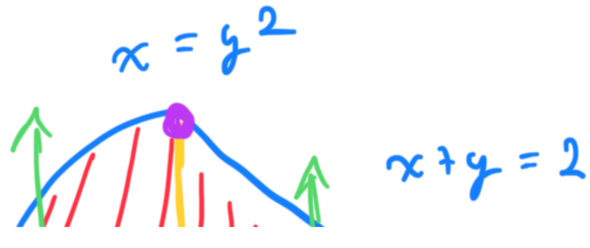
$$\int_{\text{enter } x=y}^{\text{exits } x=1} f(x,y)\,dx \quad \text{then} \quad \int_{y=0}^{y=1}\int_{\text{enter } x=y}^{\text{exits } x=1} f(x,y)\,dx\,dy$$

can do integral, same answer.

ex: Integrate a function $f(x,y)$ over

Write it in 2-ways





$x=2$

$y=0$

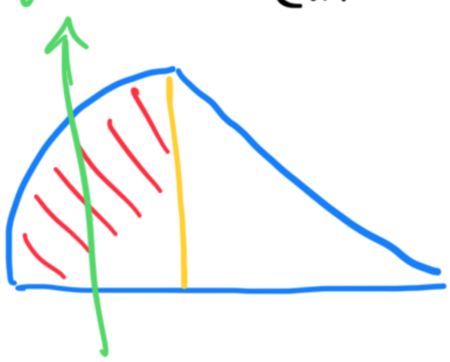
notice where you enter and exit now depends on which x-value you're at! point in purple is where curves intersect $y^2 = 2-y$

$y^2+y-2=0$ solutions $y=1$

$=(y+2)(y-1)=0$ and $y=-2$

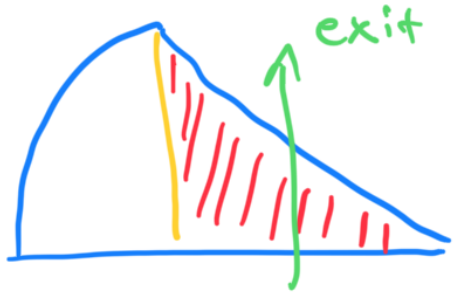
so $I = \int_0^1 \int_{\text{enter}}^{\text{exit}} \_\_\, dy\, dx + \int_1^2 \int_{\text{enter}}^{\text{exit}} \_\_\, dy\, dx$

exit $y=\sqrt{x}$



enter $y=0$

exit $y=2-x$

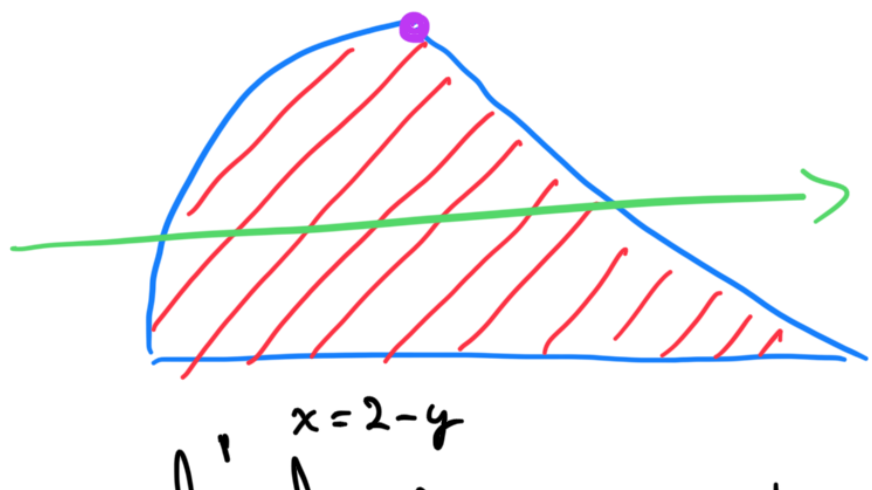


enter $y=0$

so $I = \int_0^1 \int_{y=0}^{y=\sqrt{x}} f(x,y)\, dy\, dx + \int_1^2 \int_{y=0}^{y=2-x} f(x,y)\, dy\, dx$

$(1,1)$

<u>or</u>



exits $x=2-y$

enters $x=y^2$

so $I = \int_0^1 \int_{x=y^2}^{x=2-y} f(x,y)\, dx\, dy$

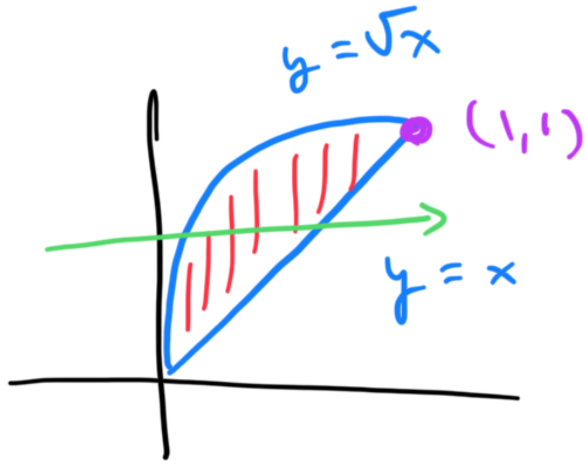
In particular if $f(x,y)\equiv 1$, then we just get the area!

e.g. $A = \int_0^1 \int_{y^2}^{2-y} dx\, dy = \int_0^1 2 - y - y^2\, dy$

$= 2y - \frac{1}{2} y^2 - \frac{1}{3} y^3 \Big|_0^1$

$= 2 - 1/2 - 1/3 = 7/6$

ex: $\int_0^1 \int_x^{\sqrt{x}} \frac{\sin(y)}{y}\, dy\, dx$



entrs $x = y^2$ exits $x = y$

$= \int_0^1 \int_{y^2}^{y} \frac{\sin(y)}{y}\, dx\, dy$

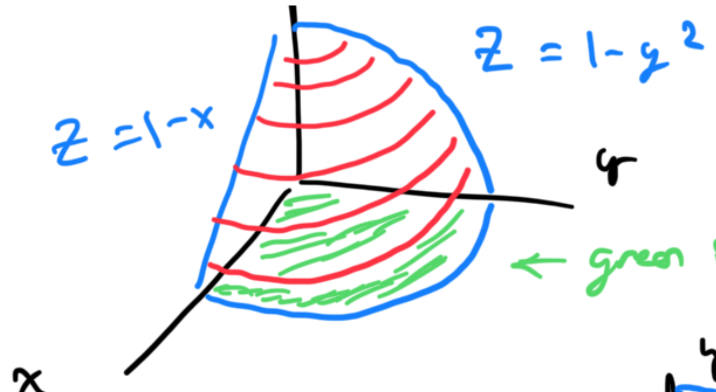
$= \int_0^1 \frac{\sin(y)}{y}\, x \Big|_{y^2}^{y}\, dy = \int_0^1 \frac{\sin(y)}{y} (y - y^2)\, dy$

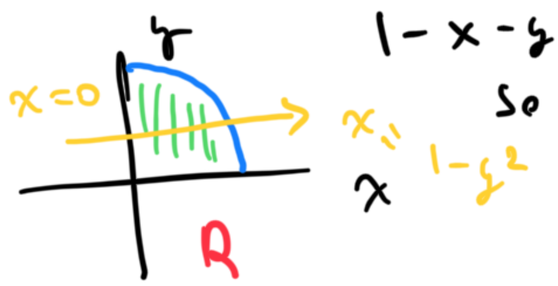
$= \int_0^1 \underbrace{\sin(y)}_{\text{simple}} - \underbrace{y \sin(y)}_{\text{integration by parts}}\, dy = 1 - \sin(1)$

two applications: Volume of a section of a surface and average value.

ex: Calculate volume determined by $z = 1 - x - y^2$ in first octant bounded by z.

1





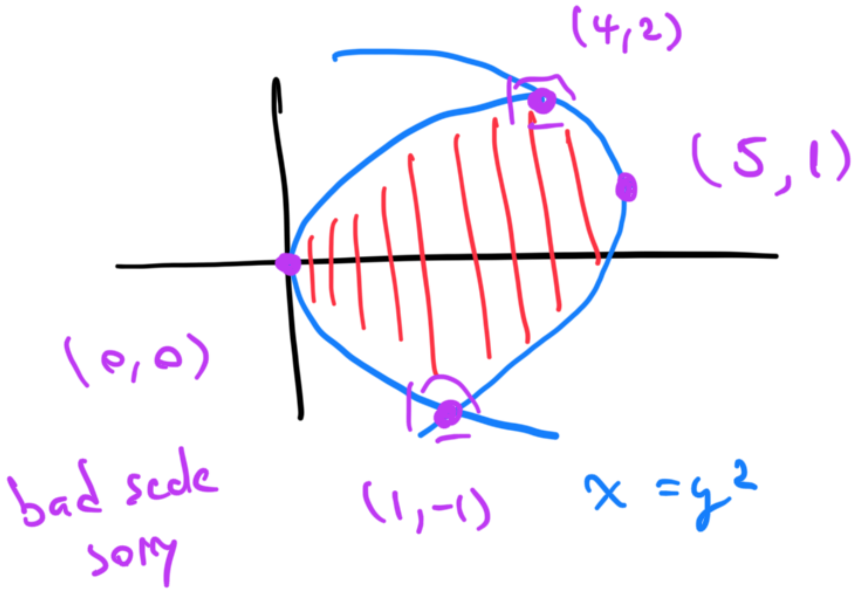
$1 - x - y^2 = 0$
so $x = 1 - y^2$

$$V = \iint_R z \, dA = \int_0^1 \int_0^{1-y^2} 1 - x - y^2 \, dx \, dy = 4/15$$

15.3. Areas by double integrals

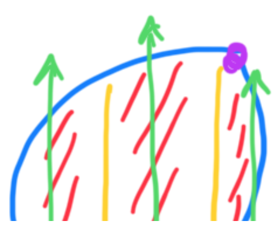
As mentioned earlier if integrate $f(x,y) \equiv 1$ on a region R, you just get Area of region.

ex: Calculate area of region defined by $x = y^2$ and $x = 5 - (y-1)^2$



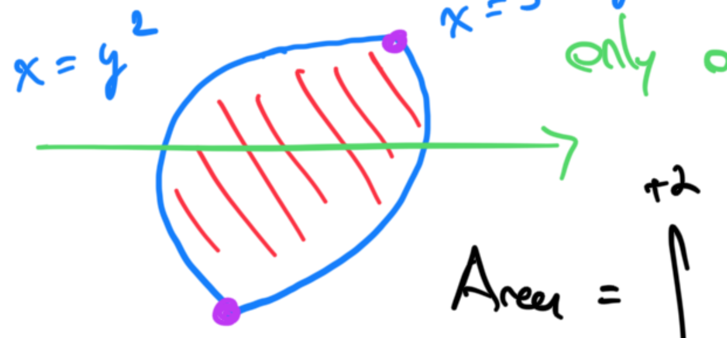
solve for boxed points by setting $y^2 = 5 - (y-1)^2$.

(i).



3 different integrals :(

(ii).



only one integral

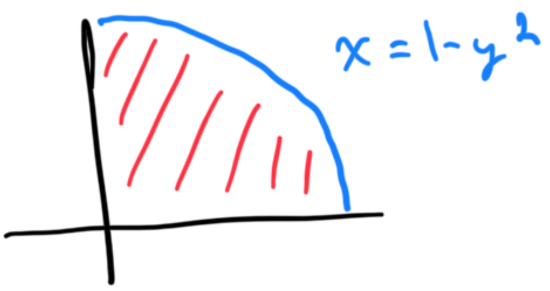
$$\text{Area} = \int_{-1}^{+2} \int_{x=y^2}^{x=5-(y-1)^2} dx\, dy$$

ex: Calculate average value of a function over region R.

$$(*) \quad \text{Avg}(f, R) = \iint_R f\, dA \Big/ \text{Area of } R$$

so for example, with $f(x,y) = 1 - x - y^2$ and R from 15.2

$$\text{Avg}(f, R) = \underbrace{\int_0^1 \int_0^{1-y^2} 1 - x - y^2\, dx\, dy}_{(i).} \Big/ \underbrace{\int_0^1 \int_0^{1-y^2} dx\, dy}_{(ii).}$$



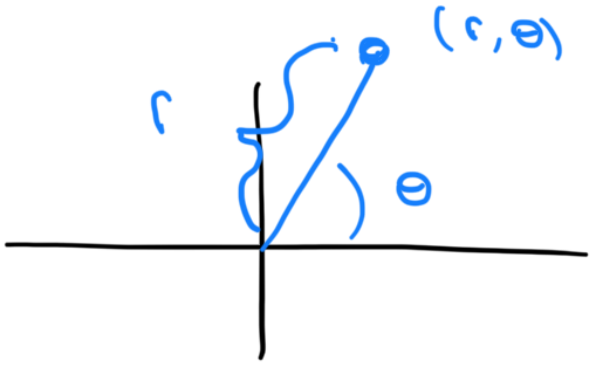
(i). Already did, 4/15

(ii). $\int_0^1 \int^{1-y^2} dx\, dy = \int^1 1 - y^2\, dy = y - \frac{1}{3}y^3 \Big|^1 = 2/$

so $\operatorname{Avg}(f, R) = \frac{4/15}{2/3} = \frac{2}{5}$

15.4: Polar Coordinate Integrals

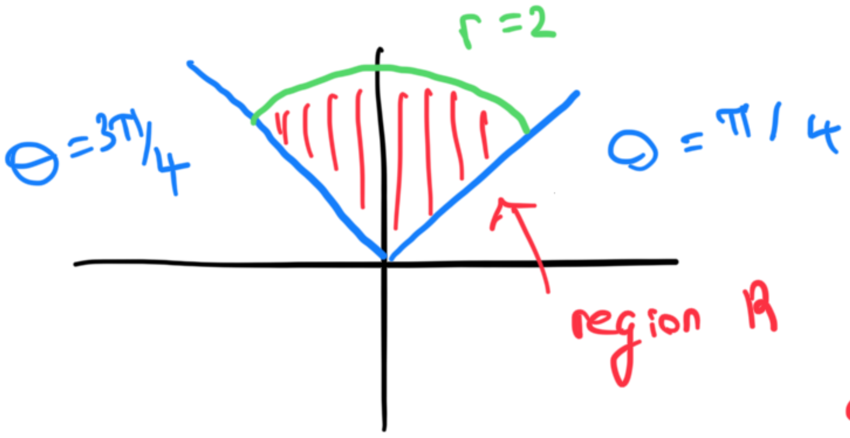
Polar coordinates are another way of specifying points in the plane instead of (x, y).



(r, θ)

One issue however is that coordinates are no longer unique. e.g.: $(1, \pi/4)$ and $(1, \pi/4 + 2\pi)$ are same point. Also $(0, \pi)$ and $(0, \pi/2)$ are same point. True for any $(0, \theta)$.

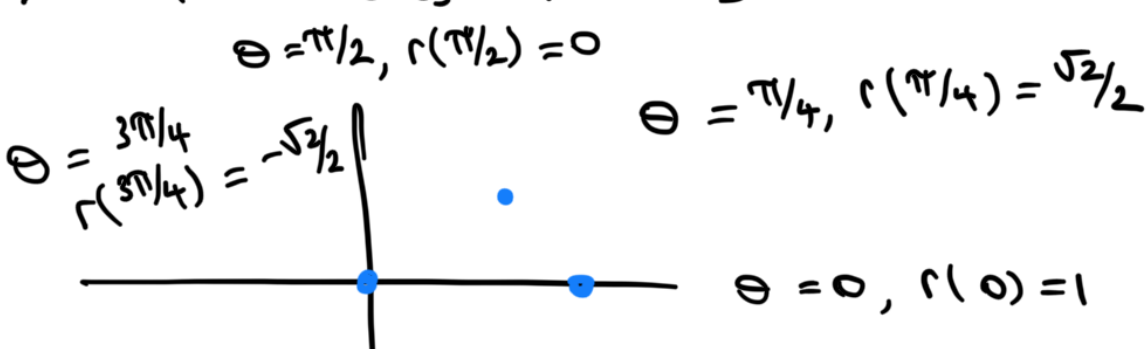
ex: What does all (r, θ) with $r = 2$ and $\pi/4 \le \theta \le 3\pi/4$ look like?



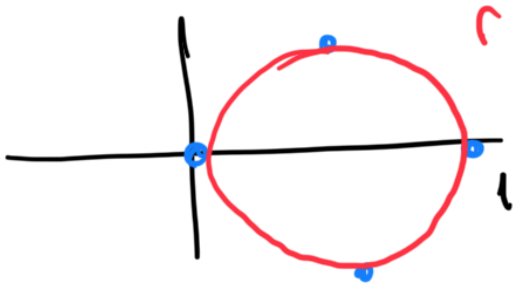
in green we get $r = 2$, $\pi/4 \le \theta \le 3\pi/4$

in red we get region $0 \le r \le 2$, $\pi/4 \le \theta \le 3\pi/4$

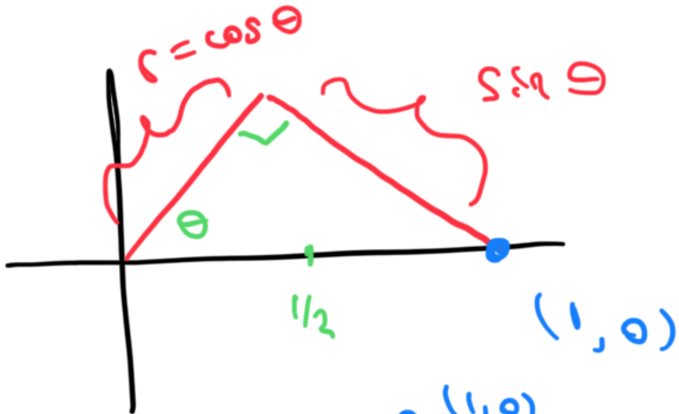
ex: What does $r = \cos\theta$ look like?

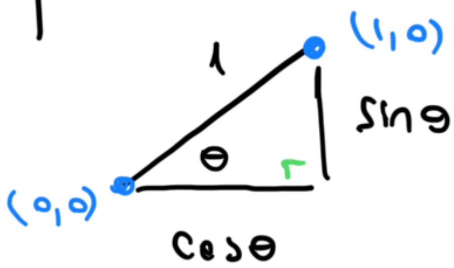


keep doing ...

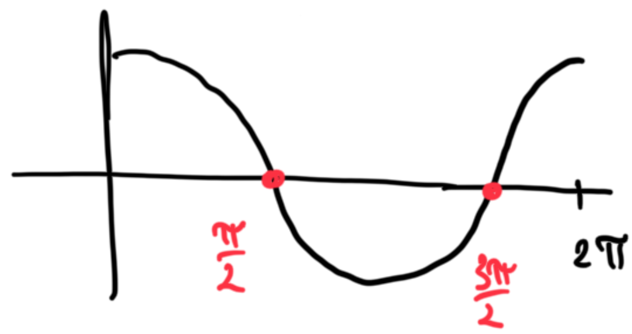


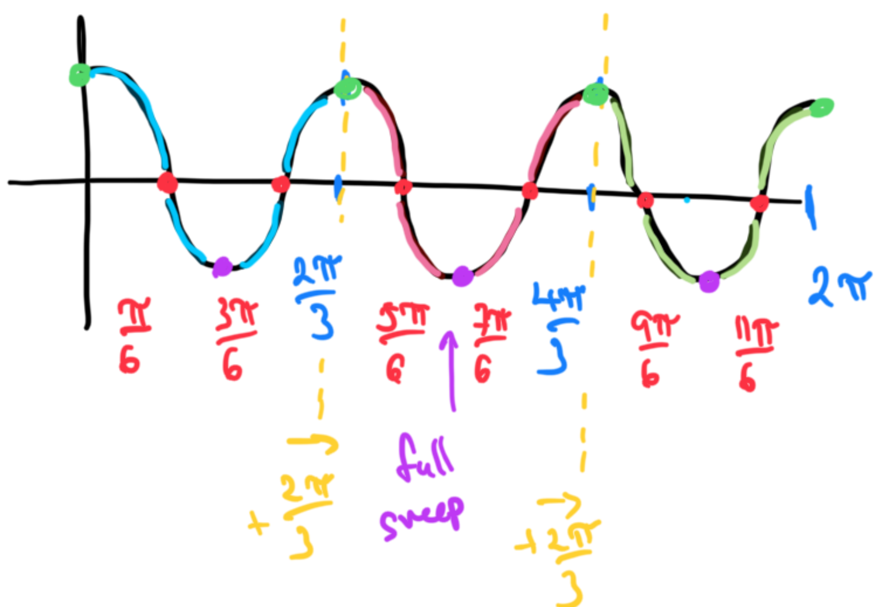
one way to see this

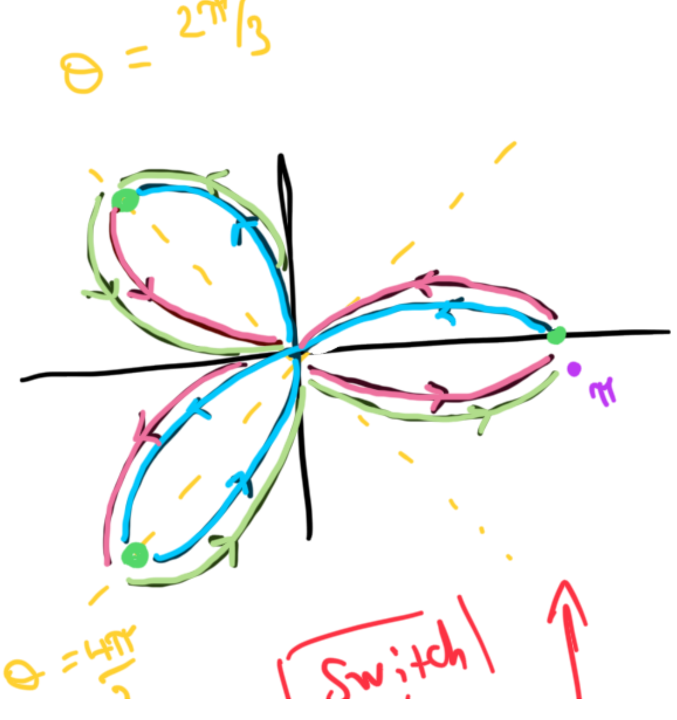




ex: $r = \cos 3\theta$



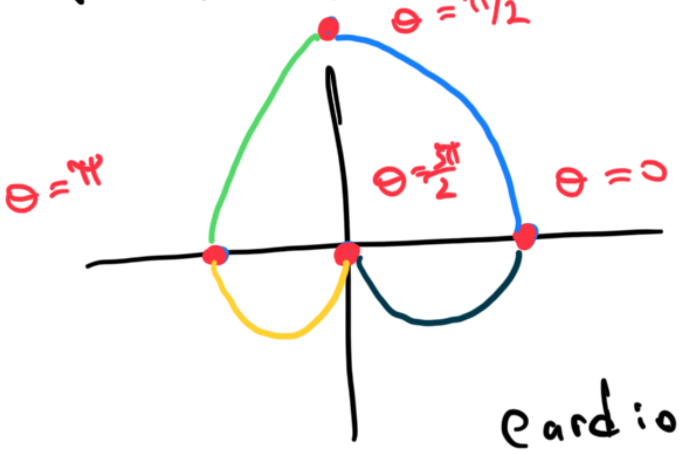


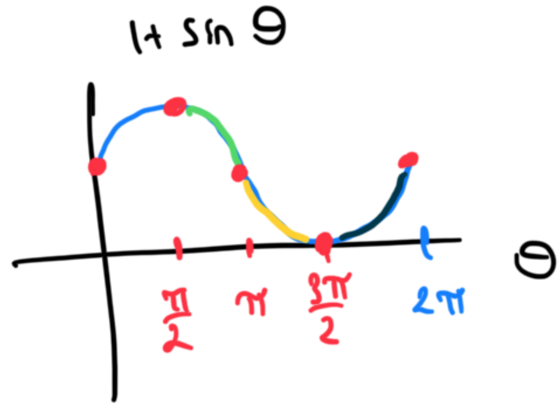


don't sweat this example too much. Just wanted to show how this looks mostly.

order

ex: $r = 1 + \sin\theta$





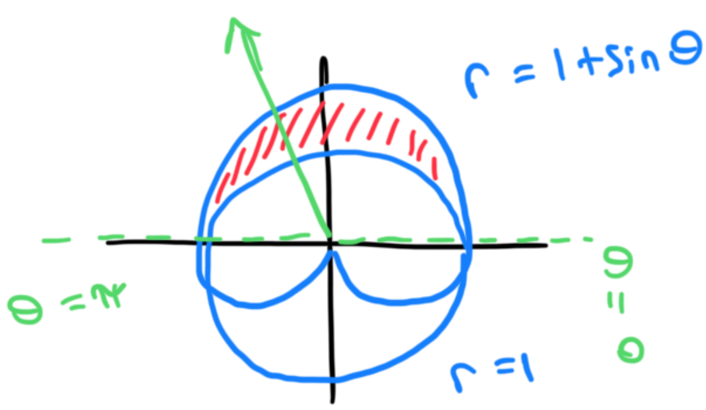
cardiod

Integrate functions $f(r,\theta)$ over regions R

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\,\underbrace{r\,dr\,d\theta}$$

$dA = r\,dr\,d\theta$

ex: Calculate area between circle radius 1 and cardiod $r = 1+\sin\theta$ as illustrated



$$\iint_R 1 \; r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_{r=1}^{r=1+\sin\theta} r\,dr\,d\theta$$

enters at $r=1$
exits at $r = 1+\sin\theta$

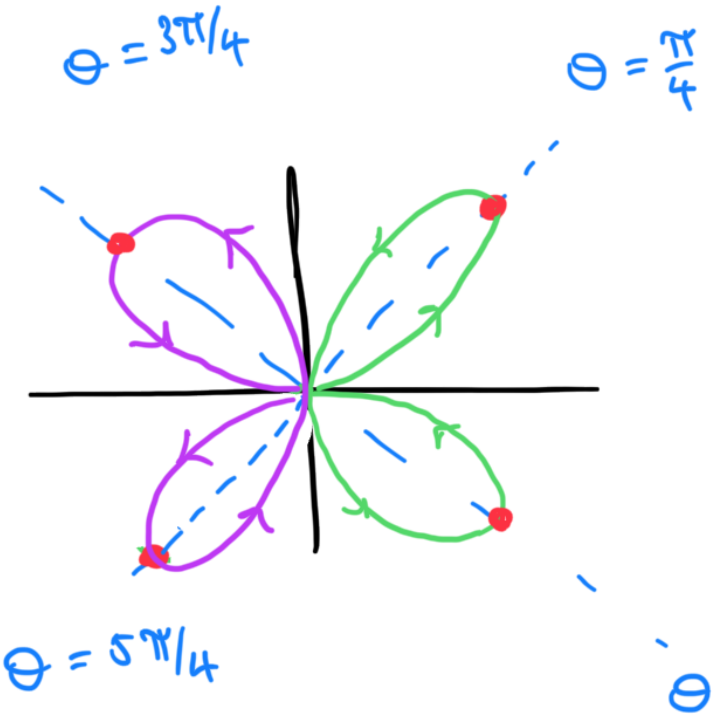
$$= \int_0^{\pi} \left(\frac{1}{2}r^2\right)\Bigg|_1^{1+\sin\theta} d\theta$$

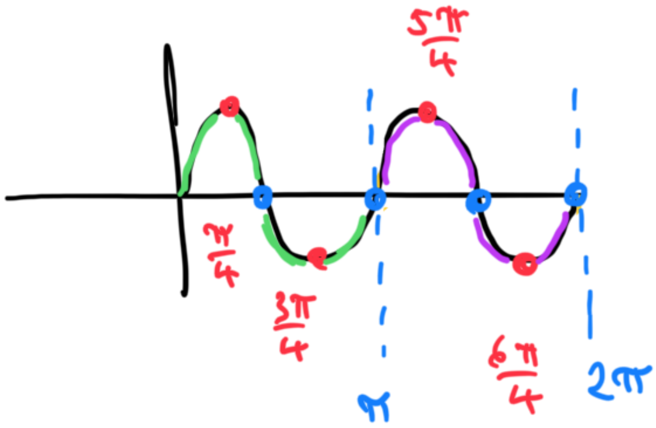
$$= \frac{1}{2}\int_0^{\pi}\left[(1+\sin\theta)^2 - 1^2\right] d\theta$$

$$= \frac{1}{2}\left(\int_0 2\sin(\theta) + \sin^2\theta \, d\theta \right)$$

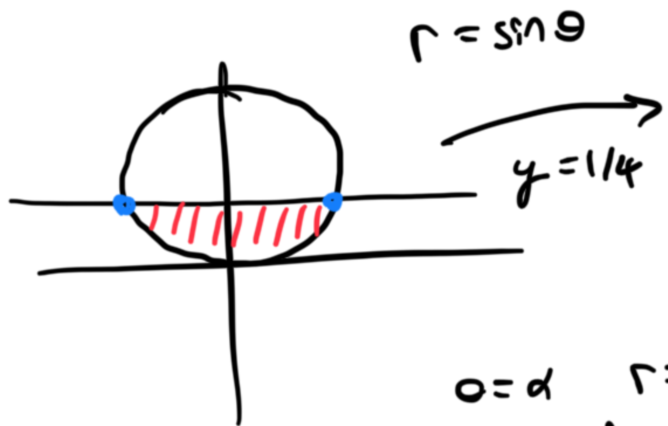
$$= \frac{1}{2}\left(\frac{8+\pi}{2} \right) = 2 + \frac{\pi}{4}$$

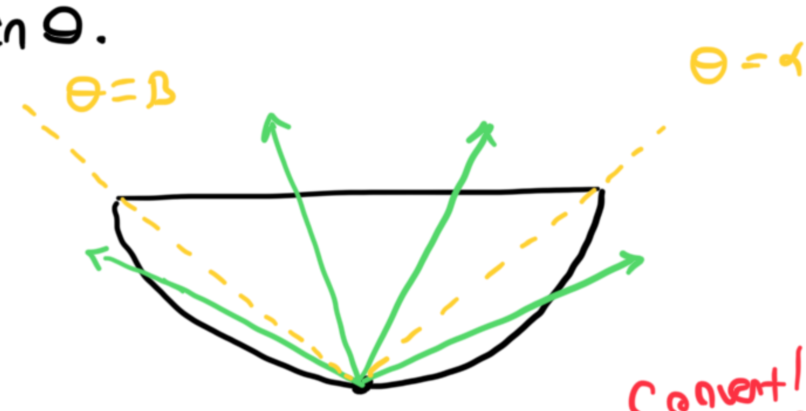
ex: $r = \sin 2\theta$





ex: Calculate area below $y = \frac{1}{4}$ above $y = 0$ and in between $r = \sin\theta$.





Convert!

$$\text{so Area} = \underbrace{\int_{\theta=0}^{\theta=\alpha} \int_{r=0}^{r=\sin\theta} r\,dr\,d\theta}_{(i)} + \underbrace{\int_{\theta=\alpha}^{\theta=\beta} \int_{r=0}^{y=1/4} r\,dr\,d\,d\theta}_{(ii)}$$

$$+ \underbrace{\int_{\theta=\beta}^{\theta=\pi} \int_{r=0}^{r=\sin\theta} r\,dr\,d\theta}_{(iii)}.$$

Cartesian to Polar and vice versa

1. $x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2+y^2}$

$\tan\theta = y/x$ (kinda)

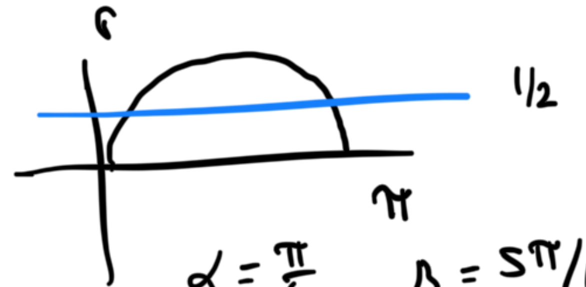
so $y = \frac{1}{4} \to r\sin\theta = \frac{1}{4} \Rightarrow r = \frac{1}{4}\csc\theta$ so

(ii) is $\displaystyle\int_{\theta=\alpha}^{\theta=\beta}\int_{r=0}^{r=\frac{1}{4}\csc\theta} r\,dr\,d\theta$

2. Find α and β. That's when $r = \frac{1}{4}\csc\theta = \sin\theta$

in other words $\frac{1}{4} = \sin^2\theta \qquad \sin\theta = \frac{1}{2}$

in $0 \le \theta \le \pi$ so



$\alpha = \frac{\pi}{6} \qquad \beta = 5\pi/6$

so (i) $\displaystyle\int_{\theta=0}^{\theta=\pi/6}\int_{r=0}^{r=\sin(\theta)} r\,dr\,d\theta$

(ii). $\displaystyle\int_{\theta=\pi/6}^{\theta=5\pi/6}\int_{r=0}^{r=\frac{1}{4}\csc\theta} r\,dr\,d\theta$

(iii). $\displaystyle\int_{\theta=5\pi/6}^{\theta=\pi}\int_{r=0}^{r=\sin(\theta)} r\,dr\,d\theta$

(i) $= \frac{1}{48}(2\pi - 3\sqrt{3})$

(ii) $= \sqrt{3}/16$

(iii) $= \frac{1}{48}(2\pi - 3\sqrt{3})$

so $\sum_i$ = Area

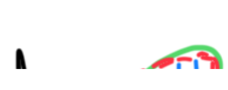
$= \frac{1}{48}(4\pi - 3\sqrt{3})$

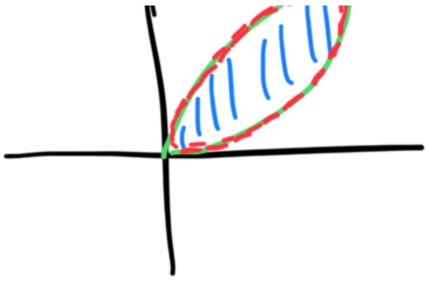
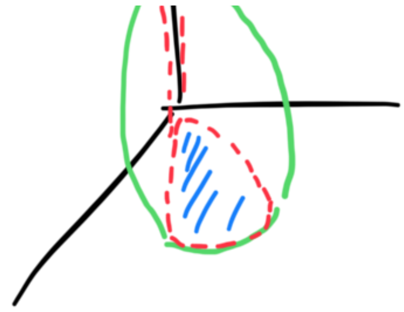
ex: Calculate volume of solid bounded by

$z = 2 - x^2 - y^2$, $z = 0$ and $r = \sin 2\theta$

in the first quadrant.

this drawing is so bad.



$$V = \iint_R z\, dA = \iint_R 2 - x^2 - y^2 \,(r\,dr\,d\theta)$$

$$= \int_0 \int_0^{\sin 2\theta} (2 - r^2)(r\,dr\,d\theta) = \int_0 r^2 - \frac{1}{4} r^4 \Big|_0^{\sin 2\theta} d\theta$$

$$= \int_0^{\pi/2} \underset{(i).}{\sin^2 2\theta} - \frac{1}{4} \underset{(ii).}{\sin^4 2\theta}\, d\theta$$

(i). $\displaystyle\int_0^{\pi/2} \sin^2 2\theta\, d\theta = \frac{\theta}{2} - \frac{1}{8}\sin(4\theta) \Big|_0^{\pi/2}$

$$= \pi/4$$

(ii). $\displaystyle\int_0^{\pi/2} \sin^4 2\theta\, d\theta = \int_0^{\pi/2} \sin^2 2\theta\,(1 - \cos^2\theta)\, d\theta$

$$= \underbrace{\int_0^{\pi/2} \sin^2 2\theta\, d\theta}_{\pi/4} - \underbrace{\int_0^{\pi/2} \sin^2 2\theta \cos^2 2\theta\, d\theta}_{(\sin 2\theta \cos 2\theta)^2}$$

$$= \left(\frac{1}{2}\sin 4\theta\right)^2$$

$$\int_0^{\pi/2} \sin^2 4\theta\, d\theta = \frac{3\pi}{16}$$

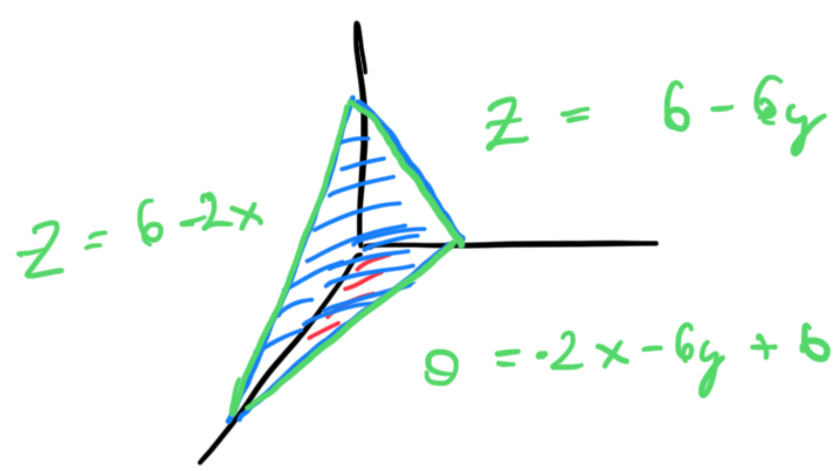
So total is

$$\frac{\pi}{4} + \left(\frac{\pi}{4} - \frac{1}{4}\left(\frac{3\pi}{16}\right)\right) = \frac{29\pi}{64}$$

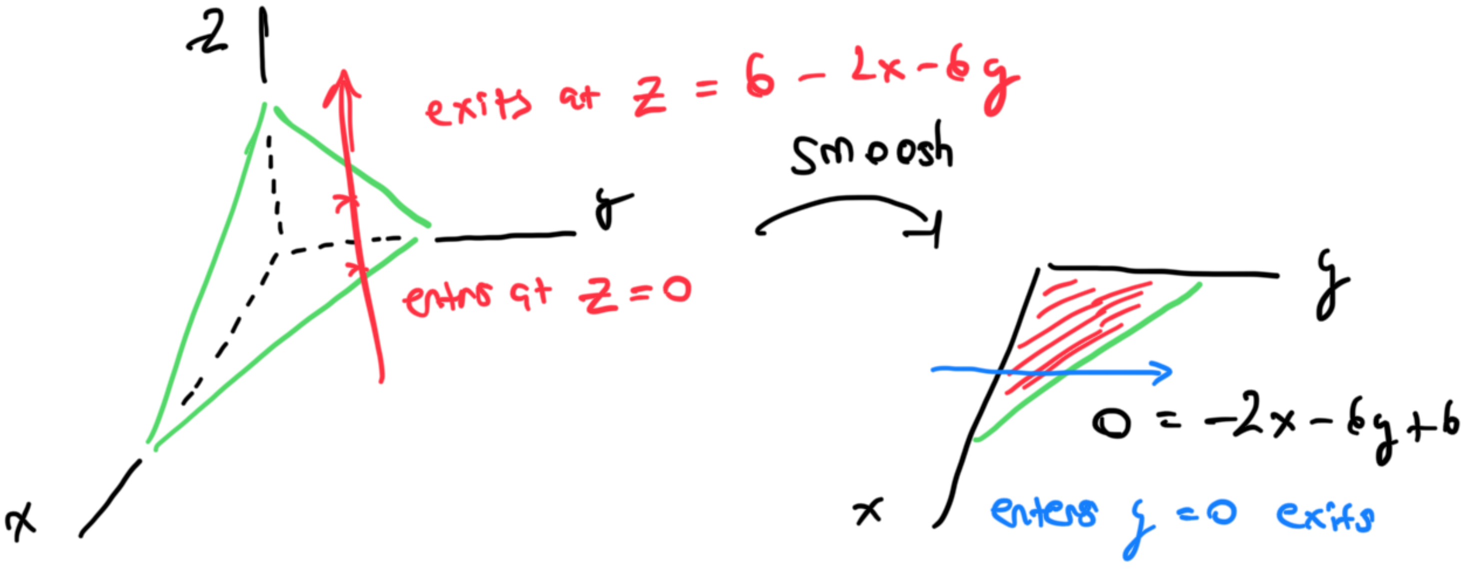
15.5: Triple Integrals in Rectangular Coordinates

Why stop at 2 variables? Why not integrate functions over space? Well that's triple integrals.

ex: Calculate the volume of the tetrahedron bounded by the 1st octant and the plane $z = -2x - 6y + 6$.

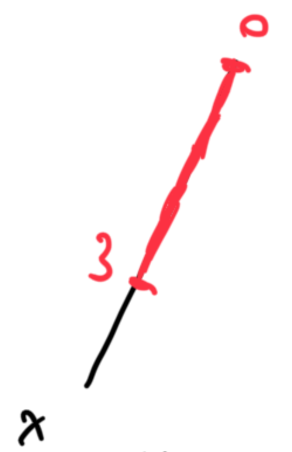


One way is really similar to what we did before.



$y = \frac{6-2x}{6} = 1 - \frac{1}{3}x$

Smeesh again



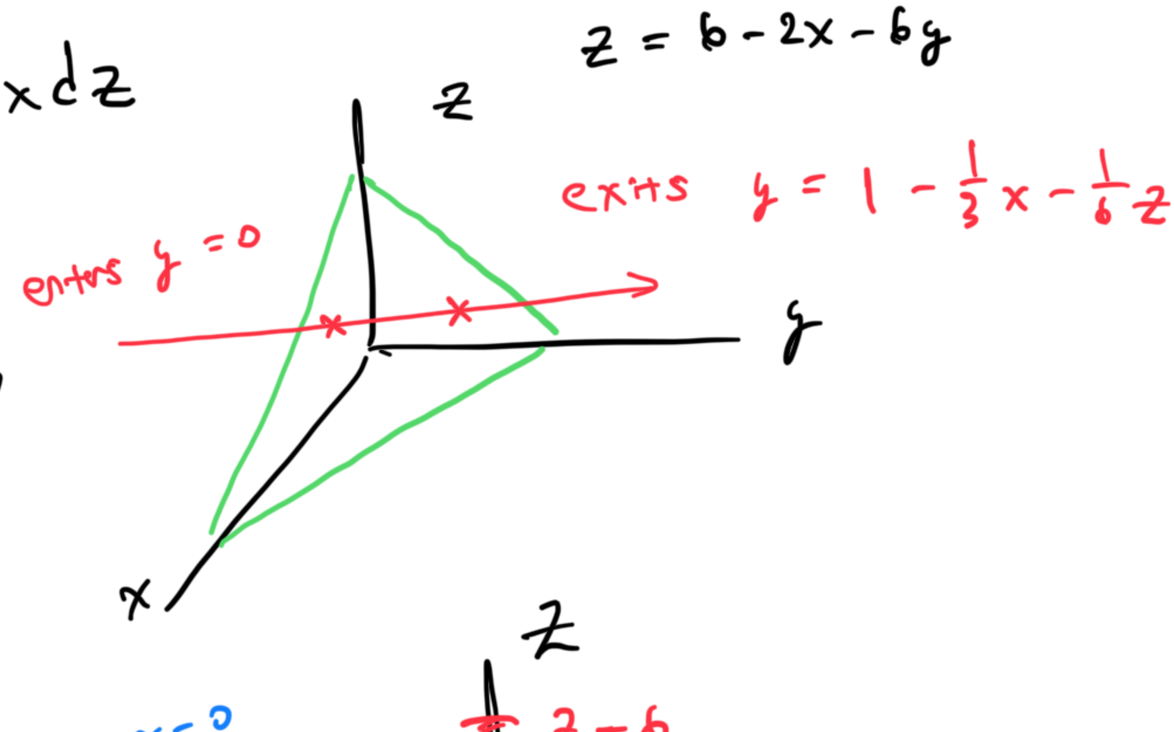
$$\text{So...}\quad V = \int_{x=0}^{x=3} \int_{y=0}^{y=1-1/3x} \int_{z=0}^{z=6-2x-6y} 1 \, dz\, dy\, dx$$

$$= 3$$

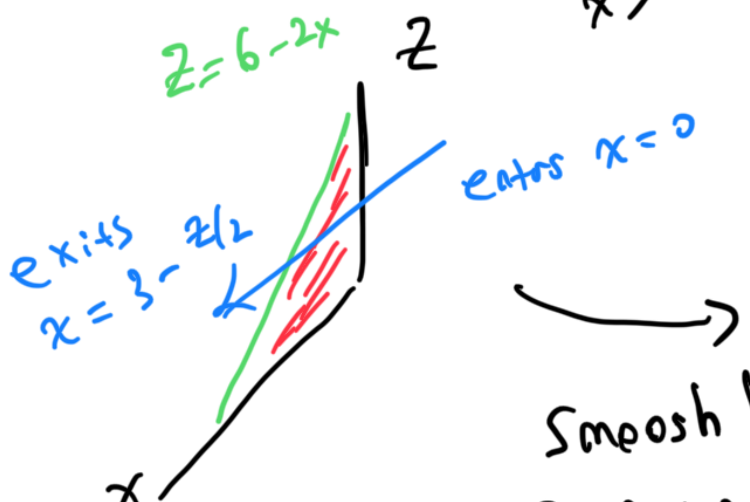
(Much easier way with cross products by the way!)

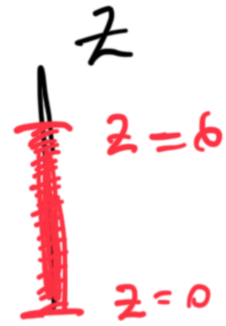
ex: Same example but now switch order to

$dy\, dx\, dz$



Smeosh



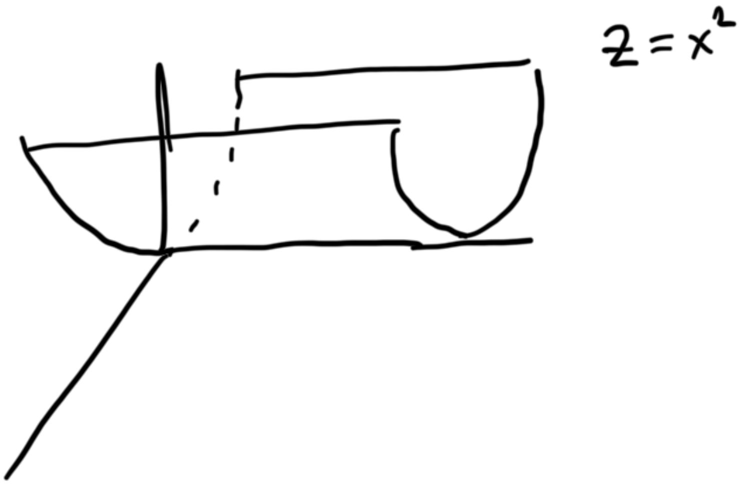


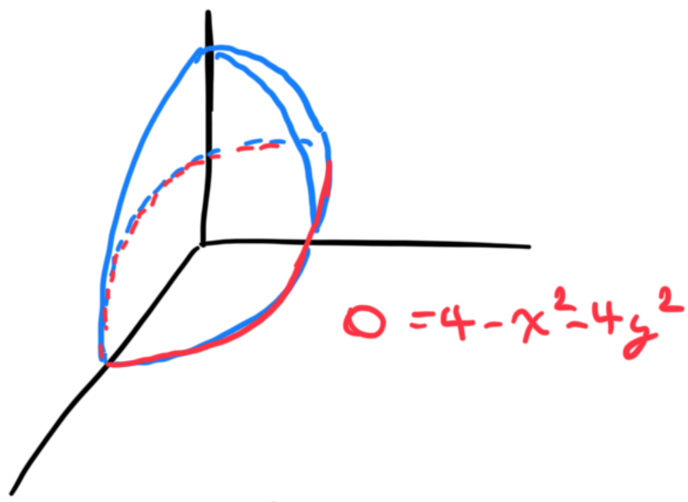
Smeosh!

$$\text{So } V = \int^{z=6} \int^{x=3-z/2} \int^{y=1-\frac{1}{3}x-\frac{1}{6}z} 1 \, dy\, dx\, dz = 3$$

$$\int_{z=0}\int_{x=0}\int_{y=0} 1 \, dy\, dx\, dz = \ldots$$

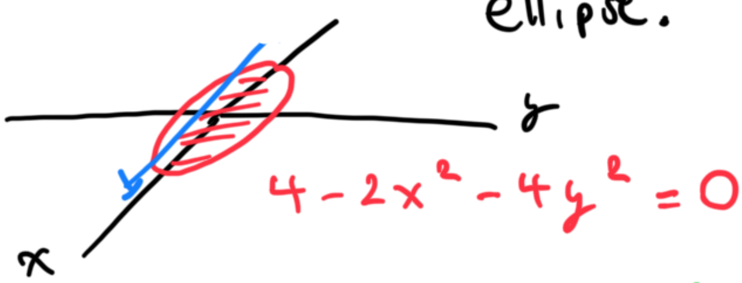
ex: Write an integral that calculates the volume of the solid region bounded by $z = x^2$ and $z = 4 - x^2 - 4y^2$



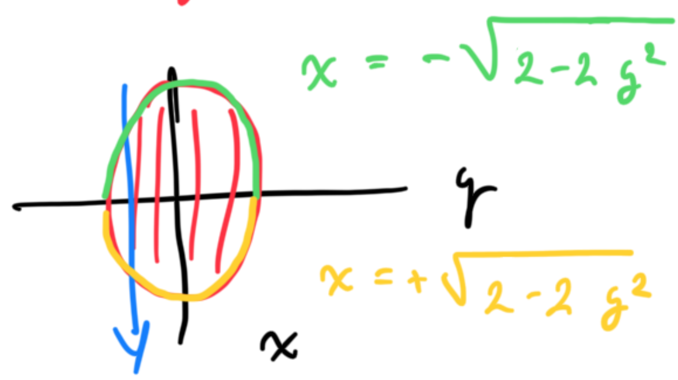


intersection is $z = x^2 = 4 - x^2 - 4y^2 = z$

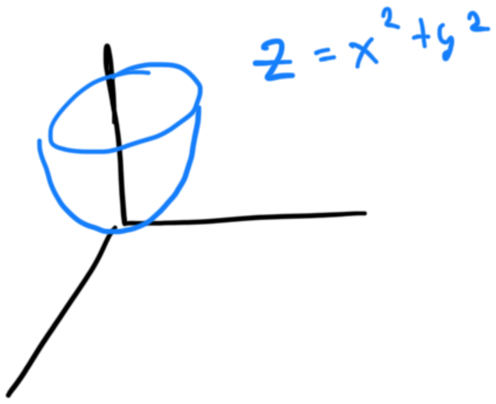
So $0 = 4 - 2x^2 - 4y^2$ which is an ellipse.

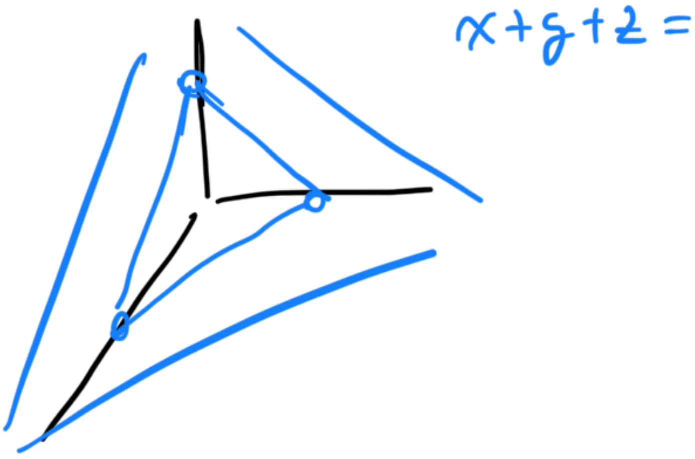


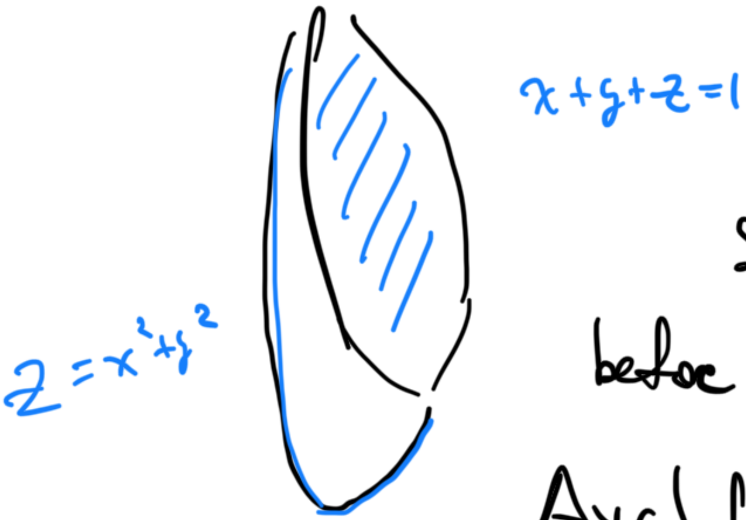
$$\text{So } V = \int_{y=-1}^{y=+1} \int_{x=-\sqrt{2-2y^2}}^{x=+\sqrt{2-2y^2}} \int_{z=x^2}^{z=4-2x^2-4y^2} dz\, dx\, dy$$



ex: Find average value of function $f(x,y,z) = x^2 + y^2$ on the solid region bounded by the plane $x + y + z = 1$ and the parabola $z = x^2 + y^2$.







So average value just like before

$$\text{Avg}(f, \text{Solid } S) = \frac{\iiint_S f\, dV}{\iiint_S dV}$$

$$= \iiint_S f\, dV \text{ (i).} \Big/ \text{Volume of } S \text{ (ii).}$$

(i).

$$x + y + z = 1 \quad \text{and} \quad z = x^2 + y^2$$

$$x + y + x^2 + y^2 = 1$$

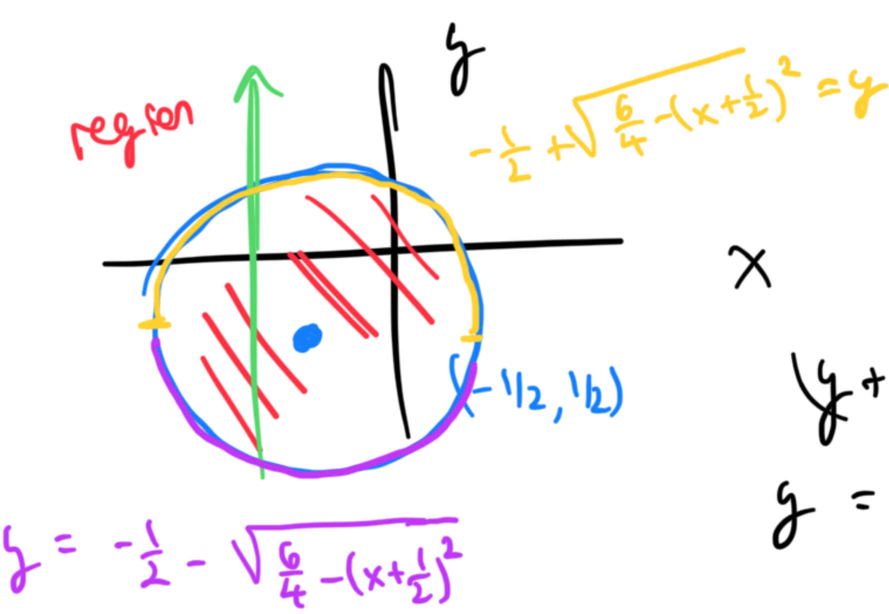
$$(x^2 + x) + (y^2 + y) = 1$$

Circle centered at $(-1/2, -1/2)$

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 1 + \frac{1}{4} + \frac{1}{4}$$

$$= 6/4 \quad \text{radius } \frac{\sqrt{6}}{2}$$

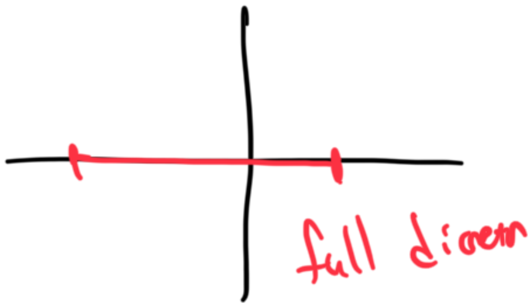
So



$$\left(y + \frac{1}{2}\right)^2 = \frac{6}{4} - \left(x + \frac{1}{2}\right)^2$$

$$y = -\frac{1}{2} \pm \sqrt{\frac{6}{4} - \left(x + \frac{1}{2}\right)^2}$$

Smoosh



Solve $y = -\frac{1}{2} \pm \sqrt{\frac{6}{4} - (x+\frac{1}{2})^2} = -\frac{1}{2}$

So $x = \frac{1}{2}(-1 \pm \sqrt{6})$

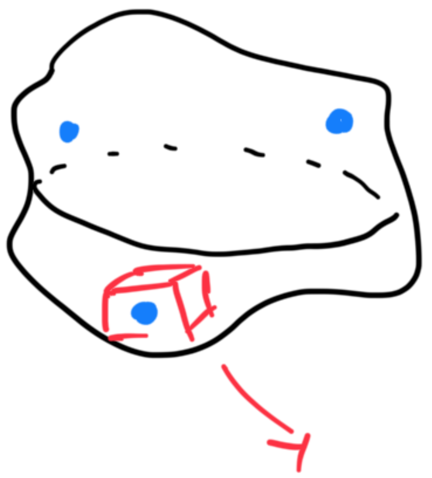
So $$(\text{i}) = \int_{x=\frac{1}{2}(-1-\sqrt{6})}^{x=\frac{1}{2}(-1+\sqrt{6})} \int_{y=-\frac{1}{2}-\sqrt{\frac{6}{4}-(x+\frac{1}{2})^2}}^{y=-\frac{1}{2}+\sqrt{\frac{6}{4}-(x+\frac{1}{2})^2}} \int_{z=x^2+y^2}^{z=1-x-y} (x^2+y^2)\, dz\, dy\, dx$$

(ii). Replace integrand above x^2+y^2 with 1 to get volume. Do not evaluate.

15.6: Mass and Moments of Inertia

Let's say have a solid body whose density modeled by $\delta(x,y,z)$

$\delta(q) = 2\ \text{kg}/\text{m}^3$



$\delta(p) = 10\ \text{kg}/\text{m}^3$

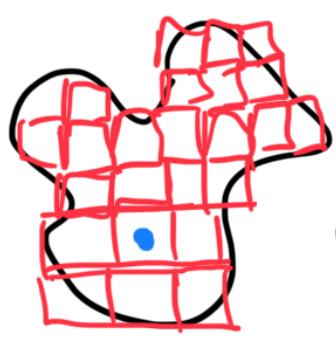
mass of cube $\approx \delta(p)\,\Delta V$

where ΔV = volume of small cube.

So $\text{Mass} \approx \sum_{\text{cubes}} \delta(p_i)\,\Delta V_i \xrightarrow{\text{limit}} \iiint_B \delta(p)\,dV$

where B is the body in question.

Similar picture in 2D



$\text{Mass} \approx \sum_{\text{squares}} \delta(p_i)\,\Delta A_i$

$= \delta(p)\,\Delta A$

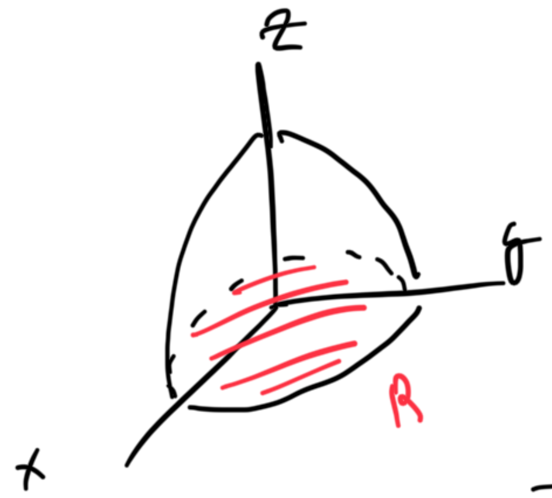
in limit $\text{Mass} = \iint_R \delta(p)\,dA$

ex: Calculate mass of body B

bounded by $z = 4 - x^2 - y^2$ and $z = 0$

Subject to density function $\delta(x, y, z) = x^2y^2$

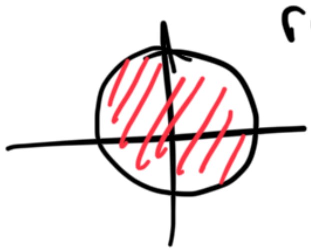
$$M = \iiint_B \delta(p)\, dV = \iint_R \int_{z=0}^{z=4-x^2-y^2} x^2y^2 \, dz \, dA$$



$$= \iint_R x^2y^2(4-x^2-y^2)\, dA$$

to polar coordinates...

radius 2



so $$M = \int_{r=0}^{r=2}\int_{\theta=0}^{\theta=2\pi} (r\cos\theta)^2(r\sin\theta)^2(4-r^2)\,\underbrace{r\,dr\,d\theta}_{dA}$$

$$= \int_0^2\int_0^{2\pi} \cos^2\theta \sin^2\theta \,(r^5)(4-r^2)\, dr\, d\theta$$

$$= \left(\int_0^2 r^5(4-r^2)\, dr\right)\left(\int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta\right)$$

$$= \left(\int_0^2 4r^5 - r^7 \, dr\right)\left(\frac{1}{4}\int_0^{2\pi} \sin^2 2\theta \, d\theta\right)$$

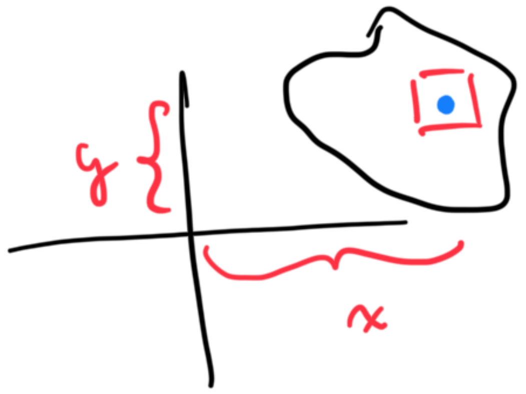
$$= \left(\frac{4}{6}r^6 - \frac{1}{8}r^8\Big|_0^2\right)\left(\frac{1}{4}\int_0^{2\pi} \left(\frac{1}{2}(1-\cos 2\theta)\right) d\theta\right)$$

$$= \left(\frac{4}{6}\cdot 48 - \frac{1}{8}128\right)\left(\frac{1}{8}\right)(2\pi) = \left(\frac{32}{3}\right)\left(\frac{1}{8}\right)(2\pi)$$

$$= \frac{8}{3}\pi$$

Center of Mass :

Integral that takes into account distance and mass relative to an origin.



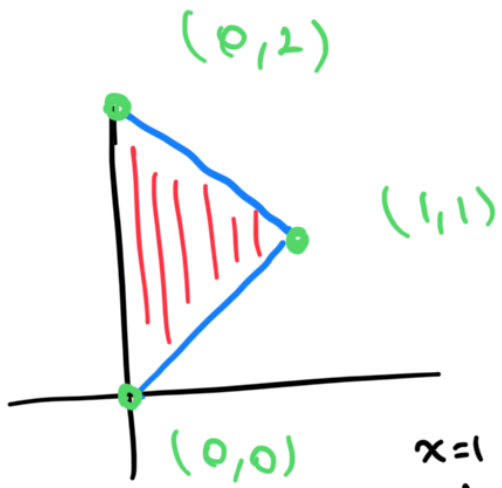
$$\sum_{\text{squares}} x_i \delta(p_i) \Delta A_i$$

$$\sum_{\text{squares}} y_i \delta(p_i) \Delta A_i$$

in limit converge to $M_y = \iint_R x \, \delta(p) \, dA$

$$M_x = \iint_R y \, \delta(p) \, dA$$

ex:



$\delta(x,y) = x^2 y$

$$M_y = \iint_R x \, \delta(x,y) \, dA$$

$$= \int_{x=0}^{x=1} \int_{y=x}^{y=2-x} x(x^2 y) \, dy \, dx$$

$$= \int_0^1 x^3 \int_{y=x}^{y=2-x} y \, dx = \frac{1}{2} \int_0^1 x^3 \left[(2-x)^2 - x^2 \right] d$$

$$= \frac{1}{2} \left(1/5 \right) = 1/10$$

Similarly, $M_x = \iint_R y(x^2 y) \, dy \, dx$

$$= \int^1 \int^{2-x} x^2 y^2 \, dy \, dx = \int^1 \ldots$$

$$\int_0^1 \int_x^{2-x} x y^2 \, dy \, dx = \frac{1}{3}\int_0^1 x\left((2-x)^3 - x^3\right) dx$$

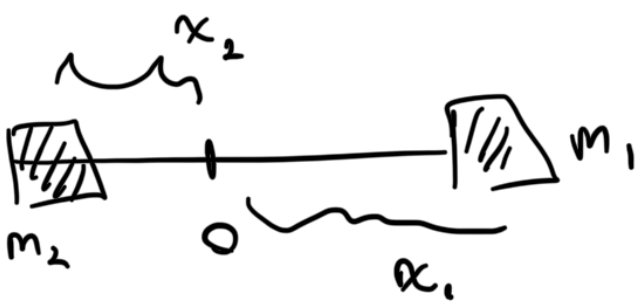
$$= (1/3)(8/15) = 8/45$$

Center of mass is where if you balance the object, it will remain still, under ideal circumstances.

$$\bar{x} = M_y / M \qquad \bar{y} = M_x / M$$

Can think of M_x, M_y as average of weighted mass-distances.

Called moments of mass.



$$\bar{x} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}$$

ex: Above example we only need $M = \int_0^1 \int_x^{2-x} x^2 y \, dy \, dx$

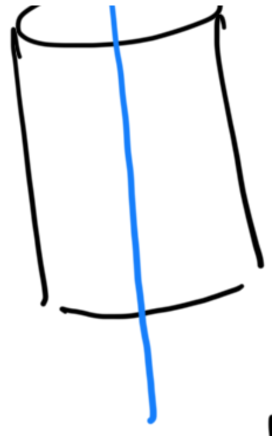
$$= \frac{1}{2}\int_0^1 x^2\left((2-x)^2 - x^2\right) dx = (1/2)(1/3) = 1/6$$

so center of mass is at $\left(\frac{M_y}{M}, \frac{M_x}{M}\right)$

$$(\bar{x}, \bar{y}) = \left(\frac{1/10}{1/6}, \frac{8/45}{1/6}\right) = \left(\frac{3}{5}, \frac{16}{15}\right)$$

Moment of inertia:

Consider a spinning mass about a



line, e.g. z-axis. Assume body spinning constant rotational speed ω.

then for a mass spinning r-away from line, we have

$KE = \frac{1}{2} m v^2$, ω = rotational velocity

$\omega r = v$

so KE of small mass $= \frac{1}{2} m (\omega r)^2$.

Summing total energy $\sum \frac{1}{2} m_i (\omega r_i)^2$

$\approx \sum \frac{1}{2} \delta(p_i) r_i^2 \omega^2 dV$

we get $KE = \frac{1}{2} \iiint_B \delta r^2 \omega^2 dV$

$v = r\omega$ bc

$\frac{ds}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega$

$$= \frac{1}{2}\omega^2 \underbrace{\iiint_B (x^2+y^2)\delta(p)\, dV}$$

I_z = moment of inertia about line L, in our case z

so $KE = \frac{1}{2}\omega^2 I_z$

You can think of I_z as mass. It takes $\frac{1}{2}mv^2$ energy to get a body m in motion.

It takes $\frac{1}{2}\omega^2 I_z$ to get something to spin about z-axis at angular velocity ω.

Similarly... $I = \iiint (x^2+z^2)\delta\, dV$

$$I_x = \iiint_B (y^2+z^2)\,\delta\, dV$$

$$I_y = \iiint_B (x^2+z^2)\,\delta\, dV$$

In 2 dimensions ...

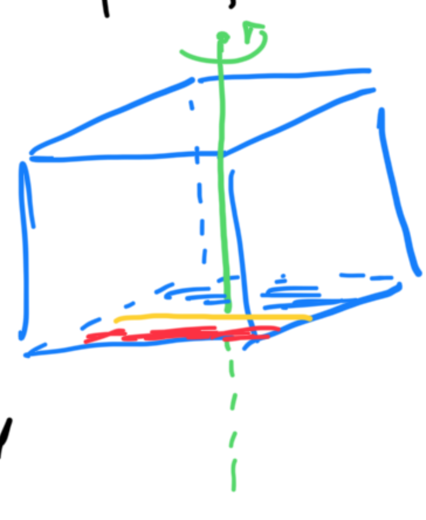
$$I_x = \iint_R y^2 \delta\, dA \qquad I_y = \iint_R x^2 \delta\, dA$$

$$I_0 = \iint_R (x^2+y^2)\,\delta\, dA$$

New in this context, object spinning around a point $(0,0)$ so $KE = \frac{1}{2}\omega^2 I_0$

ex: Calculate moment of inertia of a cube with vertices at $(\pm 1, \pm 1, 0)$ and $(\pm 1, \pm 1, 1)$ with density $\delta(x,y,z) = |xz|$.

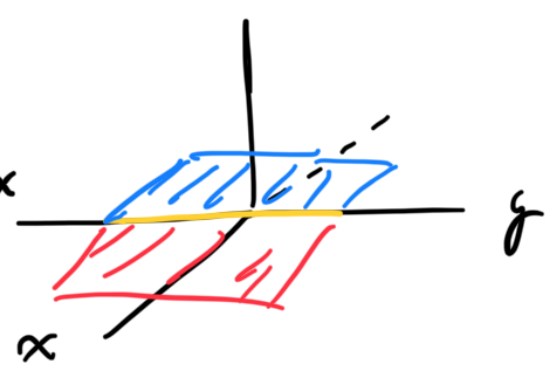
$$I_z = \iiint_B (x^2+y^2)\,\delta\, dV$$



$$= \iiint_{B \text{ with } x \le 0} (x^2+y^2)\,\delta\, dV + \iiint_{B \text{ with } x \ge 0} (x^2+y^2)\,\delta\, dV$$

$$= \int_{x=-1}^{x=0} \int_{y=-1}^{y=+1} \int_{z=0}^{z=1} (x^2+y^2)(-xz)\, dz\, dy\, dx$$

(i)



$$+\int_{x=0}^{x=1}\int_{y=-1}^{y=+1}\int_{z=0}^{z=1}(x^2+y^2)(xz)\,dz\,dy\,dx$$

(ii)

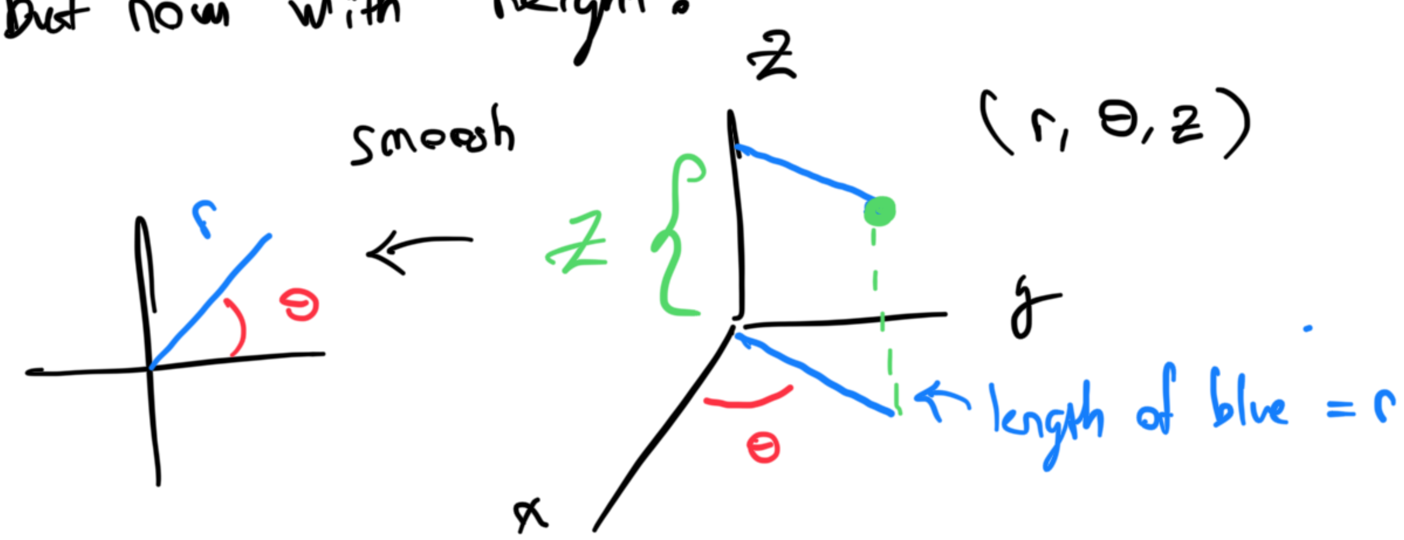
(i). $\displaystyle\int_{-1}^{0}\int_{-1}^{1}\frac{1}{2}(x^2+y^2)(-x)\,dy\,dx$ (ii). $5/12$

$$=\int_{-1}^{0}\int_{-1}^{1}\frac{1}{2}(-x^3-y^2x)\,dy\,dx$$

$$=\int_{-1}^{0}\left(-x^3-\frac{x}{3}\right)dx = 5/12 \quad \text{so } I_z = 10/12 = 5/6$$

15.7 : Integrals in Spherical and Cylindrical coordinates

Cylindrical coordinates: A point P in $\mathbb{R}^3$ is determined by (r, θ, z). Just polar coordinates but now with height.



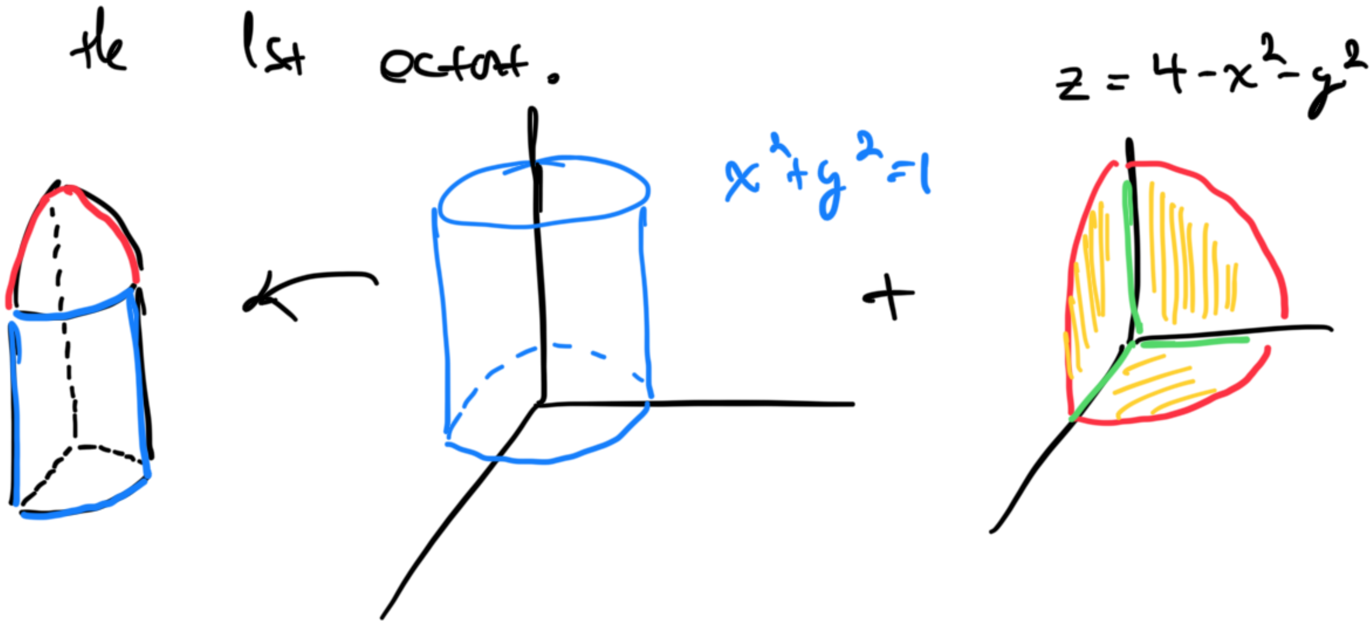
You can equate cartesian and cylindrical like before.

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

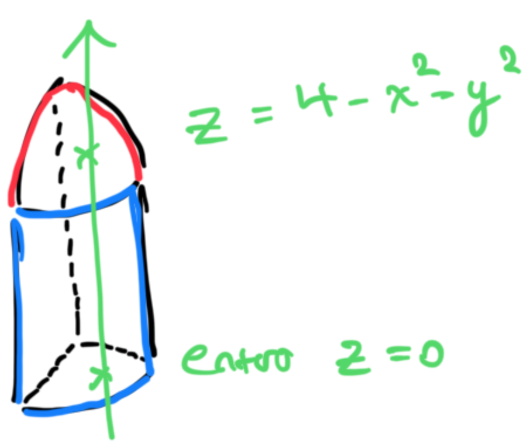
$r^2 = x^2 + y^2 \quad \tan\theta = y/x$ (where defined)

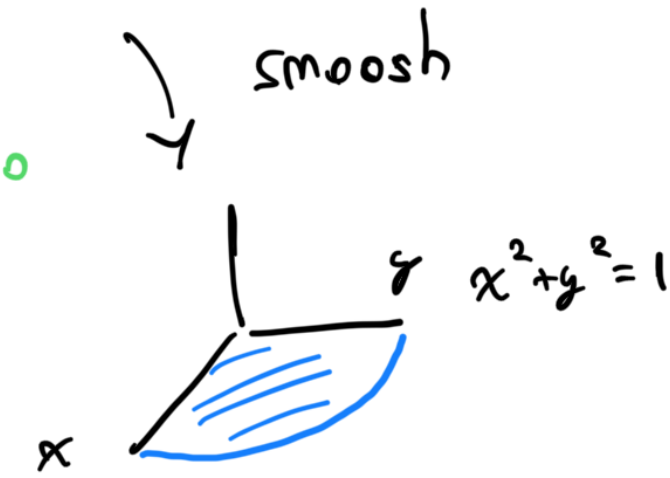
Why use them? Well good for problems where you have an axis of symmetry.

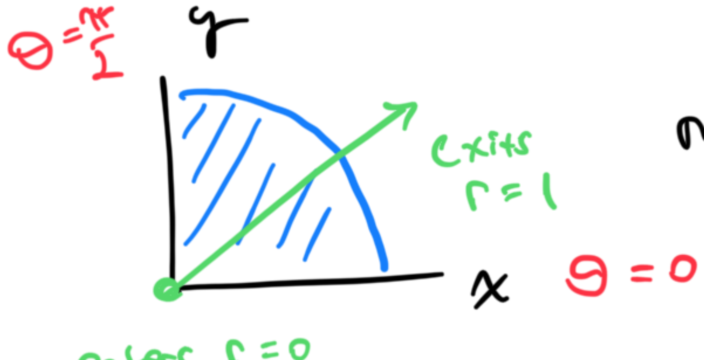
ex: Calculate volume bounded by surfaces $x^2+y^2=1$ and $z = 4-x^2-y^2$, in the 1st octant.



Find z-limits







now do polar.

$\theta = \pi/2 \quad r = 1 \qquad \pi/2 \qquad 4 - r^2$

$$V = \iiint_{\theta=0\ r=0\ z=0}^{\ \ \ z=4-x^2-y^2} dV = \iiint_{0\ 0\ 0} r\,dz\,dr\,d\theta$$

$$dV = r\,dz\,dr\,d\theta$$

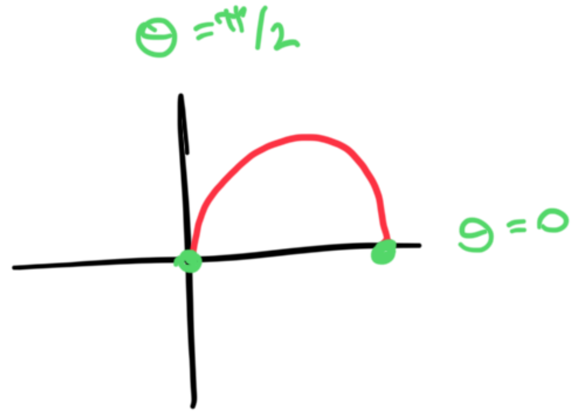
$$= \int_0^{\pi/2}\int_0^1 r(4-r^2)\,dr\,d\theta$$

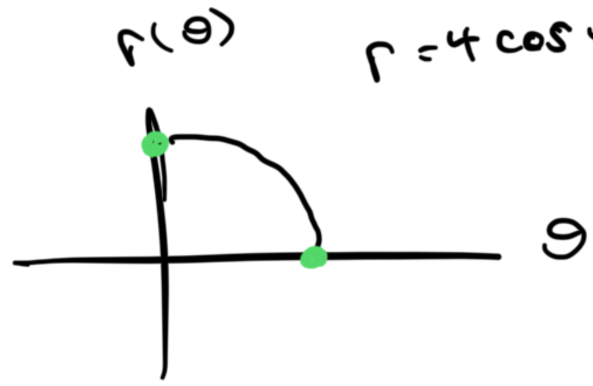
$$= \int_0^{\pi/2} 2r^2 - \frac{1}{4}r^4\Big|_0^1\,d\theta$$

$$= \int_0^{\pi/2} 7/4\,d\theta = 7\pi/8$$

ex: 15.7.11 Calculate volume

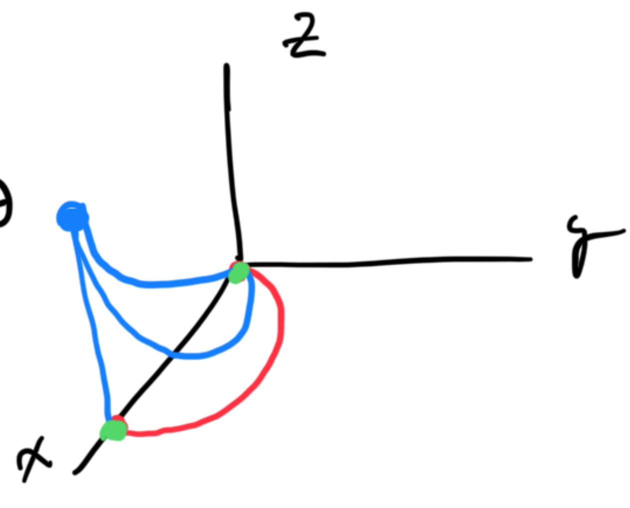
$$0 \le r \le 4\cos\theta, \quad 0 \le \theta \le \pi/2, \quad 0 \le z \le r^2$$





$r = 4\cos\theta$

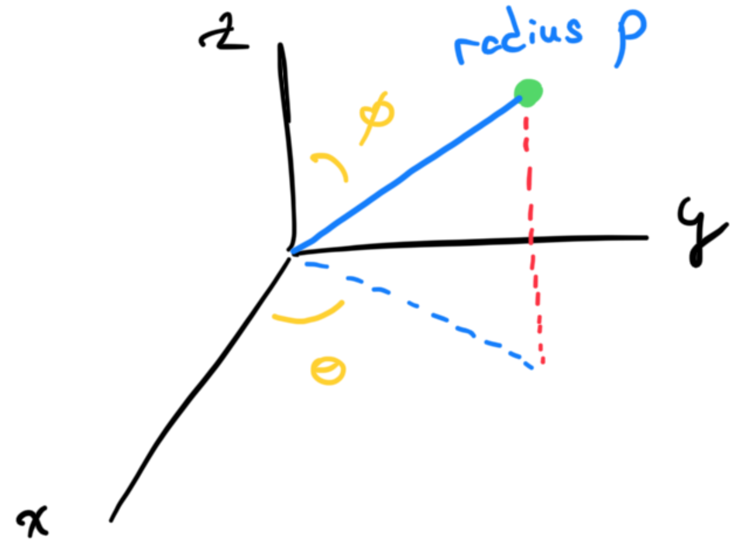
$$V = \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=4\cos\theta}\int_{z=0}^{z=r^2} r\,dz\,dr\,d\theta$$



$$V = \int_0^{\pi/2}\int_0^{4\cos\theta} r^3\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{4}(4\cos\theta)^4\,dr\,d\theta$$

$= (1/4)(48\pi) = 12\pi$

Spherical coordinates



$P = (\rho, \phi, \theta)$ or (ρ, θ, ϕ) depending on choice. Book uses (ρ, ϕ, θ)

Again have ways of going back and forth between Cartesian and spherical.

$$x = \rho \sin\phi \cos\theta \qquad \rho^2 = x^2 + y^2 + z^2$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

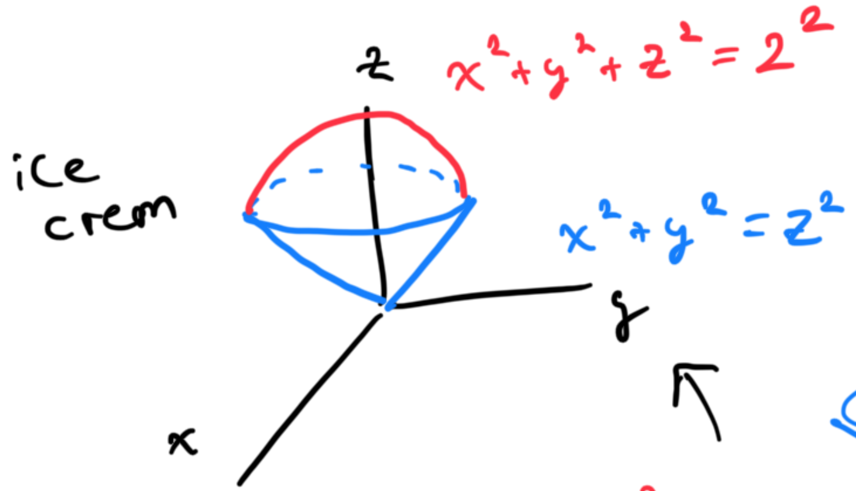
Can also relate cylindrical and spherical

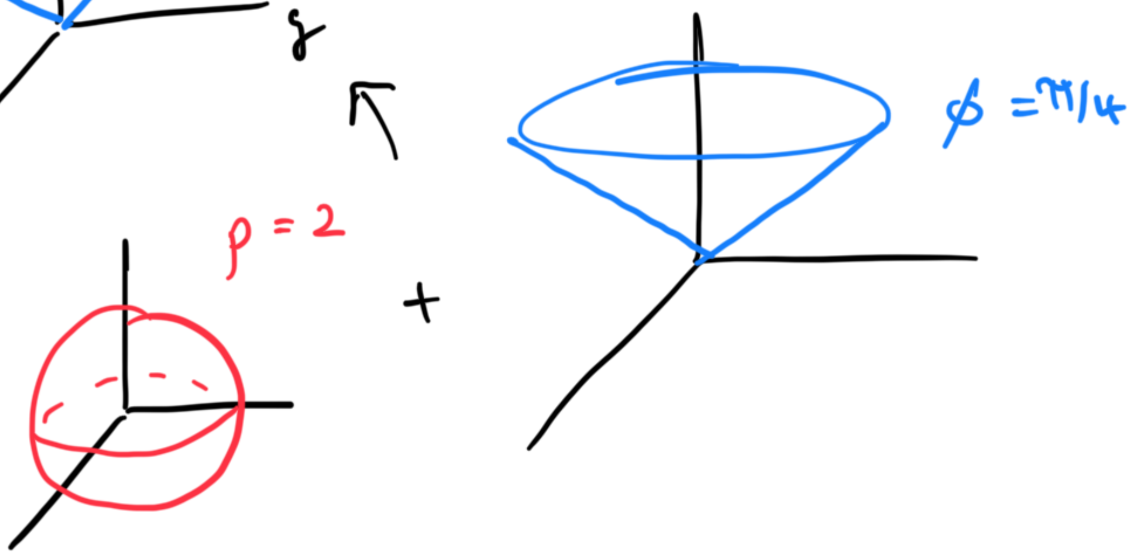
$$r = \rho \sin\phi \qquad \rho^2 = r^2 + z^2$$
$$z = \rho \cos\phi$$
$$\theta = \theta$$

Why spherical coordinates? Use when you have a point of symmetry.

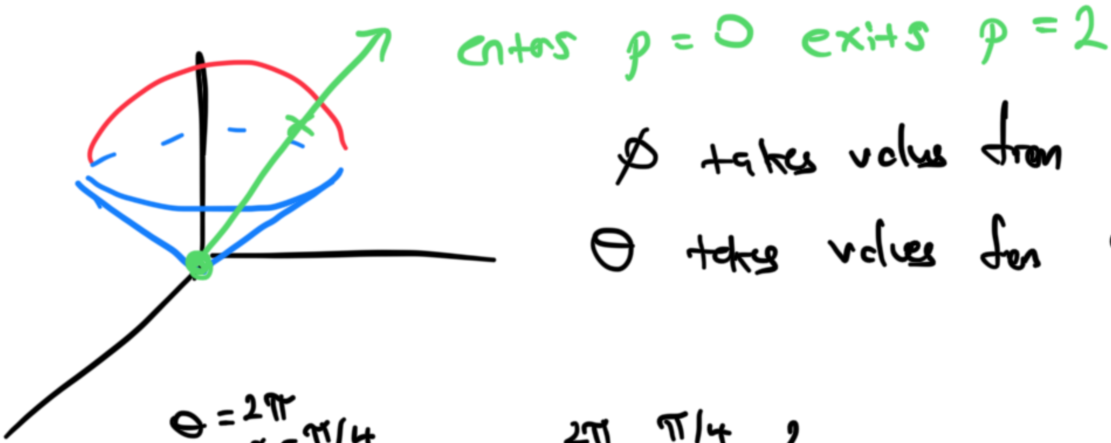
ex: Calculate volume of body bounded by by sphere of radius 2 centred at origin and the cone $x^2+y^2=z^2$ and $z \geq 0$.

$$r^2 = z^2$$
$$\rho^2 \sin^2\phi = \rho^2 \cos^2\phi$$
$$\sin^2\phi = \cos^2\phi$$
$$\sin\phi = \cos\phi$$
$$\phi = \pi/4$$





so



ϕ takes values from 0 to $\pi/4$

θ takes values from 0 to 2π

so $$V = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4}\int_{\rho=0}^{\rho=2} dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

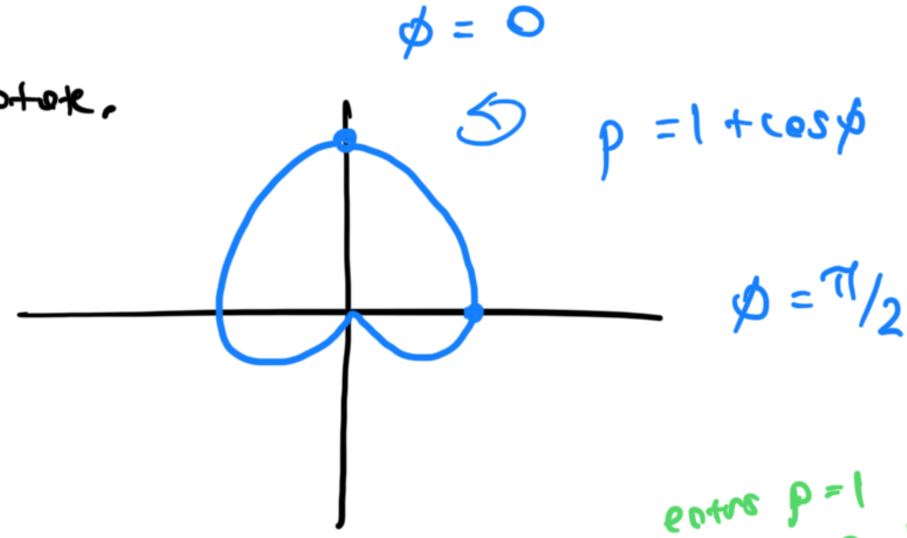
$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{8}{3} \sin\phi \, d\phi \, d\theta$$

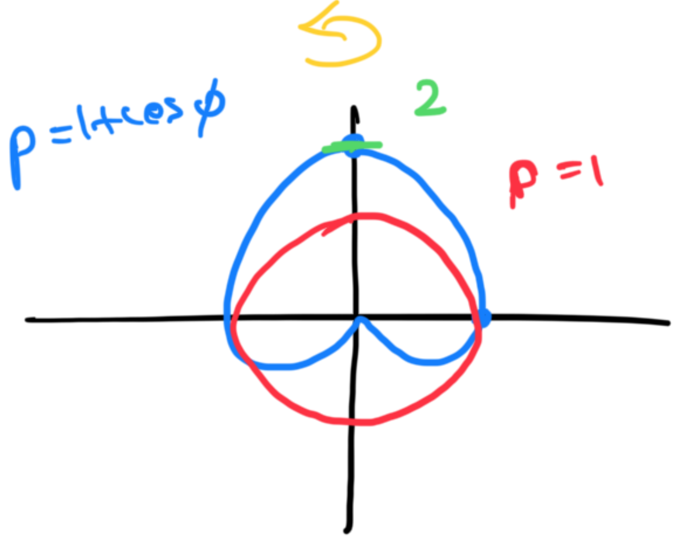
$$= \frac{8}{3}\left(1 - \frac{\sqrt{2}}{2}\right)(2\pi) = \frac{16\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)$$

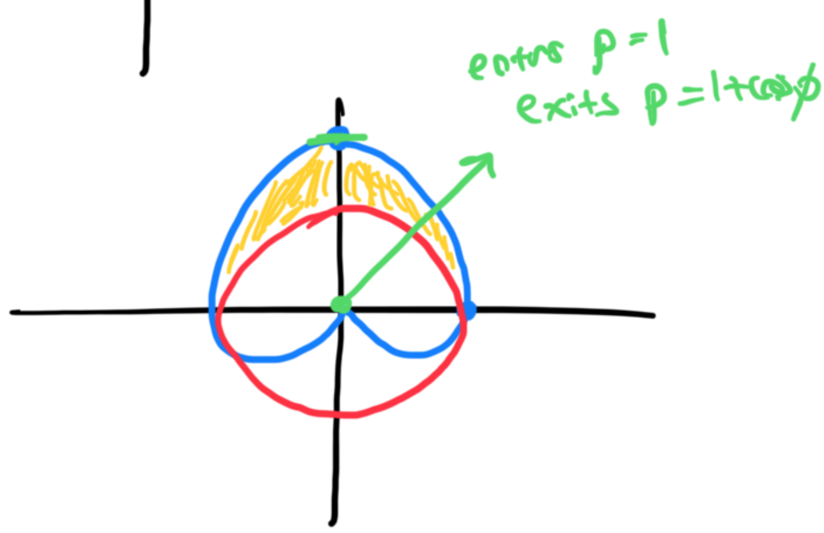
15.7.56. Find volume of solid bounded below hemisphere $\rho = 1$, $z \geq 0$, and above $\rho = 1 + \cos\phi$

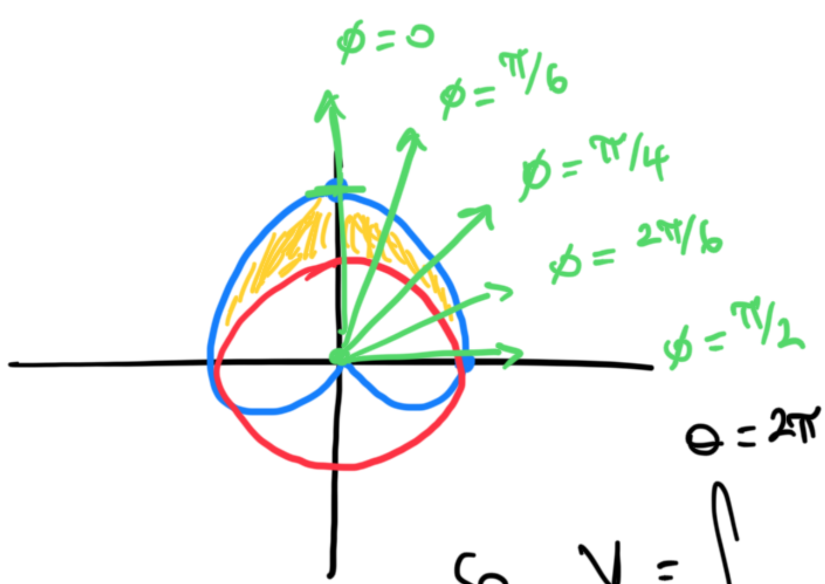
First, what does $\rho = 1 + \cos\phi$ look like?

Well first it has θ-symmetry. So draw in a plane, then rotate.









So $$V = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/2} \int_{\rho=1}^{\rho=1+\cos\phi} \underbrace{dV}_{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

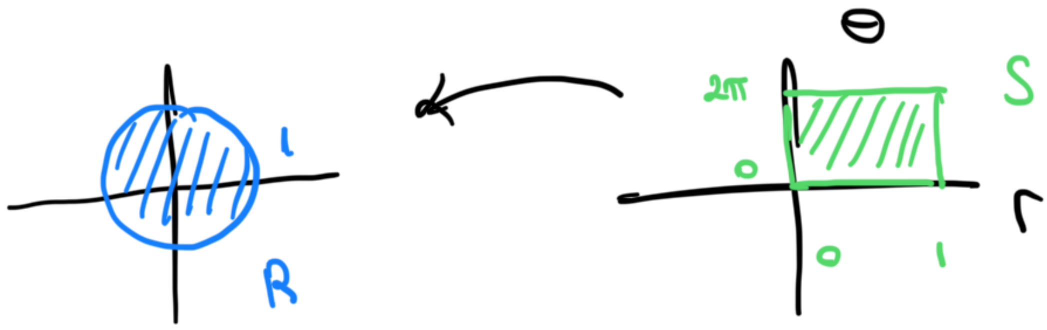
$= 11\pi/6$ (according to Mathematica)

15.8: Substitution in Multiple Integrals

When you change from one coordinate system to another, the dA or dV changes too.

You can see this in the following

$$dA = dx\,dy = r\,dr\,d\theta$$



so when

$$\iint_R f(x,y)\,dx\,dy = \iint_S f\big(x(r,\theta), y(r,\theta)\big)\,r\,dr\,d\theta$$

$$= \iint_S f(r\cos\theta, r\sin\theta)\,\underset{\uparrow ?}{r}\,dr\,d\theta$$

called the Jacobian

More generally...

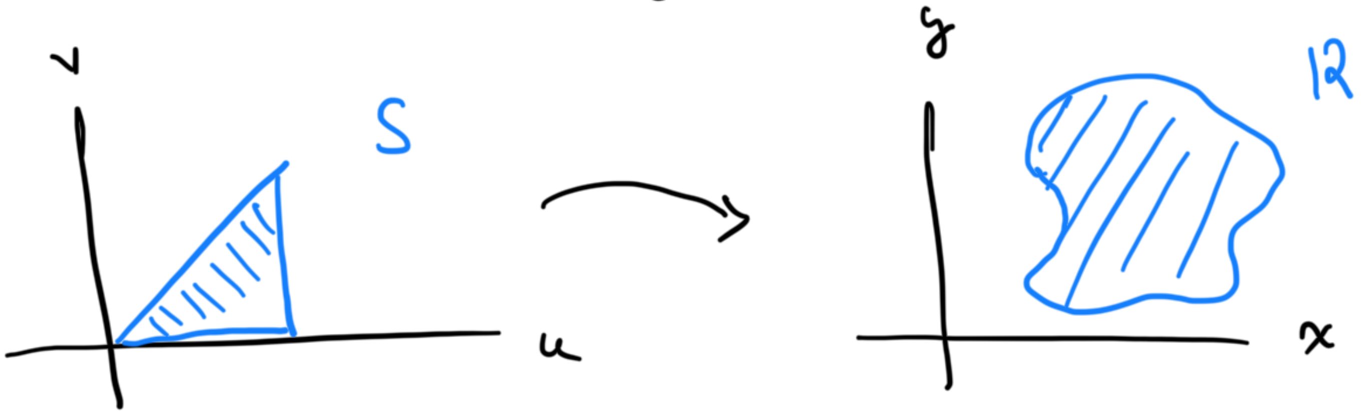
theorem: let S be in (u,v)-coordinates and say $x = g(u,v)$ $y = h(u,v)$ for some functions g, h so that

$$J(u,v) = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \frac{dx}{du}\frac{dy}{dv} - \frac{dx}{dv}\frac{dy}{du} \neq 0$$

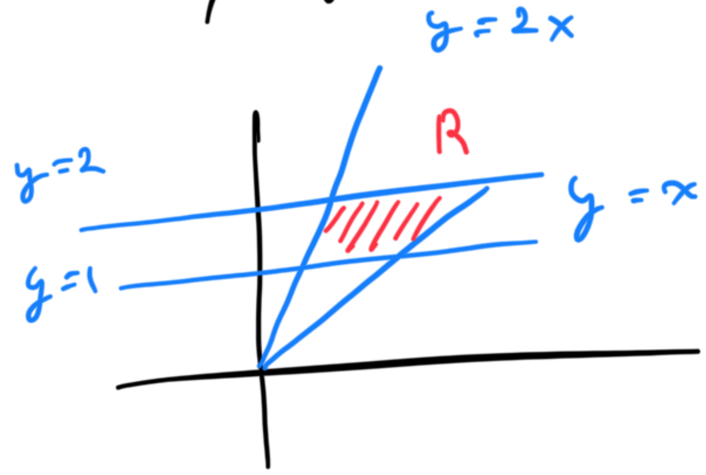
on S

let R denote image of S under (g, h). Then

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(g(u,v), h(u,v))\,|J(u,v)|\,du\,dv$$



ex: Calculate integral of $\frac{x}{y} + \frac{y}{x}$ over region bounded by $y = 2x$, $y = x$, $y = 1$ and $y = 2$

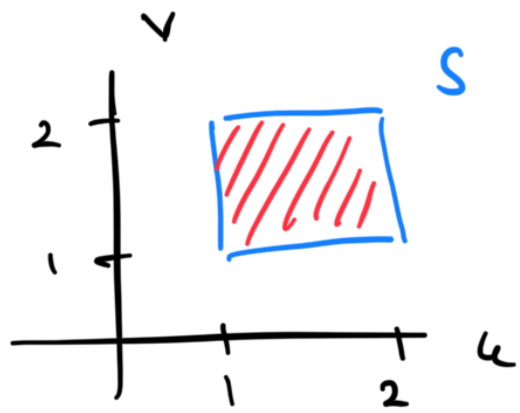


$u = \frac{y}{x}$ $\quad v = y$

then solve for x and y.

$y = v$ so $x = \frac{y}{u} = \frac{v}{u}$

What is the region?



(i). $y = 1 \sim v = 1$

(ii). $y = 2 \sim v = 2$

(iii). $y = x \sim \frac{y}{x} = 1 \sim u = 1$

(iv). $y = 2x \sim \frac{y}{x} = 2 \sim u = 2$

Jacobian: $\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} -v/u^2 & 1/u \\ 0 & 1 \end{vmatrix} = -v/u^2$

$| du \quad \partial y \; |$

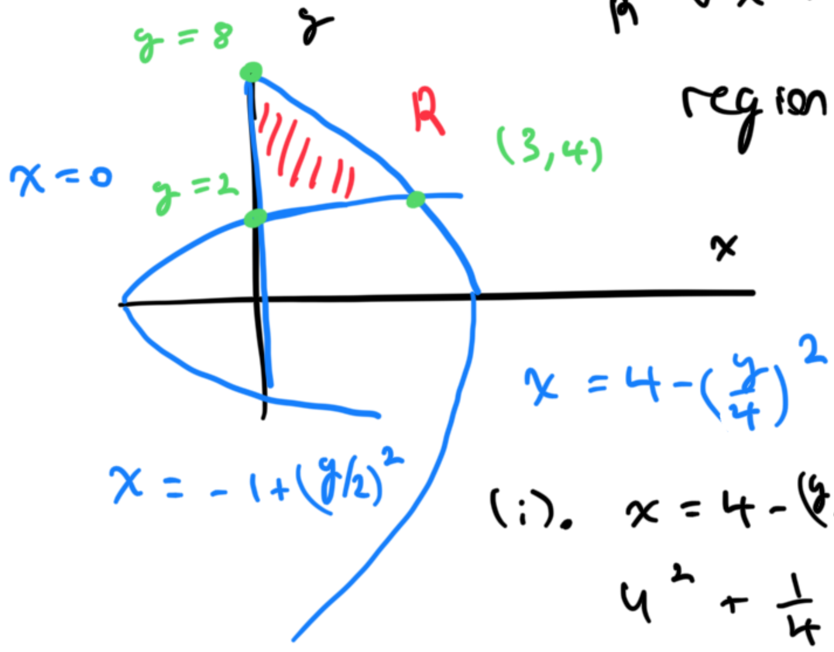
So $\displaystyle\iint_R \frac{x}{y} + \frac{y}{x} \, dA = \iint_S \left(v + \frac{1}{v}\right) \left| \frac{-v}{u^2} \right| dA'$

$$= \int_1^2 \int_1^2 \left(v + \frac{1}{v}\right) \frac{v}{u^2} \, du \, dv = \int_1^2 \int_1^2 \frac{(v^2 + v)}{u^2} \, du \, dv$$

$= 2\,3/12$ (according mathematica)

(do this ex last!)

ex: Integrate $\displaystyle\iint_R \frac{1}{\sqrt{x^2+y^2}} \, dA$ where R



region bound by $x = 4 - (y/4)^2$

$x = (y/2)^2 - 1$ and $x = 0$

using $x = u^2 - v^2$ and $y = 2uv$

(i). $x = 4 - (y/4)^2 \sim u^2 - v^2 = 4 - \left(\frac{1}{2}uv\right)^2$

$u^2 + \frac{1}{4}u^2v^2 - v^2 - 4 = 0$

$(u^2 - 4) + v^2\left(\frac{1}{4}u^2 - 1\right) = 0$

$(u^2 - 4)\left(1 + \frac{1}{4}v^2\right) = 0$

so $u = -2$ or $u = 2$. because

$x \geq 0$ and $x = u^2 - v^2$, we know

$u \geq 0$, so $u = 2$ //

(ii). $x = -1 + (y/2)^2 \sim u^2 - v^2 = -1 + (uv)^2$

$u^2 - u^2v^2 - v^2 + 1 = 0$

$u^2(1 - v^2) + (1 - v^2) = 0$

$(u^2+1)(1-v^2)=0$ so

$1=v$ or $-1=v$. bc $x \geq 0$

$u \geq 0$ like before. Also $y \geq 0$

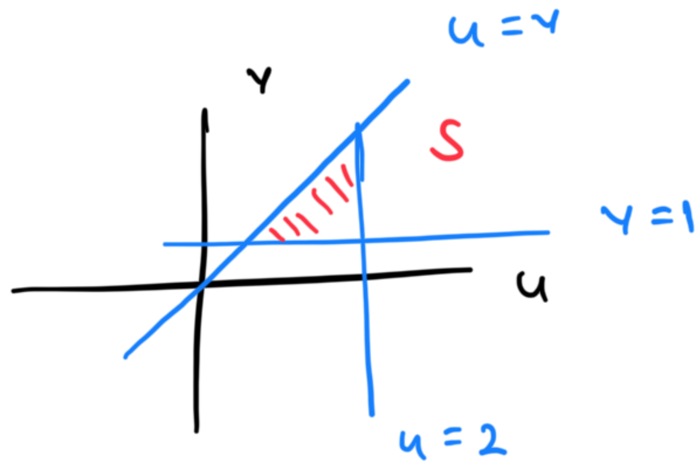
and $y = 2uv$ so $v \geq 0$. Thus

$v = +1$ //

(iii). $x = 0 \sim u^2 - v^2 = 0$ so $u = v$ or $u = -v$.

bc all $y \geq 0$ need $u = v$. //

So S bondd by



Jacobian: $J(u,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$

Integrand: $\sqrt{x^2+y^2} = \sqrt{(u^2-v^2)^2 + (2uv)^2}$

$= \left(u^4 - 2u^2v^2 + v^4 + 4u^2v^2\right)^{1/2}$

$= \left(u^4 + 2u^2v^2 + v^4\right)^{1/2} = u^2 + v^2$

so $I = \iint_S \underbrace{\left(\frac{1}{u^2+v^2}\right)}_{\text{integrand}} \underbrace{(4u^2+4v^2)}_{\text{Jacobian}}\, du\, dv = 4 \iint_S du\, dv$

$= 4\left(\tfrac{1}{2}\right) = 2$

ex: Integrate $\sqrt{1+y^3}$ over region bondd

by $y = 0$, $x = 2y^2$, $x = 1 + y^2$ using

$x = u + v^2$

$y = v$

(i). $y = 0 \sim v = 0$ //

(ii). $x = 2y^2 \sim u + v^2 = 2v^2$

so $u = v^2$ //

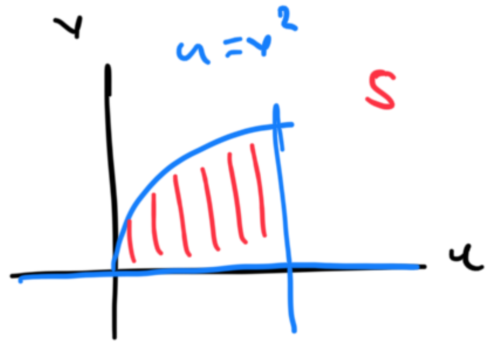
(iii). $x = 1 + y^2 \sim u + v^2 = 1 + v^2$

so $u = 1$ //

Jacboi: $J(u,v) = \begin{vmatrix} 1 & 2v \\ 0 & 1 \end{vmatrix}$

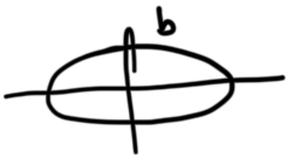
$= 1$

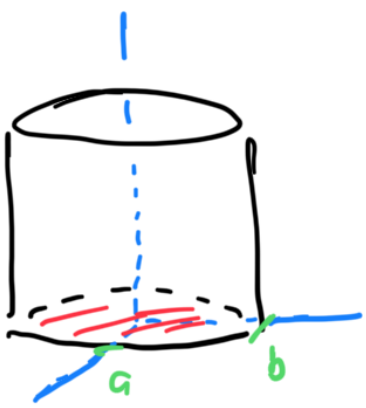
Integrd: $\sqrt{1+v^3}$

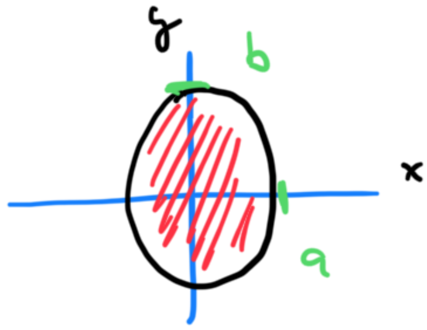


so $I = \iint_S \sqrt{1+v^3}\, 1\, du\, dv$

$= \int_0^1 \int_0^{v^2} \sqrt{1+v^3}\, du\, dv = 2(2\sqrt{2}-1)/9$

ex: Calculate volume of an elliptic cylinder of base  and height h.

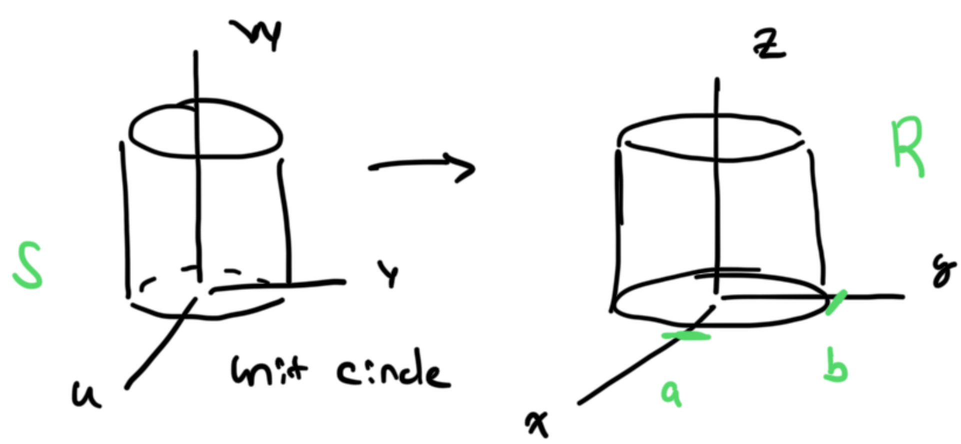




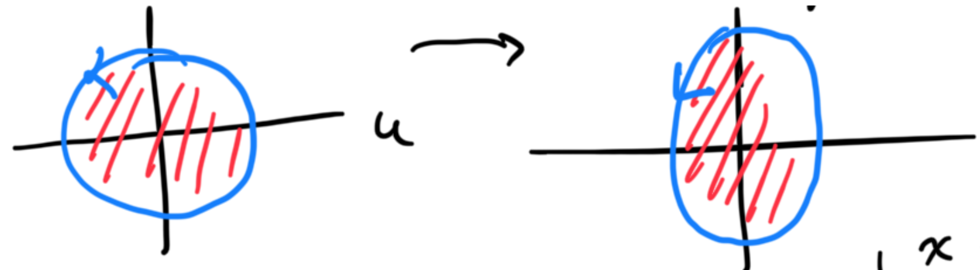
$a u = x$

$b v = y$ then

$w = z$



v y



$u^2+v^2=1 \longrightarrow \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$

$$\text{Jacobian}(u,v,w) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= ab$$

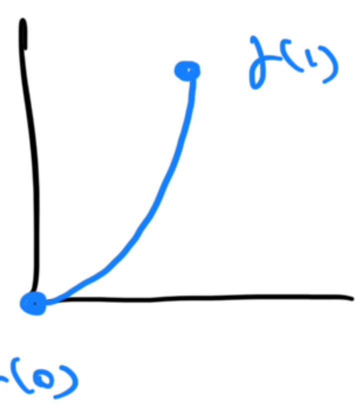
so $\iiint_R dV = \iiint_S ab\, dV'$ $\qquad dV' = du\,dv\,dw$

$= ab \iiint_S dV'$ ← volume cylinder height h unit circle base

$= \pi abh$

16.1: Line integrals of scalar functions

Motivate by example of mass of a wire in $\mathbb{R}^2$. Say we have $\delta(x,y) = x$ kg/m and have a curve by $\gamma(t) = (t, t^2)$ for $0 \le t \le 1$. So want the mass of a small part of the wire



so mass

$\gamma(t_i)$ $\qquad \approx \delta(\gamma(t_i))\big(\underbrace{\text{length from } \gamma(t_{i+1}) \text{ and } \gamma(t_i)}_{\Delta s_i}\big)$

$$\text{Mass} \approx \underbrace{\sum_i}_{\text{bunch of pieces}} \delta(\gamma(t_i))\,\Delta s_i \xrightarrow{\text{limit}} \int_0^1 \delta(\gamma(t))\,ds$$

$$ds = \sqrt{1+(2t)^2}\,dt$$

So $M = \int_0^1 t \cdot 1 \left(\sqrt{1+4t^2}\right) dt$

$$= \int_1^5 \frac{1}{8}\sqrt{u}\,du$$

$$= \frac{1}{12}\left(5\sqrt{5}-1\right)$$

$u = 1+4t^2$

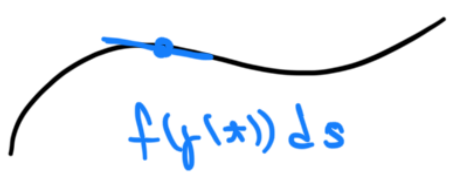
$du = 8t\,dt$

$\frac{1}{8}du = t\,dt$

$u(0)=1$

$u(1)=5$

So in general, if C is a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, we calculate 'line integrals' of a scalar function f (even if C isn't a line).



$$\int_C f\,ds = \int_a^b f(\gamma(t))\,|\vec{v}(t)|\,dt$$

where $\gamma(t)$ goes from $\gamma(a)$ to $\gamma(b)$.

ex: Evaluate $f(x,y,z) = x^2+y^2+z^2$ along the line segment from $(0,0,0)$ to $(1,1,1)$.

$\gamma(t) = (t,t,t)$ from $t=0$ to $t=1$.

$$ds = \sqrt{1^2+1^2+1^2}\,dt = \sqrt{3}\,dt$$

so $\int f\,ds = \int^1 (t^2+t^2+t^2)\sqrt{3}\,dt$

$$= 3\sqrt{3}\,\frac{t^3}{3}\Big|_0^1 = \sqrt{3}$$

ex: Same problem but $\alpha(t) = (\sin t, \sin t, \sin t)$ for $t=0$ to $t = \pi/2$.

$(f\circ\alpha)(t) = 3\sin^2 t \qquad ds = \sqrt{3\cos^2 t}\; dt$

$ds = \sqrt{3}\,|\cos t|\, dt$

so $$\int_\alpha f\, ds = \int_0^{\pi/2} 3\sin^2 t\,\sqrt{3}\cos t\; dt$$

$$= \int_0^1 3\sqrt{3}\, u^2\, du$$

$u = \sin t \qquad u(0)=0$

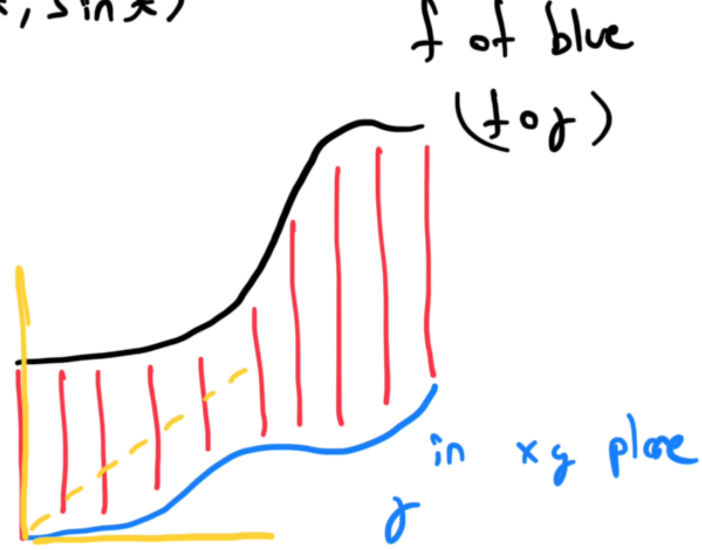
$du = \cos t\, dt \qquad u(\pi/2)=1$

$$= \sqrt{3}$$

Moral: Doesn't matter how you parametrize the path, so choose most convenient!

ex: let $z = x^2+y^2$ integrate function over path $\gamma(t) = (\cos t, \sin t)$

In general:



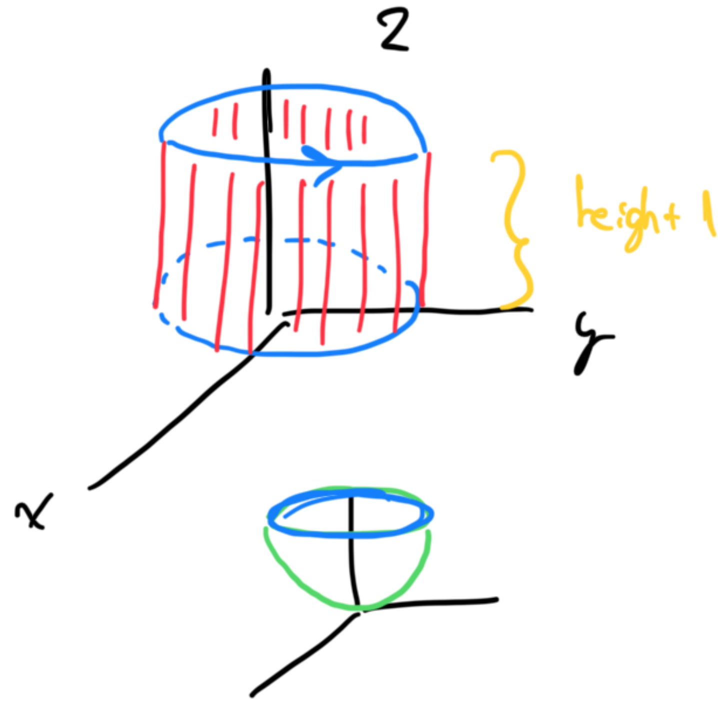
$$\text{Area} = \int_\gamma f\, ds.$$

in our exmple $f(\gamma(t)) = \cos^2 t + \sin^2 t$

$$ds = \sqrt{\sin^2 t + \cos^2 t}\, dt$$

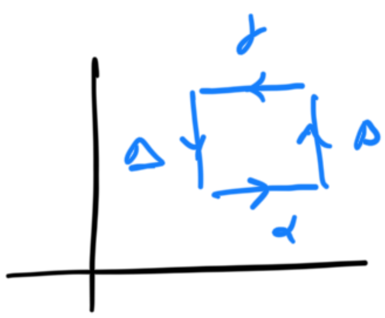
so $ds = dt$

so $\text{Area} = \int_0^{2\pi} 1\, dt = 2\pi$



ex: Say have a $\delta(x,y) = \dfrac{x}{y}$.

Want to calculate mass of square with vertices $(1,1), (2,1), (1,2), (2,2)$.



(i). $\alpha(t) = (1+t, 1)$ $\quad ds = dt$

(ii). $\beta(t) = (2, 1+t)$ $\quad ds = dt$

(iii). $\gamma(t) = (2-t, 2)$ $\quad ds = dt$

(iv). $\Delta(t) = (1, 2-t)$ $\quad ds = dt$

all $0 \le t \le 1$

(i). $\delta(\alpha(t)) = \dfrac{1+t}{1} = 1+t$

$$\int_0^1 (1+t)\, dt = 3/2$$

(ii). $(\delta \circ \beta)(t) = \dfrac{2}{1+t}$

$$\int_0^1 \frac{2}{1+t}\, dt = \ln(4)$$

(iii). $(\delta \circ \gamma)(t) = \dfrac{2-t}{2}$

$$\int_0^1 1 - \frac{t}{2}\, dt = 3/4$$

(iv). $(\delta \circ \Delta)(t) = \dfrac{1}{2-t}$

$$\int_0^1 \frac{1}{2-t}\, dt = \ln(2)$$

$M = M_1 + M_2 + M_3 + M_4 = 9/4 + 3\ln(2)$

so $M = \ldots$

ex: lets say have wire $\gamma(t) = (\cos t, \sin t, t)$ from $t=0$ to $t=2\pi$. Also say that $\delta(x,y,z) = (x^2+y^2)e^z$. Write integrals for the various moments of inertia of the wire relative to each of the coordinate axes. Do not calculate

$$I_x = \int_\gamma (y^2+z^2)\delta \, ds \qquad ds = \sqrt{1+1}\, dt = \sqrt{2}\, dt$$

$$I_y = \int_\gamma (x^2+z^2)\delta \, ds$$

$$I_z = \int_\gamma (x^2+y^2)\delta \, ds$$

$$I_x = \int_0^{2\pi} (\sin^2 t + t^2)(\cos^2 t + \sin^2 t)e^t \sqrt{2}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} (\sin^2 t + t^2)e^t \, dt$$

$$I_y = \int_0^{2\pi} (\cos^2 t + t^2)(\cos^2 t + \sin^2 t)e^t \sqrt{2}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} (\cos^2 t + t^2)e^t \, dt$$

$$I_z = \int_0^{2\pi} (\cos^2 t + \sin^2 t)(\cos^2 t + \sin^2 t)e^t \sqrt{2}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} e^t \, dt$$

ex: Same path as before but assume constant density δ. Calculate center of mass of density δ. Calculate [illegible]

the wire. Intuitively $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \pi)$.

$$M_x = \int_\gamma x \delta \, ds = \delta \int_0^{2\pi} (\cos t) \sqrt{2} \, dt = 0$$

$$M_y = \int_\gamma y \delta \, ds = \delta \int_0^{2\pi} (\sin t) \sqrt{2} \, dt = 0$$

$$M_z = \int_\gamma z \delta \, ds = \delta \int_0^{2\pi} t \sqrt{2} \, dt = \delta 2 \sqrt{2} \pi^2$$

$$M = \int_\gamma \delta \, ds = \delta \int_0^{2\pi} \sqrt{2} \, dt = \delta 2\pi \sqrt{2}$$

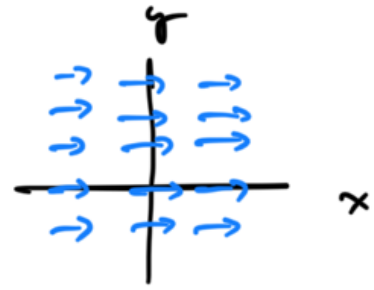
so $$(\bar{x}, \bar{y}, \bar{z}) = \frac{(0, 0, \delta 2 \sqrt{2} \pi^2)}{\delta 2\pi \sqrt{2}}$$

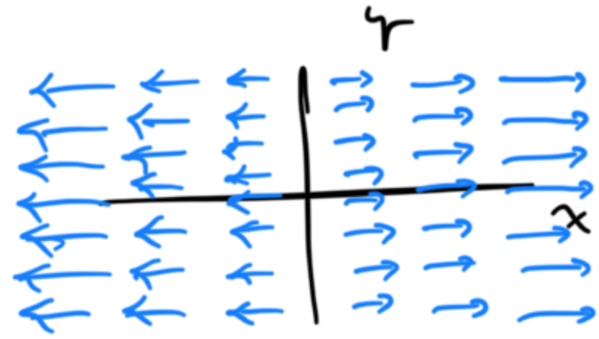
$$= (0, 0, \pi)$$

16.2: Line integrals, Vector Fields, Work, Circulation, Flux.

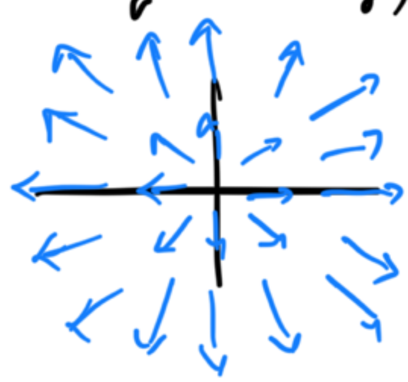
A vector field is a map that associates each point (x,y) a vector $\vec{F}(x,y)$. Can visualize them by drawing a bunch of arrows.

e.g.: $\vec{F}(x,y) = (1,0)$ e.g. $\vec{F}(x,y) = (x, 0)$





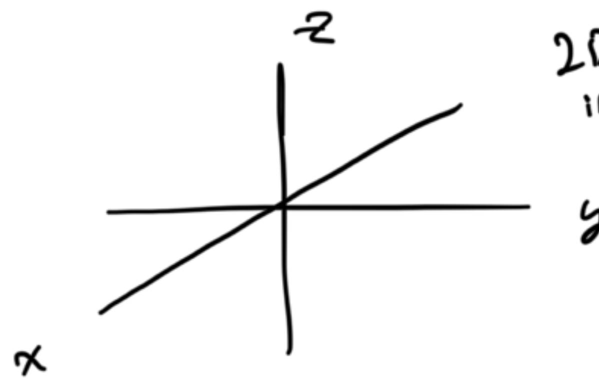
e.g. $\vec{F}(x,y) = (x,y)$



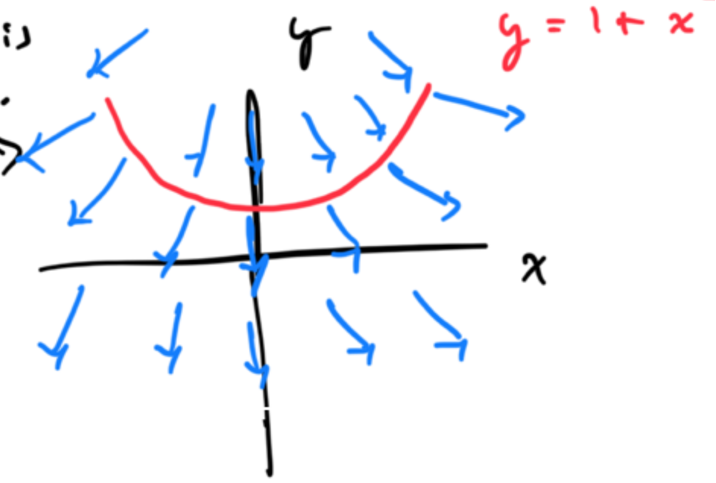
Why? : Velocity field of wind map, Electric field of particles, magnetic fields, Force fields more generally, etc.

ex: let $f(x,y,z) = x^2 - y$ then

$$\nabla f(x,y,z) = (2x, -1, 0)$$

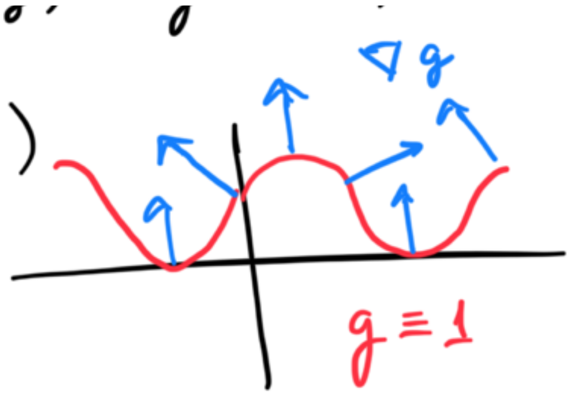


2D-slice is informative.

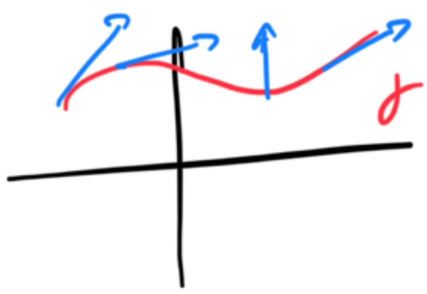


example of a gradient vector field. Take your favorite function e.g. $g(x,y) = y - \sin(x)$

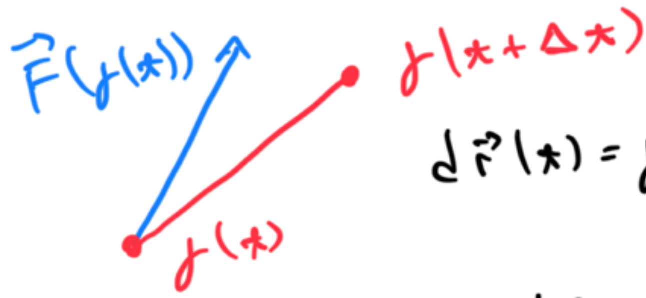
and $\nabla g(x,y) = (-\cos(x), 1)$



Let's say have a force vector field $\vec{F}(x,y)$. Say you have a curve γ. Want to calculate how much work it takes to do $\vec{F}(x,y)$ along $\vec{\gamma}$.



at infinitesimal level...



$$d\vec{r}(t) = \gamma(t+\Delta t) - \gamma(t)$$

$$W(\gamma(t)) \approx \vec{F}(\gamma(t)) \cdot d\vec{r}(t)$$
$$\approx \vec{F}(\gamma(t)) \cdot (\gamma(t+\Delta t) - \gamma(t))$$

So $$\text{Work} = \int_\gamma \vec{F} \cdot d\vec{r} = \int_a^b \underbrace{\vec{F}(\gamma(t)) \cdot \frac{d\vec{r}}{dt}(t)\, dt}_{\text{we did last section.}}$$

ex: Consider $\vec{F}(x,y,z) = (x-y, x+y, z)$ and calculate work along straight line from $(0,0,0)$ to $(1,1,0)$.

$\gamma(t) = (t, t, 0)$ so $\frac{d\vec{r}}{dt} = (1,1,0)$

$\vec{F}(\gamma(t)) = (t-t, t+t, 0) = (0, 2t, 0)$

So $W = \int_0^1 (0, 2t, 0) \cdot (1,1,0)\, dt = \int_0^1 2t\, dt = 1\, Nm$ //

Gives rise to notion of a <u>line integral</u> (even though not always lines).

def: let $\vec{F}$ be a vector field and let γ be a smooth curve parametrized by $\vec{r}(t)$. Define

$$\int_\gamma \vec{F} \cdot \vec{T}\, ds = \int_\gamma \vec{F} \cdot \frac{d\vec{r}}{ds}\, ds = \int_\gamma \vec{F} \cdot d\vec{r}$$ //

where s is arc length parameter and $\vec{T}$ is unit tangent vector.

ex: let $\vec{F}(x,y,z) = (y^2, 2xy, -1)$.

Calculate line integral of $\vec{F}$ over the curve $\gamma(t) = (t, t^2, 1)$ from $(0,0,1)$ to $(1,1,1)$.

$\vec{F}(\gamma(t)) = (t^4, 2t^3, -1)$

$\frac{d\vec{r}}{dt} = (1, 2t, 0)$ $\quad t = 0$ to $t = 1$.

So $\int_\gamma \vec{F} \cdot d\vec{r} = \int_0^1 (t^4, 2t^3, -1) \cdot (1, 2t, 0)\, dt$

$= \int_0^1 t^4 + 4t^4 + 0\, dt = \int_0^1 5t^4\, dt$

$= 1$

You may see differential form version that looks like

$$\int_\gamma M\,dx + N\,dy + P\,dz$$

which is just done in the following manner.

ex: Calculate $\int_\gamma x\,dx + y\,dy + z\,dz$

along the curve $\gamma(t) = (t, t^2, t^3)$ from $t=0$ to $t=1$.

$$\begin{aligned} x &= t & dx &= dt \\ y &= t^2 & dy &= 2t\,dt \\ z &= t^3 & dz &= 3t^2\,dt \end{aligned}$$

so ..

$$I = \int_0^1 t\,dt + (t^2)(2t)\,dt + (t^3)(3t^2)\,dt$$

$$= \int_0^1 (t + 2t^3 + 3t^5)\,dt = \frac{1}{2} + \frac{2}{3} + \frac{3}{6}$$

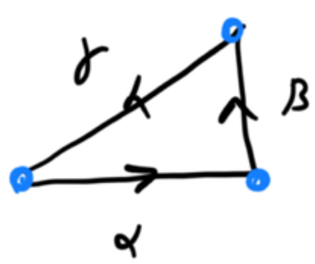
$$= 3+4+3/6 = 10/6 = 5/3$$

def: if $\gamma(t)$ is a smooth curve and $\vec{F}$ is a velocity field, e.g. velocity of water, then the <u>flow</u> of $\vec{F}$ along γ is given by

$$\text{Flow} = \int_\gamma \vec{F} \cdot d\vec{r}$$

this is called a <u>flow integral</u>. If γ starts and ends at the same point, then we call the integral the <u>circulation</u>. Denoted $\oint$ or $\oint$ depending on orientation.

ex: let $\vec{F}(x,y) = (x+y)\hat{i} + (x-y)\hat{j}$ and let γ be the triangle with vertices at $(0,0)$ $(1,0)$ and $(1,1)$. Write out integrals to calculate the circulation if path is traversed counter clockwise.



$$\alpha = (t, 0)$$
$$\beta = (1, t)$$
$$\gamma = (1-t, 1-t)$$

α: $d\vec{r} = (1,0)\,dt$ $\quad \vec{F}\cdot d\vec{r} = (t, t)\cdot(1,0)\,dt$

β: $d\vec{r} = (0,1)\,dt$ $\quad \vec{F}\cdot d\vec{r} = (1+t, 1-t)\cdot(0,1)\,dt$

γ: $d\vec{r} = (-1,-1)\,dt$ $\quad \vec{F}\cdot d\vec{r} = (2-2t, 0)\cdot(-1,-1)\,dt$

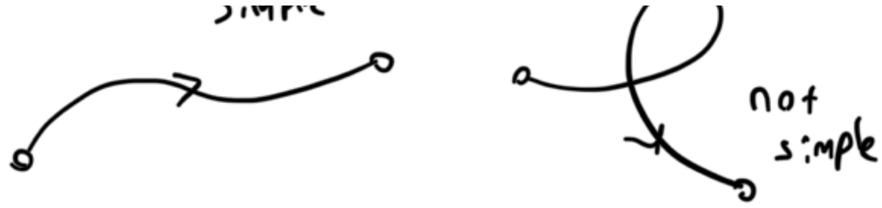
So $\displaystyle\int_\alpha \vec{F}\cdot d\vec{r} = \int_0^1 t\,dt = 1/2$

$\displaystyle\int_\beta \vec{F}\cdot d\vec{r} = \int_0^1 1-t\,dt = 1/2$ $\quad$ So $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = 1/2 + 1/2 - 1 = 0$

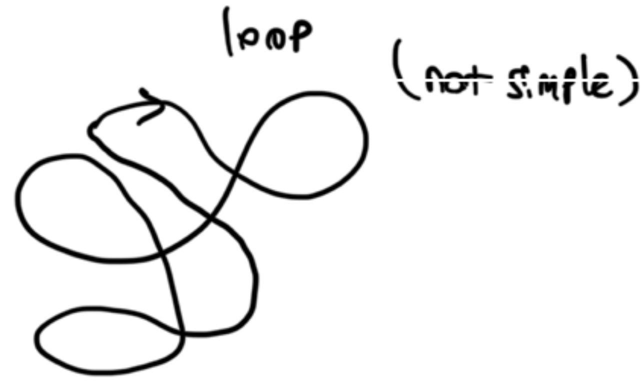
$\displaystyle\int_\gamma \vec{F}\cdot d\vec{r} = \int_0^1 -(2-2t)\,dt = -1$

Flux integrals.

A curve in $\mathbb{R}^2$ is called simple if it does not cross itself.

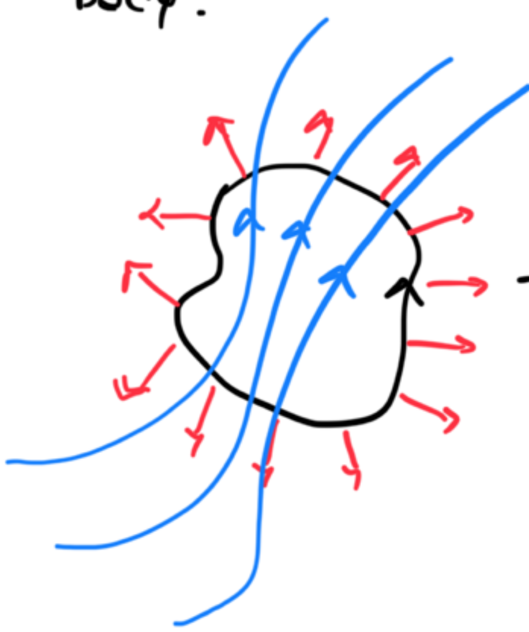


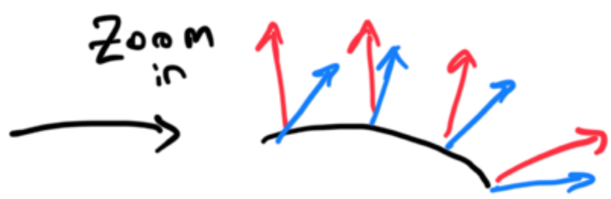
A curve is called a loop if it starts and ends at the same point.



Sometimes interested in calculating how much enters a body and exits a body.

e.g.





$\vec{F} \cdot \vec{n}$

$\vec{n}$ is unit

$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}$$

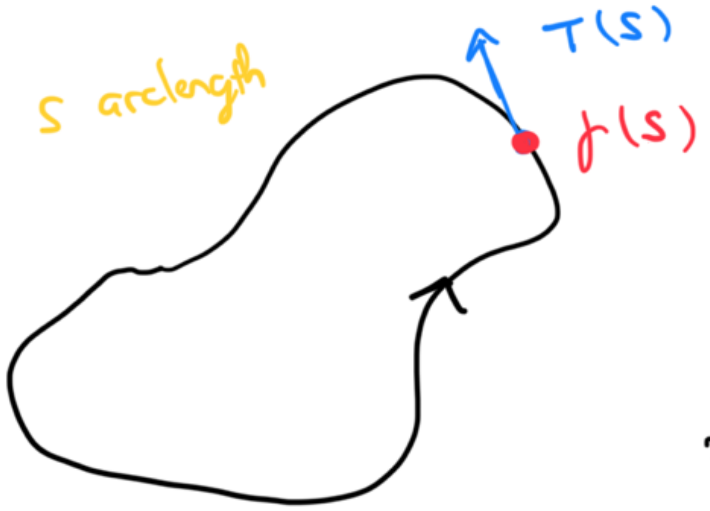
Want to add all $\vec{F} \cdot \vec{n}$ along boundary so get

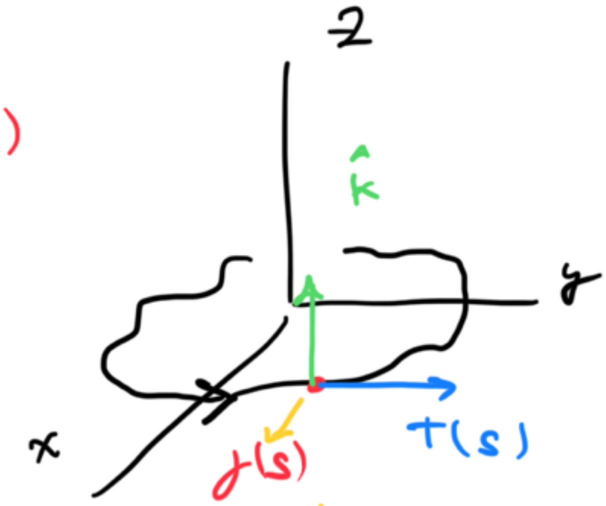
def: let C be simple closed curve in the plane and consider vector field $\vec{F}$. We define the flux of $\vec{F}$ across C to be

Flux of $\vec{F}$ across $C = \int_C \vec{F} \cdot \vec{n}\, ds$

where $\vec{n}$ is unit outward normal to C.

Question: How to get unit normal?





$T \times \hat{k}$ = outward if $\gamma(s)$ counterclockwise

In coordinates if $\gamma(s) = (x(s), y(s), 0)$ then

$$T(s) = \left(\frac{dx}{ds}, \frac{dy}{ds}, 0\right)$$

$$\vec{n}(s) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{\imath}\left(\frac{dy}{ds}\right) - \hat{\jmath}\left(\frac{dx}{ds}\right) + 0\hat{k}$$

so $\vec{n}(s) = \left(\frac{dy}{ds}, -\frac{dx}{ds}, 0\right)$ then

$$\vec{F}(x,y) \cdot \vec{n}(s) = M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$$

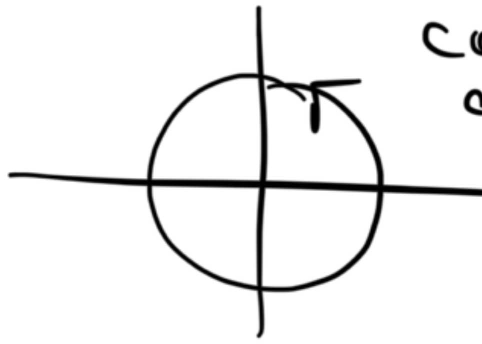
$$\vec{F}(x,y) = (M(x,y), N(x,y))$$

So...

$$\text{Flux of } \vec{F} = \oint_C M\, dy - N\, dx$$

ex: let $\vec{F}(x, y) = (-y, x)$. Evaluate the flux of $\vec{F}$ across the unit circle.

$\gamma(t) = (\cos t, \sin t)$



so $dx = -\sin t \, dt$

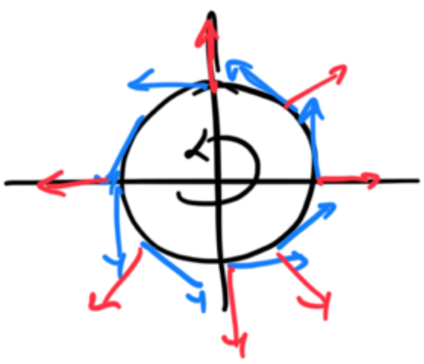
$dy = \cos t \, dt$ and $\vec{F}(\cos t, \sin t)$

$= (\underbrace{-\sin t}_{M}, \underbrace{\cos t}_{N})$

thus $\oint_\gamma \vec{F} \cdot \vec{n} \, ds = \int_0^{2\pi} \underbrace{(-\sin t)}_{M} \underbrace{(\cos t) \, dt}_{dy} = 0$

$= \underbrace{(\cos t)}_{N} \underbrace{(-\sin t) \, dt}_{dx}$

why?



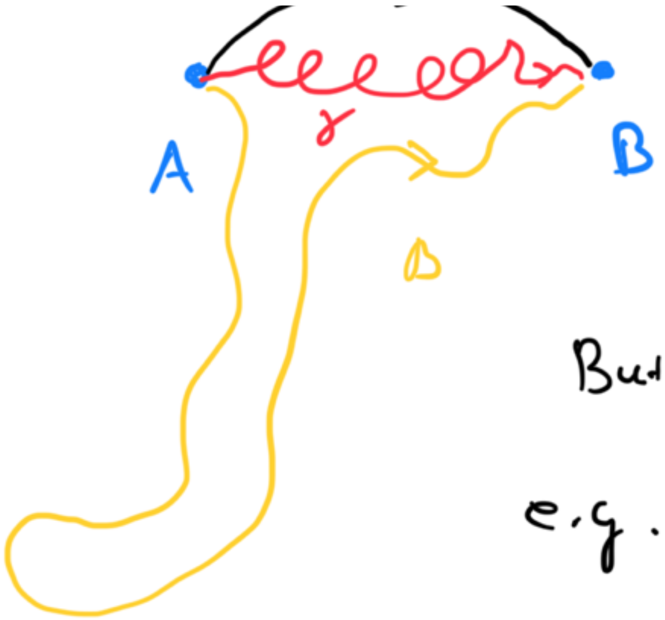
nothing entering or exiting!

16.3: Path independence, Conservative Fields, and Potential Functions

Typically if integrating vector field along a path, it depends on which path.

typically

$\int \vec{F} \cdot d\vec{r} \neq \int \vec{F} \cdot d\vec{r}$



$$\int_\alpha \vec{F}\cdot d\vec{r} \neq \int_\beta \vec{F}\cdot d\vec{r}$$

$$\neq \int_\gamma \vec{F}\cdot d\vec{r}$$

But sometimes it happens!

e.g. $\vec{F} = (0, -g)$ gravity.

let γ be any path from $(0,0)$ to $(0,1)$.

$(x(t), y(t)) = \gamma(t)$ with $\gamma(0) = A$ $\gamma(1) = B$

then $d\vec{r} = (dx, dy) = (x'(t)\,dt,\ y'(t)\,dt)$

so $$W = \int_\gamma \vec{F}\cdot d\vec{r} = \int_0^1 -g\,\frac{dy}{dt}\,dt$$

$$= -g\left(y(1) - y(0)\right) = -g(0-1)$$

$$\approx 10\ \text{Nm}$$

def: let $\vec{F}$ be a vector field such that for any two points A and B, and any two paths α, β connecting them we have

$$\int_\alpha \vec{F}\cdot d\vec{r} = \int_\beta \vec{F}\cdot d\vec{r}$$

We say $\vec{F}$ is a <u>conservative</u> vector field and

$\int_\alpha \vec{F}\cdot d\vec{r}$ is <u>path independent</u>.

Why did above example work and how can you construct more? Here's a definition and a consequence.

def: if $\vec{F}$ is a vector field such that $\vec{F} = \nabla f$ for some scalar function f, we call f a potential function for $\vec{F}$.

e.g. gravitational potential, electrical potential.

Why these nice? Because we have

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_\gamma \nabla f \cdot d\vec{r} = f(B) - f(A)$$

where $\gamma(0) = A$ and $\gamma(1) = B$.

In particular, $\oint_\gamma \vec{F} \cdot d\vec{r} = 0$ for any closed loop γ.

Word of warning: Assumptions are made on domain of $\vec{F}$, domain of f, derivatives, and paths.

- We require $\vec{F}$ to be defined on a connected region containing path γ.

e.g.

$\leftarrow$ $\vec{F}$ defined on all of this.

- Same for f

- Want curves to be piecewise smooth

Wario! or Mario!

- In future theorems we'll need simply connected which is a connected region 'without holes.'

e.g.

simply connected

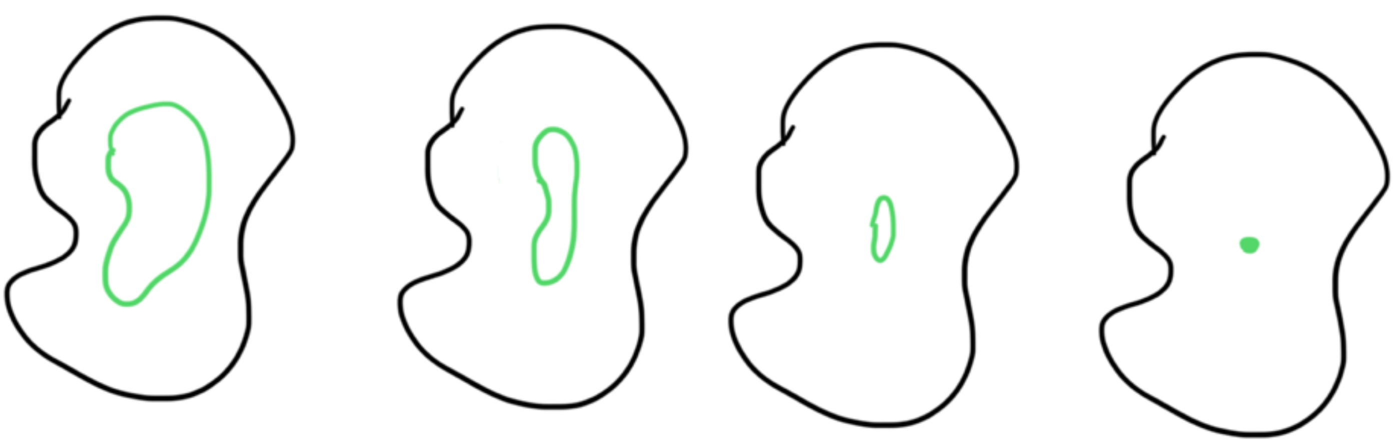
not simply connected

whole solid ↓

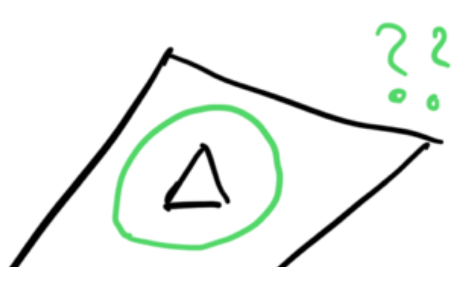
simply connected

whole solid not simply connected.

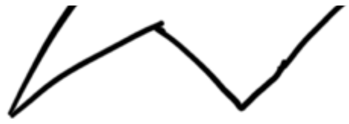
idea is every loop can be shrunk to a point.



simply connected



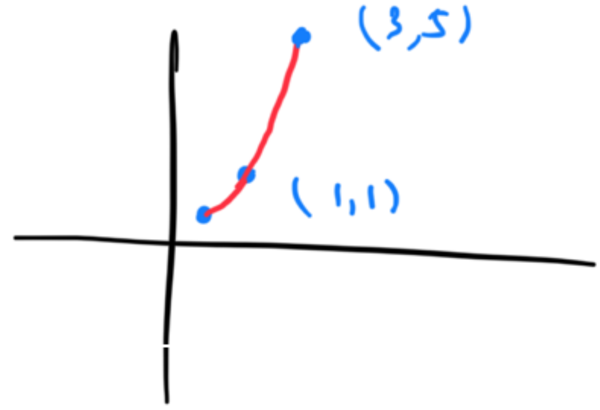
not simply connected.



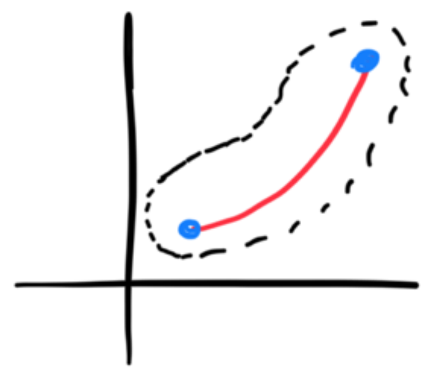
ex: Integrate $\vec{F} = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right)$

along the parabola connecting the points

$(1,1)$, $(2,2)$ and $(3,5)$, from $(1,1)$ to $(3,5)$



notice $\vec{F}$ is 'bad' only at $(0,0)$. We can contain entire parabola in a region that avoids $(0,0)$.



good!

May apply theorem. Note $f(x,y) = \ln(x^2+y^2)$

then $\nabla f = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right)$ so

$$\int_\gamma \nabla f \cdot d\vec{r} = f(B) - f(A) = \ln(3^2+5^2) - \ln(1^2+1^2)$$

$$= \ln(34) - \ln(2)$$

$$= \ln(17)$$

We saw that if $\vec{F}$ comes from a potential function i.e. $\vec{F}$ is a gradient field means path integrals are independent of path and only depend on endpoints. Under mild assumptions on domain the converse is true.

theorem: let $\vec{F}$ be a vector field whose component functions are continuous on an open connected region D. Then $\vec{F}$ is conservative iff $\vec{F}$ is a gradient field. (i.e. can solve $\vec{F} = \nabla f$ for some function f).

Corollary: $\oint_\gamma \vec{F} \cdot d\vec{r} = 0$ for all loops in domain D iff $\vec{F}$ is conservative on D.

Summarize as... let D be open domain and $\vec{F}$ be 'nice'.

$\vec{F} = \nabla f$ on D for some f iff $\vec{F}$ conservative on D iff $\oint_\gamma \vec{F} \cdot d\vec{r} = 0$ for all loops in D

Two obvious questions:

1. How do you test if a vector field is conservative?

2. How do you find a potential function if $\vec{F}$ is conservative?

theorem: let $\vec{F} = (M, N, P)$ all M, N, P functions of x, y, z, on simply connected domain D.

Assume nice differentiability. Then $\vec{F}$ conservative iff $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$ = curl of $\vec{F}$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k} = \vec{0}$$

(should mention in 2D same deal sort of)

$\vec{F} = (M, N)$ then on simply connected domain $\vec{F}$ conservative iff $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

(Just like we did before!)

ex: Determine if

$$\vec{F} = \left(\cos(y)e^{yz^2},\ e^{yz^2} x z^2 \cos(y) + \ln(z) - e^{yz^2} x \sin(y),\ 1 + \frac{y}{z} + 2e^{yz^2} x y z \cos(y)\right)$$ is conservative

on $z>0$. Now $\vec{F}$ defined everywhere on $z>0$ simply connected, so curl test will suffice.

You can do these calculations and find out that

$\nabla \times \vec{F} = \vec{0}$ so by the theorem it's conservative.

What potential function? Options.

1. $\cos(y) e^{yz^2}$ wrt to x ← easiest.

2. $e^{yz^2} x z^2 \cos(y) + \ln(z) - e^{yz^2} x \sin(y)$ wrt to y

3. $1 + \frac{y}{z} + 2e^{yz^2} xyz \cos(y)$ wrt to z

$$f(x,y,z) = x\cos(y) e^{yz^2} + h(y,z).$$

$$\frac{df}{dy} = \left(-x\sin(y) + x\cos(y) z^2\right) e^{yz^2} + \frac{dh}{dy}$$

$$= -x\sin(y) e^{yz^2} + x\cos(y) z^2 e^{yz^2} + h_y$$

$$= -x\sin(y) e^{yz^2} + xz^2\cos(y) e^{yz^2} + \ln(z)$$

so $\frac{dh}{dy} = \ln(z)$ so $h(y,z) = y\ln(z) + j(z)$.

$$\frac{df}{dz} = x\cos(y) e^{yz^2} \cdot 2yz + \frac{y}{z} + j'(z)$$

$$= 1 + \frac{y}{z} + 2e^{yz^2} xyz\cos(y)$$

so $j'(z) = 1 \Rightarrow j(z) = z + C$

thus potential function is

$H(x,y,z) = x\cos(y)\,e^{\cdots} + y\ln(\cdots) \cdots$
constant C'.

Simply connected hypothesis is nessary

ex: $\vec{F}(x,y) = \left(\underbrace{-y/_{x^2+y^2}}_{M},\ \underbrace{x/_{x^2+y^2}}_{N} \right)$

then $\dfrac{dM}{dy} - \dfrac{dN}{dx} = 0$ (can check).

However domain not simply connected.

Moreover if $\int_\gamma \vec{F}\cdot d\vec{r}$ with γ unit circle,

get $d\vec{r} = (-\sin(t), \cos(t))\,dt$ and

$(\vec{F}\circ\gamma)(t) = \left(\dfrac{-\sin t}{\cos^2 t + \sin^2 t}, \dfrac{\cos t}{\cos^2 t + \sin^2 t} \right)$

so $\vec{F}\cdot d\vec{r} = 1\,dt$ and

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 1\,dt = 2\pi \text{ so}$$

not conservative.

No potential function exists!

Differential forms

Recall we had integrals of the form

$\int M\,dx + N\,dy$ (2D) and

$\int M\,dx + N\,dy$ (2D)

$\int M\,dx + N\,dy + P\,dz$ (3D)

Quantities $M\,dx + N\,dy$ or $M\,dx + N\,dy + P\,dz$ are called differential forms (more specifically they are called 1-forms) You can integrate them over curves like we've been doing.

A particularly nice type of differential form is an exact differential form.

An exact differential form $\omega = M\,dx + N\,dy + P\,dz$ is one where you can find a function f so that

$$\omega = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \quad \text{so}$$

$$M = \frac{\partial f}{\partial x} \qquad N = \frac{\partial f}{\partial y} \qquad P = \frac{\partial f}{\partial z}$$

It's nice because we have for any path γ,

$$\int_\gamma \omega = \int_\gamma M\,dx + N\,dy + P\,dz = \int_\gamma df$$

$$= \int_\gamma \nabla f \cdot d\vec{r} = f(B) - f(A)$$ where A is starting point of γ and B is the end.

To test if a diff form

$$\omega = M\,dx + N\,dy + P\,dz$$ is exact

on a simply connected domain we use the

Same test as before,

theorem: $\omega = M\,dx + N\,dy + P\,dz$ is exact on a simply connected domain iff

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

(in 2D $\omega = M\,dx + N\,dy$ need $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$)

ex: Show $\omega = (2x+y)\,dx + x\,dy + 2z\,dz$ is exact and then find a function f so that $df = \omega$.

1. Exactness just calculate and note domain is all of $\mathbb{R}^3$.

2. $\frac{\partial f}{\partial x} = 2x+y \qquad \frac{\partial f}{\partial y} = x \qquad \frac{\partial f}{\partial z} = 2z$

why not

$$f(x,y,z) = xy + g(x,z)$$

$$\frac{\partial f}{\partial x} = y + \frac{\partial g}{\partial x} = y + 2x \quad \text{so} \quad \frac{\partial g}{\partial x} = 2x$$

$$\text{so} \quad g(x,z) = x^2 + h(z)$$

$$\frac{\partial f}{\partial z} = 0 + 0 + h'(z) = 2z \quad \text{so} \quad h(z) = z^2 + C$$

$$f(x,y,z) = x^2 + xy + z^2 + C$$

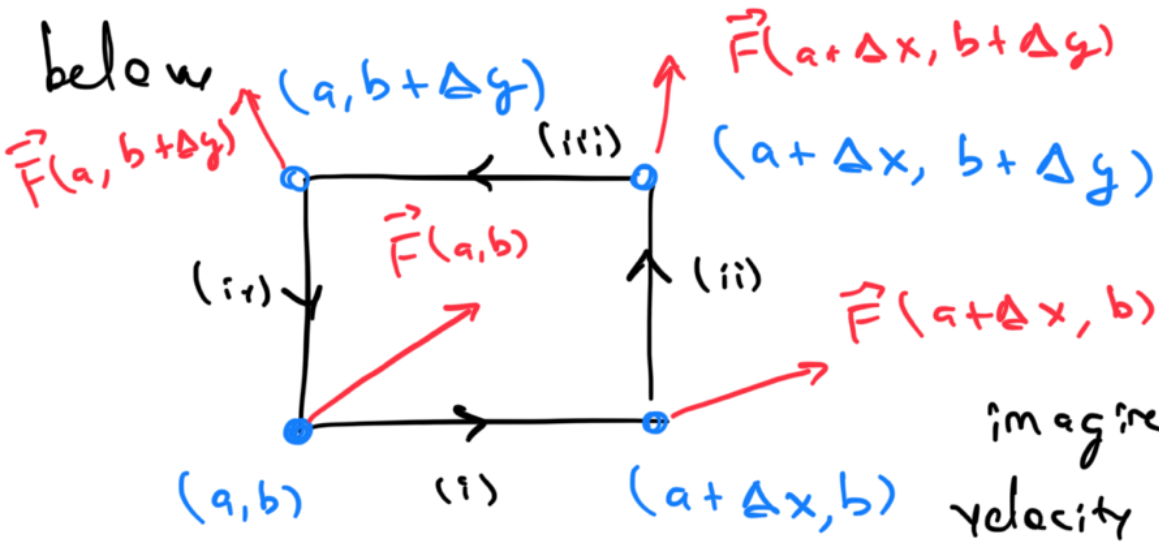
So $f(x,y,z) = x + xy + z$.

If want $\int_\gamma \omega$ when γ goes from $(0,0,0)$ to $(1,1,1)$ then

$$\int_\gamma \omega = f(1,1,1) - f(0,0,0)$$
$$= 3 \ //$$

16.4: Green's Theorem in the Plane

Consider a force field $\vec{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$ in the plane. Consider a small rectangle below



imagine $\vec{F}$ is a velocity field of a fluid.

Wish to calculate velocity along the rectangle.

(i). $\vec{F}(a,b)\cdot d\vec{r} = M(a,b)\,\Delta x$

(ii). $\vec{F}(a+\Delta x, b)\cdot d\vec{r} = N(a+\Delta x, b)\,\Delta y$

(iii). $\vec{F}(a, b+\Delta y)\cdot d\vec{r} = M(a, b+\Delta y)(-\Delta x)$

(iv). $\vec{F}(a,b)\cdot d\vec{r} = N(a,b)(-\Delta y)$

(i) + (iii)

$= [M(a, b+\Delta y) - M(a,b)](-\Delta x)$

$\approx -\dfrac{dM}{dy}(a,b)\,\Delta x\,\Delta y$

(ii) + (iv)

$= [N(a+\Delta x, b) - N(a,b)]\,\Delta y$

$\approx \dfrac{dN}{dx}(a,b)\,\Delta x\,\Delta y$

so $\sum_i (*) = \left(\dfrac{dN}{dx} - \dfrac{dM}{dy}\right)(a,b)\,\Delta x\,\Delta y$

circulation around rectangle =

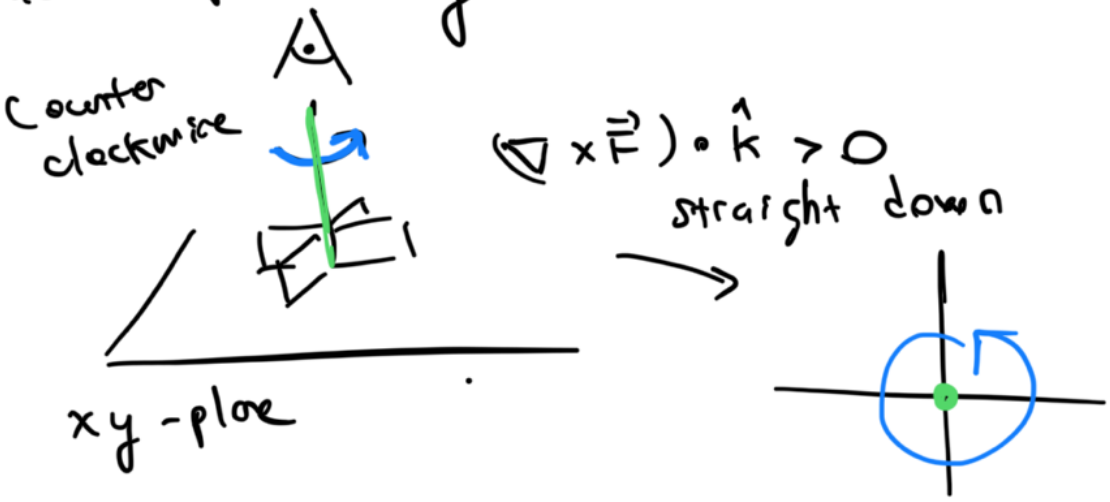
area of rectangle $= \Delta x \Delta y$

Call $\frac{dN}{dx} - \frac{dM}{dy}$ circulation density

$= (\nabla \times \vec{F}) \cdot \hat{k}$, $\vec{F} = (M, N, 0)$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d_x & d_y & d_z \\ M & N & 0 \end{vmatrix}$$

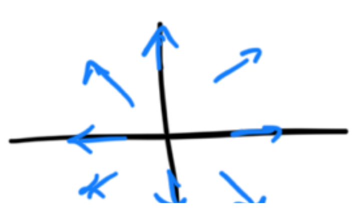
= measurement how fast a paddle spins subject to $\vec{F}$.



ex: Calculate the circulation density of the below vector fields

1. $\vec{F} = (x, y)$
2. $\vec{F} = (y, -x)$
3. $\vec{F} = (0, x)$
4. $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$

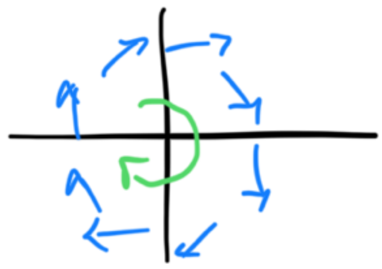
1. $(\nabla \times \vec{F}) \cdot \hat{k} = \frac{d}{dx} y - \frac{d}{dy} x = 0$



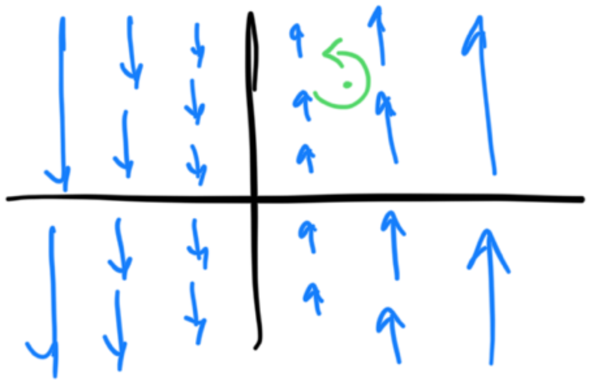
no circulation!

2. $(\nabla \times \vec{F}) \cdot \hat{k} = \frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y) = -2 < 0$

clockwise < 0



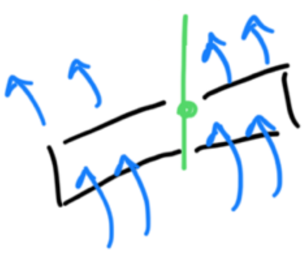
3. $(\nabla \times \vec{F}) \cdot \hat{k} = \frac{\partial}{\partial x}(x) = 1 > 0$



4. $(\nabla \times \vec{F}) \cdot \hat{k} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = 0$!

this is weird, this is just like (2), so what gives? Note $\vec{F}$ not defined at $(0,0)$, but also, if you put a paddle in water, it'd be like this

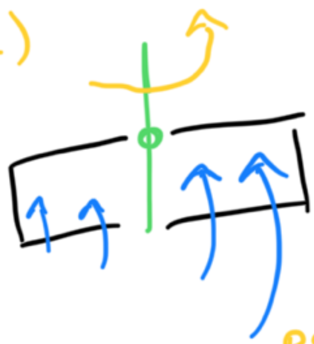
(4)



both hitting at just right forces to keep the thing from moving bc $|\vec{F}| = 1$.

vs.

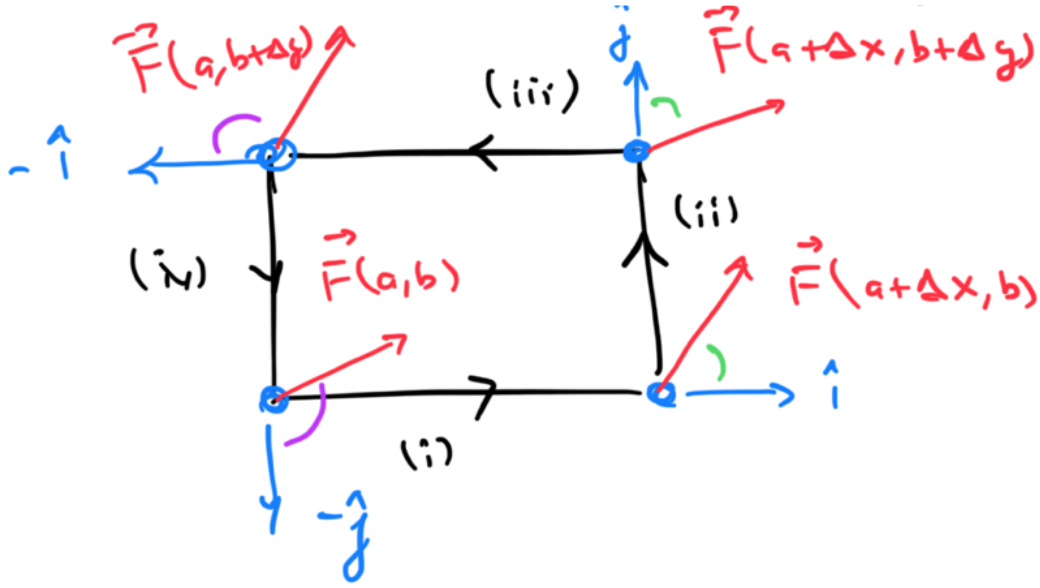
(2)



as you move farther out there's more force, so the paddle will spin bc $|\vec{F}|^2 = r^2$

Divergence

Again consider $\vec{F} = (M, N)$ now want to measure how much a fluid is leaving a small area.



(i). $\vec{F}(a,b)\cdot\vec{n}\,ds = -N(a,b)\,\Delta x$

(ii). $\vec{F}(a+\Delta x,b)\cdot\vec{n}\,ds = M(a+\Delta x,b)\,\Delta y$

(iii). $\vec{F}(a,b+\Delta y)\cdot\vec{n}\,ds = N(a,b+\Delta y)(+\Delta x)$

(iv). $\vec{F}(a,b)\cdot\vec{n}\,ds = M(a,b)(-\Delta y)$

$$\text{(i)}+\text{(iii)} = \big(N(a,b+\Delta y) - N(a,b)\big)\Delta x \approx \frac{dN}{dy}\Delta y\,\Delta x$$

$$\text{(ii)}+\text{(iv)} = \big(M(a+\Delta x,b) - M(a,b)\big)\Delta y \approx \frac{dM}{dx}\Delta x\,\Delta y$$

so $\sum_i (\cdot) = \left(\frac{dM}{dx} + \frac{dN}{dy}\right)\Delta x\,\Delta y$

= flux of $\vec{F}$ across box

so Avg flux across area

$$= \left(\frac{dM}{dx} + \frac{dN}{dy}\right)(a,b)$$

<u>flux density</u>

or

<u>divergence</u>

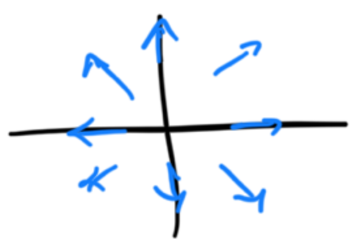
Also denoted by $\nabla\cdot\vec{F} = \frac{dM}{dx} + \frac{dN}{dy}$

measures how much stuff is expanding or contracting at a point.

ex: Calculate flux of vector fields below.
Explain results with pictures.

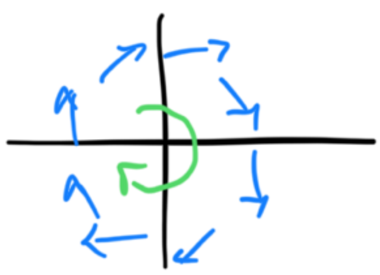
1. $\vec{F} = (x, y)$
2. $\vec{F} = (y, -x)$
3. $\vec{F} = (0, x)$
4. $\vec{F} = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right)$

1.



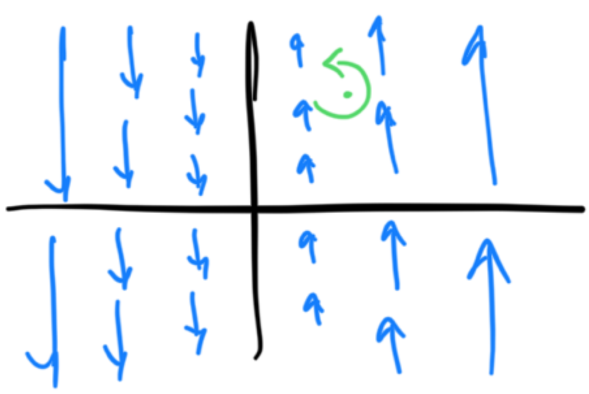
We expect $\nabla \cdot \vec{F} > 0$. In fact,
$\nabla \cdot \vec{F} = \frac{d}{dx}(x) + \frac{d}{dy}(y) = 2 > 0$.

2.



expect $\nabla \cdot \vec{F} = 0$.
$\frac{d}{dx}(-y) + \frac{d}{dy}(x) = 0$.
Just rotating.

3.



$\frac{d}{dx}(0) + \frac{d}{dy}(x) = 0$

only expanding in a single direction. Kinda like how the area of a line is 0.

4. Similar to 2. No expansion/contraction.
Calculation on your own.

Green's Theorem

let γ be a simple closed curve positively oriented (counter clockwise). let $\vec{F} = M\hat{i} + N\hat{j}$ and $\vec{T}$ and $\vec{n}$ denote unit tangent / outward unit normal to γ. let R denote inside of γ. Then

$$\oint_\gamma \vec{F}\cdot\vec{T}\,ds = \oint_\gamma M\,dx + N\,dy = \iint_R \underbrace{(\nabla\times\vec{F})}_{\text{curl}}\,dA$$

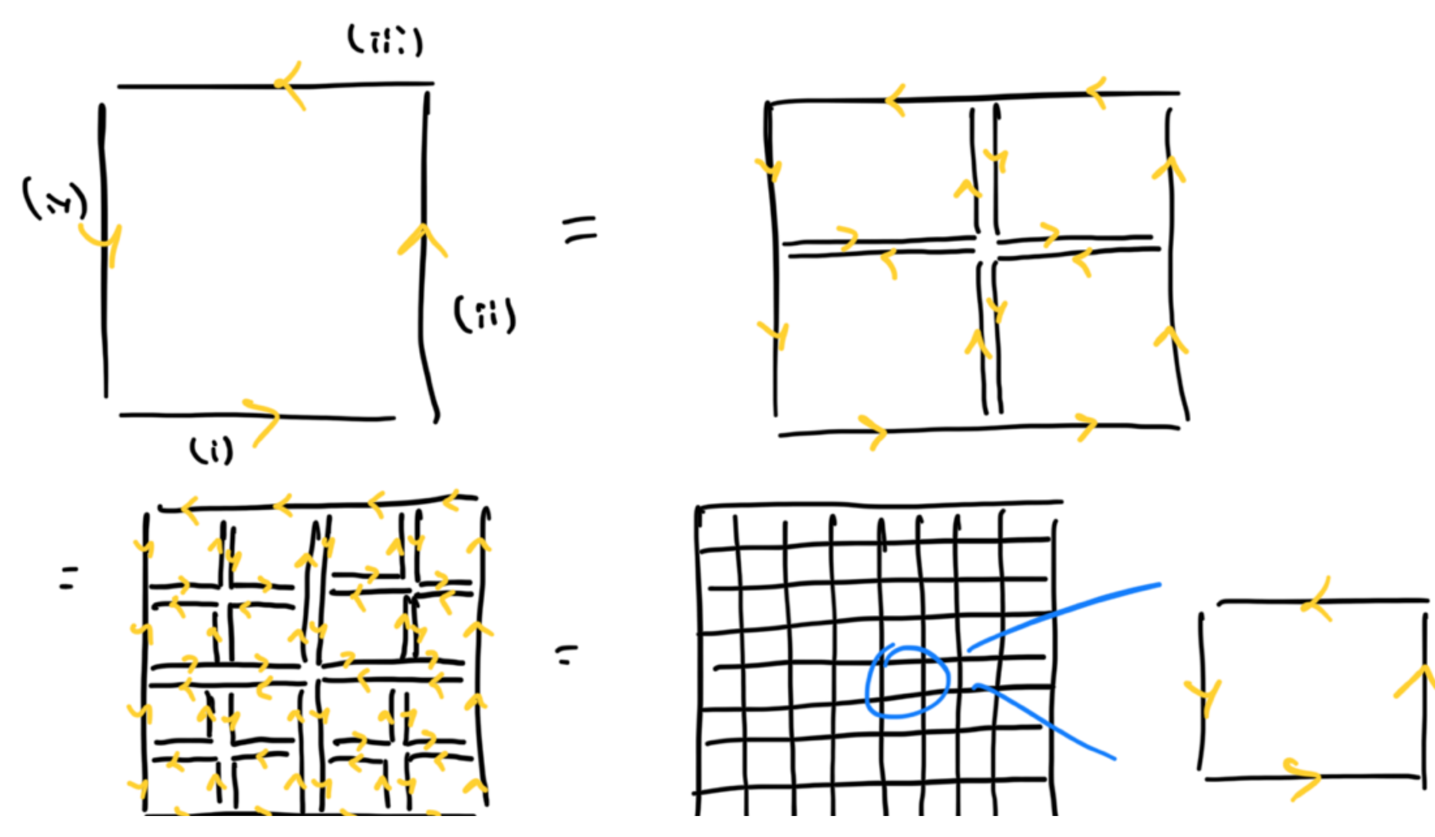
$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \qquad \text{version 1}$$

Also..

$$\oint_\gamma \vec{F}\cdot\vec{n}\,ds = \oint_\gamma M\,dy - N\,dx = \iint_R \underbrace{(\nabla\cdot\vec{F})}_{\text{divergence}}\,dA$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA \qquad \text{version 2}$$

Why $\displaystyle\oint \vec{F}\cdot\vec{T}\,ds = \iint_R (\nabla\times\vec{F})\,dA$?



So $(i)+(ii)+(iii)+(iv) = \sum$ circulation of all squares

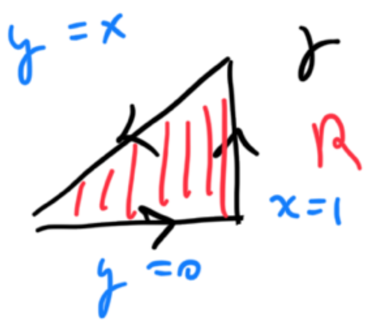
$$= \sum \frac{\text{circulation of } \square}{\text{area of } \square} \cdot \text{area} \to 0$$

$$\oint_{\partial R} \vec{F}\cdot\vec{T}\, ds \to \iint_R \nabla \times \vec{F}\, dA$$

How you get Version 2 from Version 1 is a simple trick. Think about it! (Hint: Choose the appropriate vector field)

ex: Evaluate $\int_\gamma xy\,dx + \frac{1}{2}(x^2+y^2)\,dy$

where γ is the triangle counterclockwise with vertices at $(0,0)$, $(1,0)$, $(1,1)$.



$$xy\,dx + \frac{1}{2}(x^2+y^2)\,dy$$
$$= \left(xy, \tfrac{1}{2}(x^2+y^2)\right)\cdot d\vec{r}$$
$$\vec{F} = \left(xy, \tfrac{1}{2}(x^2+y^2)\right)$$
$$= (M, N)$$

So by Green's theorem...

$$I = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (x - x)\, dA = 0$$

(However, this is no surprise bc

$xy\,dx + \frac{1}{2}(x^2+y^2)\,dy$ is an exact

diff form on simply connected domain!)

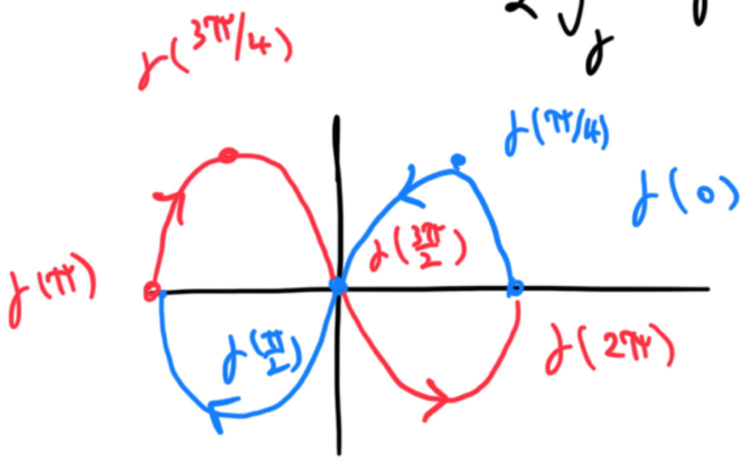
ex: Calculate the area of the region bounded by $x(t) = \cos(t)$, $y = \sin(2t)$ for $0 \le t \le 2\pi$.

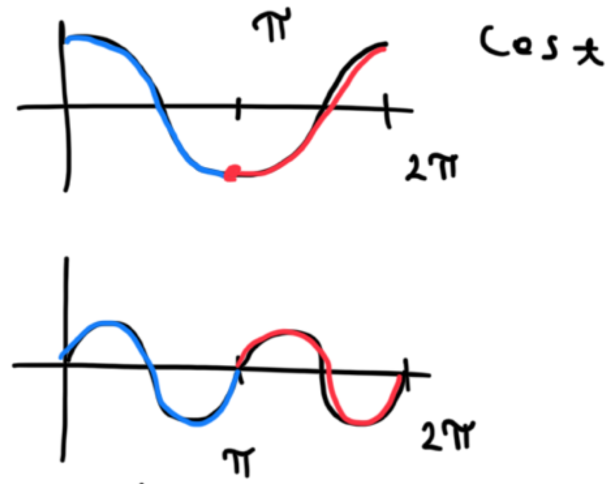
Trick: $\underset{\text{of } R}{\text{Area}} = \iint_R 1 \, dA = \iint_R \frac{1}{2} + \frac{1}{2} \, dA$

$\frac{dN}{dx} = \frac{1}{2} \qquad \frac{dM}{dy} = -\frac{1}{2} \qquad \text{so} \qquad \frac{dN}{dx} - \frac{dM}{dy} = 1$

$M = -y/2$, $N = x/2$ then Green's theorem says

$$\text{Area} = \frac{1}{2} \oint_\gamma -y\,dx + x\,dy \;//$$





$$\text{Area} = 2\,(\text{left loop}) = 2 \oint^{\text{opp!}} -y\,dx + x\,dy$$

$$\text{Area left loop} = \oint y\,dx - x\,dy$$

$$= \int_{\pi/2}^{3\pi/2} \sin(2t)(-\sin(t))\,dt - \cos(t)(2\cos(2t))\,dt$$

$dx = -\sin(t)\,dt$

$dy = 2\cos(2t)\,dt$

$$= 8/3$$

So total area $= 16/3$ //

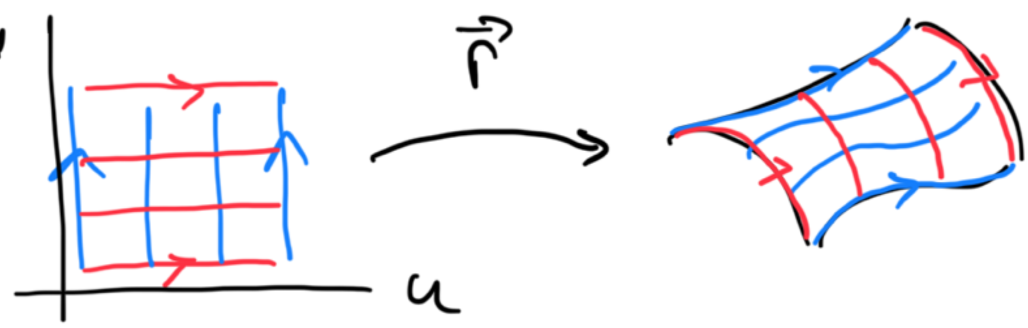
Word of warning: Need $\vec{F}$ to be 'nice' on interior of γ to apply Green's theorem. So be sure to double check it has derivatives and stuff on $\mathbb{R}$.

16.5 : Surfaces and Area

a curve is a map from $\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^2$ or $\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^3$

one-dimensional

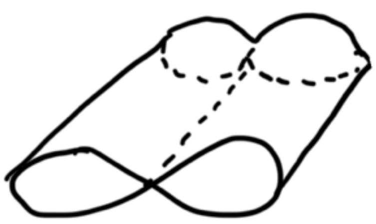
a **surface** is a map from $\mathbb{R}^2 \xrightarrow{\vec{r}} \mathbb{R}^3$



$$\vec{r}(u,v) = f(u,v)\,\hat{i} + g(u,v)\,\hat{j} + h(u,v)\,\hat{k}$$

We don't want our surface to self intersect.

so none of



← bad.

so $\vec{r}(u,v) \neq \vec{r}(u',v')$ if $(u,v) \neq (u',v')$

ex: Parametrize the cone

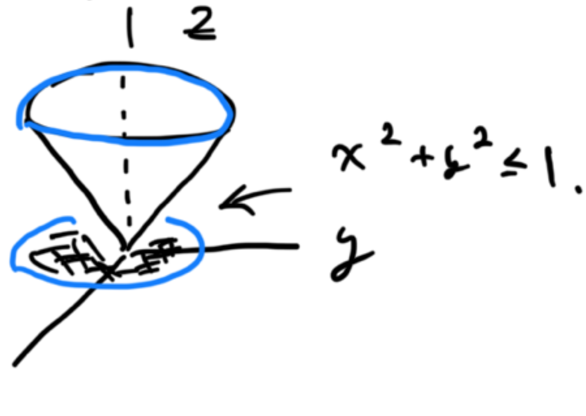
$$x^2+y^2 = z^2 \quad \text{where} \quad 0 \le z \le 1.$$

is solve for $z = \sqrt{x^2+y^2}$

One way is solve ...

so $\vec{r}(x,y) = (x, y, \sqrt{x^2+y^2})$ where

$x^2+y^2 \le 1$.



another is $\vec{S}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

bc $x^2+y^2=z^2$ so $r^2=z^2 \sim r=z$

and $(r\cos\theta, r\sin\theta)$ gives you circle.

ex: Parametrize the sphere $x^2+y^2+z^2=a^2$

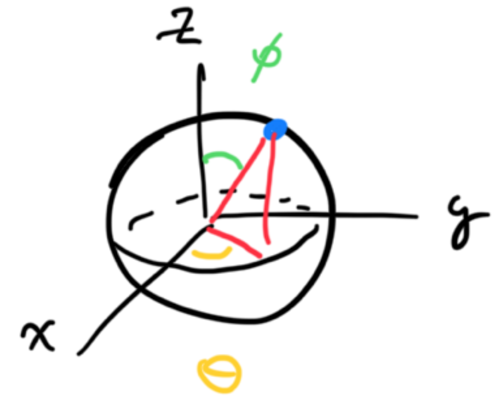
where $a>0$. (spherical co-ordinates

give this to us). $\rho = a$

Recall $x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$



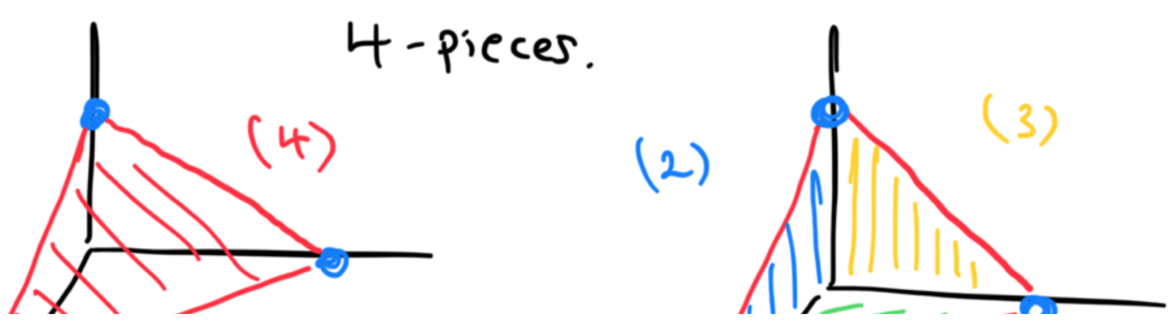
so $\vec{r}(\theta,\phi) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

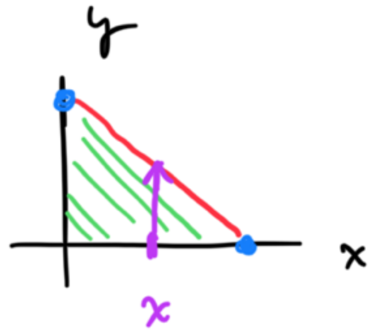
ex: Paramaterize tetrahedron with vertices at

$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

4-pieces.



1.



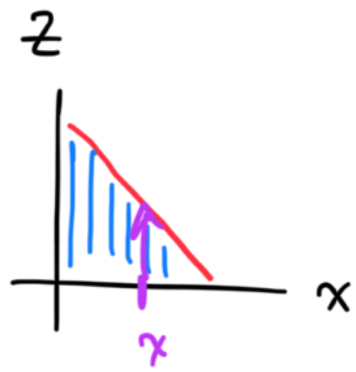
$\vec{r}(u,v) = (u, v(y))$

$0 \le u \le 1 \quad = (u, v(1-u))$

$0 \le v \le 1$

$\vec{r}(u,v) = (u, v(1-u), 0)$ in $\mathbb{R}^3$

2.



$\vec{r}(u,v) = (u, v\,z)$

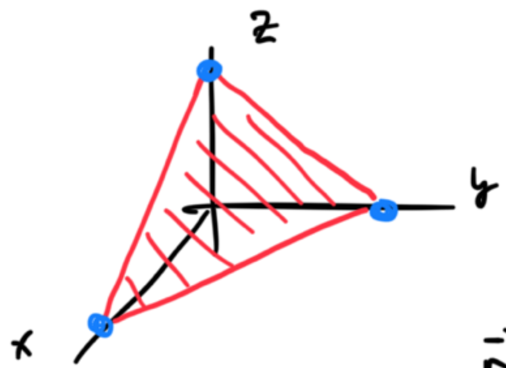
$= (u, v(1-u))$

$\vec{r}(u,v) = (u, 0, v(1-u))$

3.

$\vec{r}(u,v) = (0, u, v(1-u))$

4.



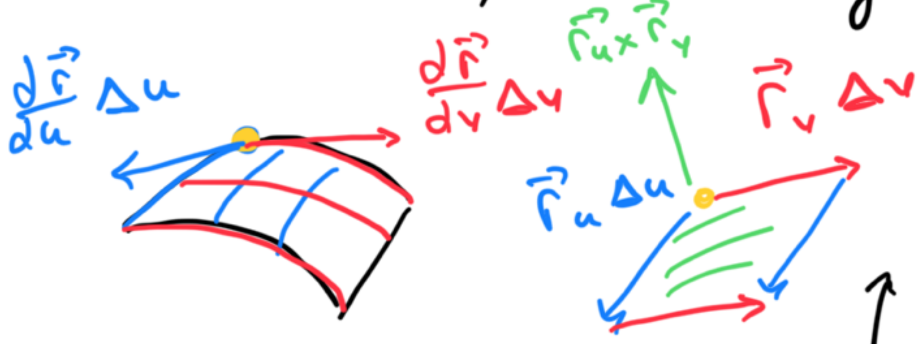
Just use (1) then add z.

$x + y + z = 1$ so

$z = 1 - x - y$

$\vec{r}(u,v) = (u, v(1-u), 1-u-v(1-u))$

Surface area: let's say $\vec{r}(u,v)$ gives a surface.



area $= |\vec{r}_u \times \vec{r}_v|\,\Delta u\,\Delta v$

outward pointing normal.

add all these,

so get Area $= \iint |\vec{r}_u \times \vec{r}_v|\, dA$

R'

ex: A sphere $x^2+y^2+z^2=a^2$

then $\vec{r}(\theta,\phi)=(a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi)$

so $\dfrac{d\vec{r}}{d\theta}=(-a\sin\phi\sin\theta,\ a\sin\phi\cos\theta,\ 0)$

$\dfrac{d\vec{r}}{d\phi}=(a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi)$

so $\vec{r}_\theta\times\vec{r}_\phi=(-a^2\cos\theta\sin^2\phi,\ -a^2\sin^2\phi\sin\theta,$
$-a^2\cos\phi\sin\phi)$

so $|\vec{r}_\theta\times\vec{r}_\phi|=a^2|\sin\phi|$ thus,

$$\text{Area}=\iint_R a^2|\sin\phi|\,dA=\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi}a^2\sin\phi\,d\phi\,d\theta$$

$$=4\pi a^2$$

ex: Calculate the surface area of the surface $z=x^2-y^2$ defined on $x^2+y^2\le1$

$\vec{S}(r,\theta)=(r\cos\theta,\ r\sin\theta,\ (r\cos\theta)^2-(r\sin\theta)^2)$

so $\dfrac{d\vec{S}}{dr}=(\cos\theta,\ \sin\theta,\ 2r\cos(2\theta))$

$\dfrac{d\vec{S}}{d\theta}=(-r\sin\theta,\ r\cos(\theta),\ -4r^2\cos\theta\sin\theta)$

$\vec{S}_r \times \vec{S}_\theta = \left(-2r^2\cos\theta,\ 2r^2\sin\theta,\ r \right)$ so

$|\vec{S}_r \times \vec{S}_\theta| = \sqrt{r^2 + 4r^4}$

$$\iint_R \sqrt{r^2+4r^4}\, dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r\sqrt{1+4r^2}\, dr\, d\theta$$

$$= \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

Sometimes surfaces are defined implicitly, e.g.

$F(x,y,c) = c$ for some constant c.

e.g. $x^2 + y^2 - z^3 = 1$

kinda

idea is assume $\nabla \vec{F}(x,y,z) \neq \vec{0}$ and also

$\nabla \vec{F}(x,y,z) \cdot \hat{k} \neq 0$ with a region.

$\neq 0$ for each

then one can write $\vec{r}(u,v) = u\,\hat{\imath} + v\,\hat{\jmath} + h(u,v)\,\hat{k}$

for <u>some</u> function $h(u,v)$. (Implicit function theorem).

then $\vec{r}_u = \left(1, 0, \frac{dh}{du}\right)$ so $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & h_u \\ 0 & 1 & h_v \end{vmatrix}$

$\vec{r}_v = \left(0, 1, \frac{dh}{dv}\right)$

$= ($

$(-h_u, -h_v, 1)$

However, $F(x,y,z) = c$ and around small region

$F(u,v, h(u,v)) = 0$ so

$$\frac{dF}{dx} \cdot \underset{\substack{\| \\ 1}}{\frac{dx}{du}} + \frac{dF}{dy} \cdot \underset{\substack{\| \\ 0}}{\frac{dy}{du}} + \frac{dF}{dz} \underset{\substack{\| \\ \frac{dh}{du}}}{\frac{dz}{du}} = 0$$ so

$$\frac{dh}{du} = \frac{-dF}{dx} \Big/ \frac{dF}{dz}.$$ Similarly,

$$\frac{dF}{dx} \underset{\substack{\| \\ 0}}{\frac{dx}{dv}} + \frac{dF}{dy} \underset{\substack{\| \\ 1}}{\frac{dy}{dv}} + \frac{dF}{dz} \underset{\substack{\| \\ \frac{dh}{dv}}}{\frac{dz}{dv}}$$ so

$$\frac{dh}{dv} = - \frac{dF}{dy} \Big/ \frac{dF}{dz}$$ so

$$\vec{r}_u \times \vec{r}_v = \left(+ F_x / F_z, + F_y / F_z, 1 \right)$$
$$= \nabla \vec{F} / F_z = \nabla \vec{F} / (\nabla \vec{F} \cdot \hat{z})$$

So $SA = \iint_R \nabla \vec{F} / (\nabla \vec{F} \cdot \hat{z})\, dA$ In fact, ...

thm: let $F(x,y,z) = c$ over a region R parallel to xy-plane. Then

$$SA = \iint_R \frac{|\nabla \vec{F}|}{|\nabla \vec{F} \cdot \hat{z}|}\, dA$$ provided $\nabla \vec{F} \cdot \hat{z} \neq 0$ on R.

Moreover, we also have

$$SA = \iint \frac{|\nabla \vec{F}|}{|\nabla \vec{F} \cdot \hat{x}|}\, dA$$ provided $\nabla \vec{F} \cdot \hat{x} \neq 0$ on R in yz-plane

and

$$SA = \iint_R \frac{|\nabla \vec{F}|}{|\nabla \vec{F} \cdot \hat{y}|}\, dA$$ provided $\nabla \vec{F} \cdot \hat{y} \neq 0$ on R in zx-plane.

Note in particular, if $z = f(x,y)$ then we have

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

bc $\nabla \vec{F} = (f_x, f_y, -1)$

One issue with this is how to evaluate $\nabla \vec{F}$ if you don't know z! (only works in really nice cases).

ex: Calculate area of cone $x^2+y^2=z^2$ bounded by $z \geq 0$ and square centred at $(0,0)$ side length 2.

$F = x^2+y^2-z^2$ so

$\nabla F = (2x, 2y, -2z)$

$\nabla F \cdot \hat{z} = -2z$ so

$$|\nabla F| / |\nabla F \cdot \hat{z}| = \sqrt{4x^2+4y^2+4z^2} / 2z$$

$$= \sqrt{x^2+y^2+z^2} / \qquad z = \sqrt{x^2+y^2}$$

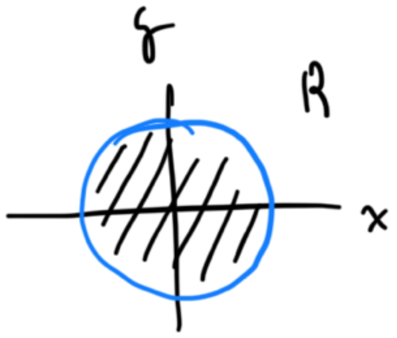
So $$= \sqrt{2x^2+2y^2} / \sqrt{x^2+y^2} = \sqrt{2}.$$

So $$SA = \iint_R \sqrt{2}\, dA = \sqrt{2}\,(\text{Area of } \square)$$

$$= 4\sqrt{2}$$

ex: Calculate surface area of $z = x^2 + y^2$ on region bounded by $0 \le x^2 + y^2 \le 1$.

$$SA = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA$$



$f_x = 2x$

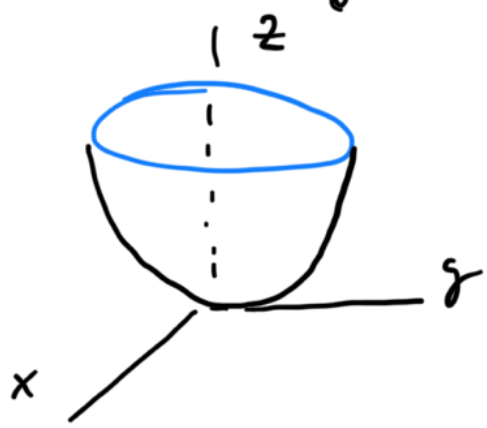
$f_y = 2y$

$$\sqrt{(2x)^2 + (2y)^2 + 1}\; dx\,dy$$

$$= \sqrt{4(x^2+y^2)+1}\; dx\,dy$$

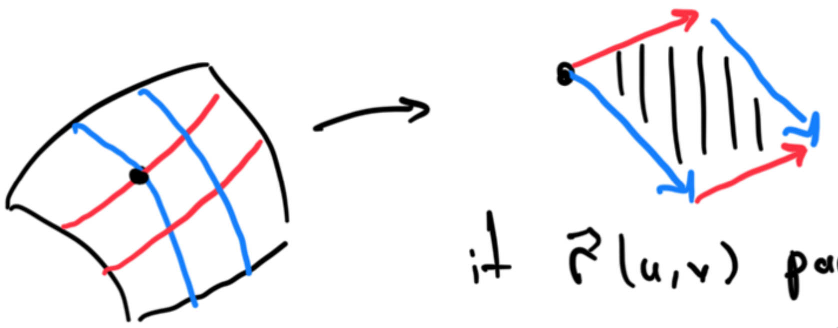
$$= \sqrt{4r^2+1}\; r\,dr\,d\theta$$

So $$SA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r\sqrt{4r^2+1}\; dr\,d\theta = \frac{\pi}{6}(5\sqrt{5}-1)$$



16.6: Surface Integrals

Motivated by physical example. Say you have surface S with non-uniform density. How do calculate its mass? Or if you need to integrate x-coordinate to find 1st moment about a plane for surface. Point being we need to integrate functions over surfaces.



differential area is denoted by $d\sigma$.

if $\vec{r}(u,v)$ parametrizes S then $d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv$ like in last section.

Ways to integrate scalar function over surface

1. Have a parametrization $\vec{r}(u,v)$, Then

$$\iint_S G \, d\sigma = \iint_R (G \circ \vec{r})(u,v) \, |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

2. If S is implicitly $F(x,y,z) = c$ and satisfies $\nabla F \cdot \hat{k} \neq 0$ on region R then

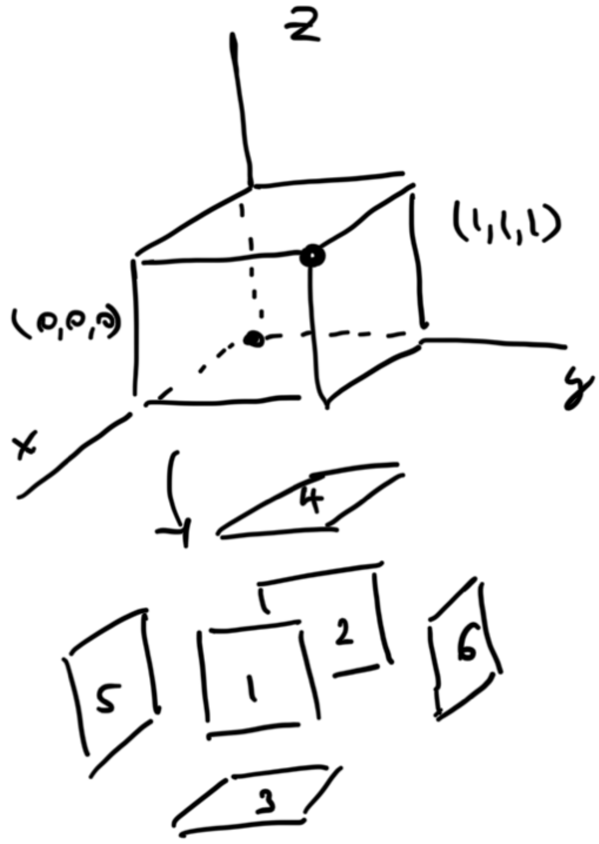
$$\iint G \, d\sigma = \iint G \frac{|\nabla F|}{|\nabla F \cdot \hat{k}|} \, dA$$

(similar statements for $\hat{\imath}$, $\hat{\jmath}$)

3. If S given explicitly as $z = f(x,y)$ then

$$\iint_S G\, d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\; dA$$

ex: let S be a cube below. Assume its mass is given by $\delta(x,y,z) = x^2yz$. Calculate its center of mass.



M_{yz} = 1st moment about yz-plane

$$= \iint_S x\delta\, d\sigma$$

M_{xz} = 1st moment about xz-plane

$$= \iint_S y\delta\, d\sigma$$

M_{xy} = 1st moment about xy-plane

$$= \iint_S z\delta\, d\sigma$$

$$M = \iint_S \delta\, d\sigma = \iint_1 \delta\, d\sigma + \iint_2 \delta\, d\sigma + \dots + \iint_6 \delta\, d\sigma$$

well 3 of them are zero, namely 5, 3, 2

because on 5, 3, 2, $\delta \equiv 0$ as either $x, y,$ or $z = 0$ on them. So $\qquad \delta = x^2yz$

$$M = \iint \delta\, d\sigma + \iint \delta\, d\sigma + \iint \delta\, d\sigma$$

$$\iint_1 \quad \iint_4 \quad \iint_6$$

δ on $1 = 1^2 yz = yz$

δ on $4 = x^2 y 1 = x^2 y$

δ on $6 = x^2 1 z = x^2 z$

$$\iint_1 \delta \, d\sigma = \int_{y=0}^{y=1} \int_{z=0}^{z=1} yz \, dz \, dy = 1/4$$

$$\iint_4 \delta \, d\sigma = \int_{x=0}^{x=1} \int_{y=0}^{y=1} x^2 y \, dy \, dx = 1/6$$

$$\iint_6 \delta \, d\sigma = \int_{x=0}^{x=1} \int_{z=0}^{z=1} x^2 z \, dz \, dx = 1/6$$

so $M = \frac{1}{4} + \frac{1}{6} + \frac{1}{6} = 7/12$ kg

M_{yz} very similar calculation.

$$M_{yz} = \iint_S x \delta \, d\sigma = \iint_1 x \delta \, d\sigma + \iint_4 x \delta \, d\sigma + \iint_6 x \delta \, d\sigma$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \text{ m kg}$$

$$M_{xz} = \iint_S y \delta \, d\sigma = \iint_1 y \delta \, d\sigma + \iint_4 y \delta \, d\sigma + \iint_6 y \delta \, d\sigma$$

$$= \frac{1}{6} + \frac{1}{9} + \frac{1}{6} = 4/9 \text{ m kg}$$

$$M_{xy} = \iint_S z \delta \, d\sigma = \iint_1 z \delta \, d\sigma + \iint_4 z \delta \, d\sigma + \iint_6 z \delta \, d\sigma$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{9} = 4/9 \text{ m kg}$$

so $(\bar{x}, \bar{y}, \bar{z}) = (1/2, 4/9, 4/9) / (7/12) = \left(6/7 \text{ m}, \frac{16}{21} \text{ m}, \frac{16}{21} \text{ m}\right)$

ex: let S be upper hemisphere of a sphere of radius a

whose density is $\delta(x,y,z) = z$

Calculate the various moments of inertia of S.

$I_x = \iint\limits_S (y^2+z^2)\,\delta\, d\sigma \quad I_y = \iint\limits_S (x^2+z^2)\,\delta\, d\sigma \quad I_z = \iint\limits_S (x^2+y^2)\,\delta\, d\sigma$

$x^2+y^2+z^2=a^2$

$$\vec{S}(\phi,\theta) = (\underset{x}{a\sin\phi\cos\theta},\ \underset{y}{a\sin\phi\sin\theta},\ \underset{z}{a\cos\phi})$$

where $0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/2.$

$$\frac{\partial \vec{S}}{\partial \phi} = (a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi)$$

$$\frac{\partial \vec{S}}{\partial \theta} = (-a\sin\phi\sin\theta,\ a\sin\phi\cos\theta,\ 0)$$

$$\vec{S}_\phi \times \vec{S}_\theta = (a^2\cos\theta\sin^2\phi,\ a^2\sin^2\phi\sin\theta,\ a^2\cos\phi\sin\phi)$$

$$|\vec{S}_\phi \times \vec{S}_\theta| = a^2|\sin\phi|$$

so $d\sigma = a^2\sin\phi\, d\phi\, d\theta$ (bc $\sin\phi \ge 0$ for $0 \le \phi \le \pi$)

so

$$I_x = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2} \underbrace{\left((a\sin\phi\sin\theta)^2 + (a\cos\phi)^2\right)}_{y^2+z^2} \underbrace{(a\cos\phi)}_{z} \underbrace{(a^2\sin\phi)\, d\phi\, d\theta}_{d\sigma}$$

$$= 3a^5\pi/4$$

$$I_y = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2} \underbrace{\left((a\sin\phi\cos\theta)^2 + (a\cos\phi)^2\right)}_{x^2+z^2} \underbrace{(a\cos\phi)}_{z} (a^2\sin\phi)\, d\phi\, d\theta$$

$$= 3a^5\pi/4$$

$$I_z = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2} \left((a\sin\phi\cos\theta)^2 + (a\sin\phi\sin\theta)^2\right)(a\cos\phi)(a^2\sin\phi)\, d\phi\, d\theta$$

$= a^5\pi/2$ note $I_z \neq I_x$ and I_y

and $I_x = I_y$

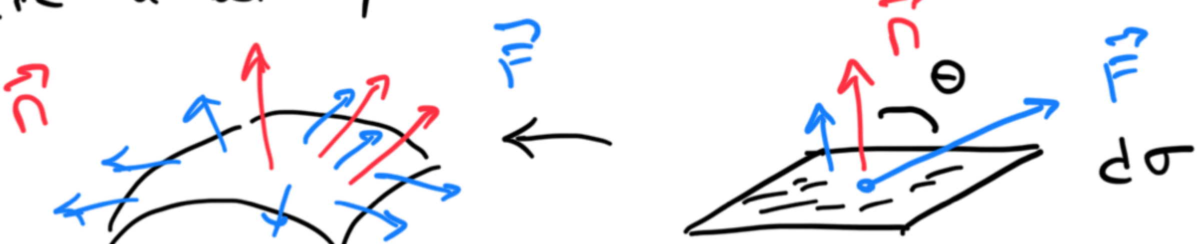
I_x resistance to

I_y resistance to

I_z resistance to

Flux Integrals

Imagine a velocity vector field over a surface



$\text{proj}_{\vec{n}} \vec{F} = \vec{F} \cdot \vec{n}$

How much of $\vec{F}$ goes through $d\sigma$

$= (\vec{F} \cdot \vec{n})\, d\sigma$

so define

Flux of $\vec{F}$ through S in direction of $\vec{n}$ $= \iint_S (\vec{F} \cdot \vec{n})\, d\sigma$ //

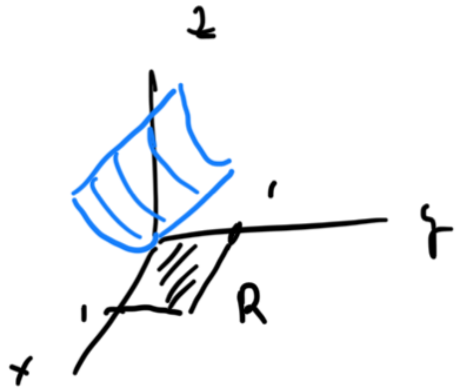
(note requires S to be <u>oriented</u> choice of a normal direction)

ex.

ex: Consider the function $z = \frac{1}{2}x^2 + y$

defined on the square $0 \le x, y \le 1$.

let S be $z = \frac{1}{2}x^2 + y$ on this region with normal pointing downward.



let $\vec{F} = (y, x, z) = y\hat{i} + x\hat{j} + z\hat{k}$. Calculate flux of $\vec{F}$ over S in direction of $\vec{n}$.

let $g(x,y,z) = \frac{1}{2}x^2 + y - z$ so $\nabla g = (x, 1, -1)$

$$|\nabla g| = \sqrt{x^2 + 2}$$

$$|\nabla g \cdot \hat{k}| = 1$$

so $d\sigma = \sqrt{x^2+2}\, dA$

$\vec{n} = \dfrac{\nabla g}{|\nabla g|} = \dfrac{(x, 1, -1)}{\sqrt{x^2+2}}$ note $\vec{n}$ has negative z so it's pointing down.

So $\vec{F} = (y, x, \frac{1}{2}x^2 + y)$ on S

$$\vec{F} \cdot \vec{n}\, d\sigma = \vec{F} \cdot \frac{\nabla g}{|\nabla g|} \frac{|\nabla g|}{|\nabla g \cdot \hat{k}|}\, dA$$

$$= \vec{F} \cdot \frac{\nabla g}{|\nabla g \cdot \hat{k}|}\, dA$$

in general, $\displaystyle\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_R \frac{\vec{F} \cdot \nabla g}{|\nabla g \cdot \hat{k}|}\, dA$ (nice to remember)

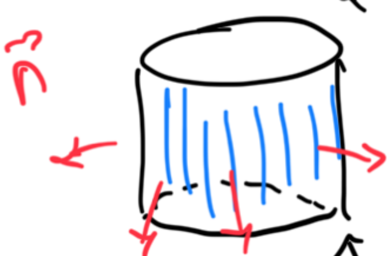
$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1} \left(xy + x - \tfrac{1}{2}x^2 - y\right) dx\, dy = 1/12$$

ex: let S be surface $x^2+y^2=4$ between $z=0$ and $z=1$, with normal pointing away from z-axis. let $\vec{F} = (x+y,\ x-y,\ x^2)$

Calculate flux of $\vec{F}$ through S in direction of $\vec{n}$.

Parametrize S via $\vec{r}(\theta, h) = (2\cos\theta, 2\sin\theta, h)$

where $0 \le \theta \le 2\pi$ and $0 \le h \le 1$.



← no cap

(I on old and don't understand that expression)

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Recall $\vec{r}_u \times \vec{r}_v$ normal to surface so (up to a sign)

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$ and

$$d\sigma = |\vec{r}_u \times \vec{r}_v|\, dA$$

$$\iint_S (\vec{F} \cdot \vec{n})\, d\sigma = \iint_R \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= \iint_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

another helpful one to know.

so $\vec{F}(\theta, h) = \left(2\cos\theta + 2\sin\theta,\ 2\cos\theta - 2\sin\theta,\ (2\cos\theta)^2\right)$

$\vec{r}_\theta = (-2\sin\theta, 2\cos\theta, 0)$ $\quad \vec{r}_h = (0, 0, 1)$

so $\vec{r}_\theta \times \vec{r}_h = (2\cos\theta, 2\sin\theta, 0)$

the obvious choice anyway!

but that was

So $\vec{F}\cdot(\vec{r}_\theta \times \vec{r}_h) = 4(\cos(2\theta) + \sin(2\theta))$

thus $\displaystyle\iint_S (\vec{F}\cdot\vec{n})\, d\sigma = \int_{h=0}^{h=1}\int_{\theta=0}^{\theta=2\pi} 4(\cos(2\theta)+\sin(2\theta))\, d\theta\, dh$

$= 0\,!$

Stoke's theorem 16.7

Recall for a vector field $\vec{F} = (M, N, P)$

we have

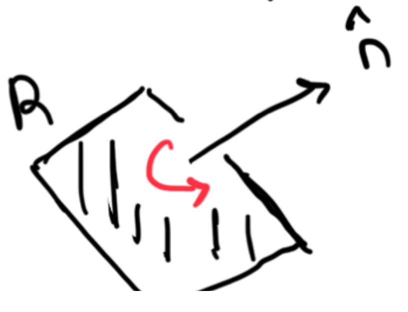
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \hat{i}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \hat{j}\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

What is it? It's kind of like a tool that knows how much spinning is happening in a region.

For example

$(\nabla\times\vec{F})(a,b,c)\cdot\hat{n}$ " $=$ " How much particle circulates in $\vec{F}$ in region $\perp$ to $\hat{n}$ at point (a,b,c) relative to induced orientation of $\hat{n}$.

(right hand rule)



$(\nabla\times\vec{F})\cdot\hat{n} > 0$ means spinning with red

$(\nabla\times\vec{F})\cdot\hat{n} < 0$ means spinning against

$(\nabla \times F)\cdot n$ ...

red

$(\nabla \times \vec{F})\cdot \hat{n} = 0$ no spin

Recall Green's theorem says that for a planar region R with boundary $\vec{\gamma}$,

we have $\oint_\gamma \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dA$

R

$$= \iint_R (\nabla \vec{F}\cdot \hat{k})\, dA$$

Stoke's theorem generalizes this to non-planar regions.

S

$$\iint_S (\nabla \times \vec{F})\cdot \hat{n}\, d\sigma = \oint_\gamma \vec{F}\cdot d\vec{r}$$

ex: let $\vec{F} = (xy, y, z)$ and S be surface defined by hemisphere $x^2+y^2+z^2=1$ and $z \geq 0$. let $\vec{n}$ point upward.

Verify Stoke's theorem for this example.

(i). $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d_x & d_y & d_z \\ xy & y & z \end{vmatrix} = (0, 0, -x) = -x\hat{k}$

$g(x,y,z) = x^2+y^2+z^2-1$

$\nabla g = (2x, 2y, 2z)$ so $\vec{n} = \frac{\nabla g}{|\nabla g|} = \frac{2(x,y,z)}{2\sqrt{x^2+y^2+z^2}}$

$\nabla g \cdot \hat{k} = 2z$

$= (x, y, z)$

but again we knew that!

$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \hat{k}|} dA = \frac{2\sqrt{x^2+y^2+z^2}}{2z} dA = \frac{1}{z} dA$$

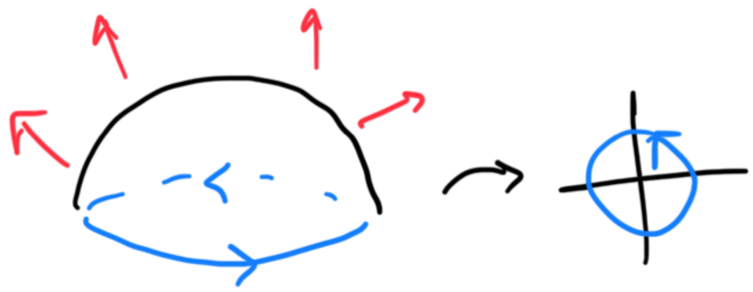
so $(\nabla \times \vec{F}) \cdot \vec{n} = -xz$ and

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma = \iint_S (-xz) \frac{1}{z} dA = -\iint_S x \, dA$$

$$= -\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/2} (\sin\phi\cos\theta)(1^2 \sin\phi \, d\phi \, d\theta)$$

$$= 0$$

(ii). $\vec{F}$ on γ



$\oint_\gamma \vec{F} \cdot d\vec{r}$, $\gamma(t) = (\cos t, \sin t, 0)$

$d\vec{r} = (-\sin t, \cos t, 0) \, dt$

$$(\vec{F} \circ \gamma)(t) = (\cos t \sin t, \sin t, 0)$$

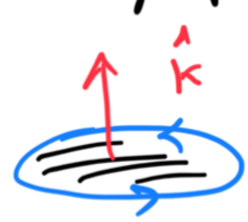
so $$\oint_\gamma \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=2\pi} (-\cos t \sin^2 t + \cos t \sin t) \, dt$$

$$= 0$$

So $$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma = \oint_\gamma \vec{F} \cdot d\vec{r}$$

Also note, let's change S to unit disk in xy-plane.

then $\hat{n} = \hat{k}$, $d\sigma = r \, dr \, d\theta$

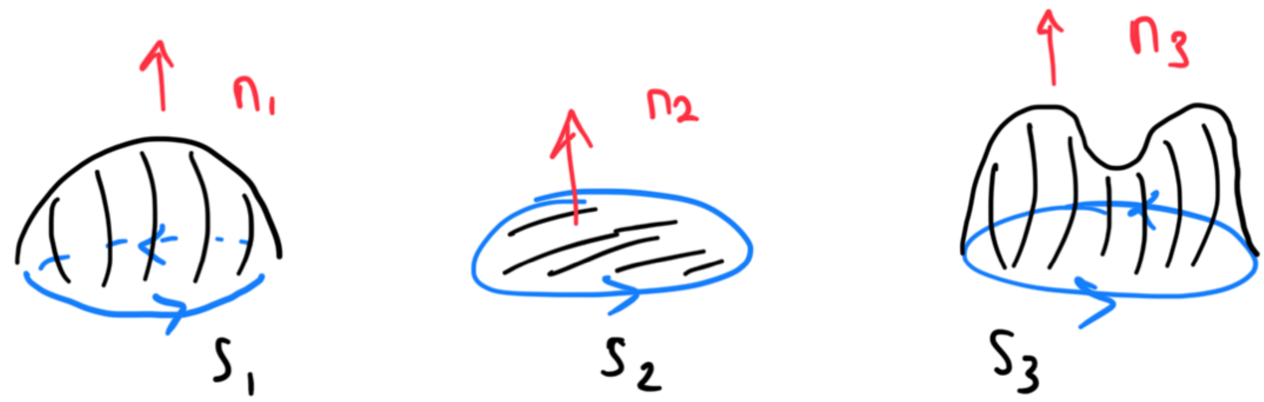


$(\nabla \times \vec{F}) \cdot \hat{n} = (0,0,-x) \cdot (0,0,1) = -x$

then $\displaystyle\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma = \iint_R (-x)\, dA$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (-r\cos\theta)\, r\, dr\, d\theta = 0$$

Observation: A bunch of surfaces bound unit circle. Here are some,



Stoke's theorem says $\displaystyle\iint_{S_1} (\nabla \times \vec{F}) \cdot n_1\, d\sigma_1$

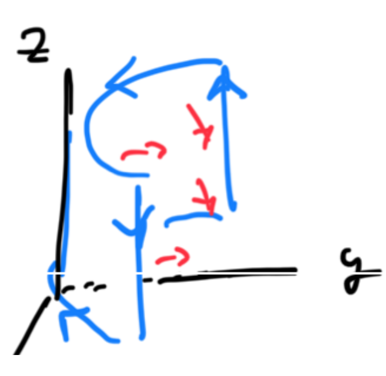
$$= \iint_{S_2} (\nabla \times \vec{F}) \cdot n_2\, d\sigma_2$$

moral of the story, choose easiest one!

$$= \iint_{S_3} (\nabla \times \vec{F}) \cdot n_3\, d\sigma_3$$

ex: Consider surface $\vec{r}(u,v) = (u, u^2, v)$ where $-1 \le u \le 1$ and $0 \le v \le \pi/2$ with normal pointing towards $+y$-axis. Verify Stoke's theorem for

$\vec{F} = (\sin(z), x, y)$



x

(i). $\vec{r}_u = (1, 2u, 0)$

$\vec{r}_v = (0, 0, 1)$

$\vec{r}_u \times \vec{r}_v = (2u, -1, 0)$ so need to take $\vec{n} = -\vec{r}_u \times \vec{r}_v$ so y-positive.

$\nabla \times \vec{F} = (1, \cos z, 1) = \hat{i} + \cos z \,\hat{j} + \hat{k}$ so

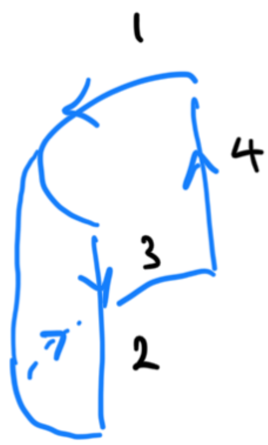
$-(\nabla \times \vec{F}) \cdot (\vec{r}_u \times \vec{r}_v) = -2u + \cos(v)$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma = \iint_R (\nabla \times \vec{F}) \cdot (-\vec{r}_u \times \vec{r}_v) \, dA$$

$$= \int_{v=0}^{v=1} \int_{u=-1}^{u=\pi/2} \left(-2u + \cos(v)\right) du \, dv$$

$$= 2$$

(ii).



1: $\alpha(t) = (t, t^2, \pi/2)$ $\quad -1 \le t \le 1$

2: $\beta(t) = (1, 1, \pi/2 - t)$ $\quad 0 \le t \le \pi/2$

3: $\gamma(t) = ((1-t), (1-t)^2, 0)$ $\quad 0 \le t \le 2$

4: $\delta(t) = (-1, +1, t)$ $\quad 0 \le t \le \pi/2$

1. $d\vec{r} = (1, 2t, 0)\, dt$

$(\vec{F} \circ \alpha)(t) = (1, t, t^2)$

$$\oint_\alpha \vec{F} \cdot d\vec{r} = \int_{-1}^{+1} (1 + 2t^2)\, dt = \frac{10}{3}$$

2. $d\vec{r} = (0, 0, -1)\, dt$

$(\vec{F} \circ \beta)(t) = (\cos t, 1, 1)$

$$\oint_\beta \vec F\cdot d\vec r = \int_0^{\pi/2} (-1)\,dt = -\pi/2$$

3. $d\vec r = (-1, -2(1-t), 0)\,dt$

$$(\vec F\circ\gamma)(t) = (0, (1-t), (1-t)^2)$$

$$\oint_\gamma \vec F\cdot d\vec r = \int_0^2 -2(1-t)^2\,dt = -4/3$$

4. $d\vec r = (0,0,1)\,dt$

$$(\vec F\circ\delta)(t) = (\sin t, -1, +1)$$

$$\oint_\delta \vec F\cdot d\vec r = \int_0^{\pi/2} (+1)\,dt = +\pi/2$$

so $$\oint_{\partial S} \vec F\cdot d\vec r = \frac{10}{6} - \frac{\pi}{2} - \frac{4}{3} + \frac{\pi}{2} = 2$$

so $$\iint_S (\nabla\times\vec F)\cdot\vec n\,d\sigma = \oint_{\partial S}\vec F\cdot d\vec r$$

One last bit about Stokes' theorem,

Need to assume $\vec F$ and $\nabla\times\vec F$ are 'nice' on surface. Theorem is false if $\vec F$ is badly behaved on surface. Also another neat bit is that

$$(\nabla\times F)\cdot\hat u\Big|_Q = \lim_{p\to 0} \frac{\iint_S(\nabla\times\vec F)\cdot\vec u\,d\sigma}{\text{Area of small disk about Q.}}$$

$$= \lim \frac{1}{\pi\rho^2}\iint(\nabla\times\vec F)\cdot\vec u\,d\sigma$$

$\hat{u} \perp$ disk

$$\lim_{\rho \to 0} \ \ldots \iint_S \ldots$$

$$= \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \oint_\gamma \vec{F} \cdot d\vec{r} = \text{infinitesimal circulation density}$$

Note this means direction of greatest circulation is $\nabla \times \vec{F}$!

Also if let $\vec{F} = \nabla f$ then we have

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, d\sigma = \oint_\gamma \vec{F} \cdot d\vec{r}$$

$$= \oint_\gamma \nabla f \cdot d\vec{r} = 0 \text{ by conservative vector field}$$

if true for any surface S then $\nabla \times \vec{F} = \vec{0}$ (can check by direct calculation too).

But this says

$$\nabla \times (\nabla f) = \vec{0}.$$

Finally if $\nabla \times \vec{F} = \vec{0}$ on a simply connected region S

then $\oint_\gamma \vec{F} \cdot d\vec{r} = 0$ where γ is boundary of S.

Thus on simply connected regions the following are equivalent.

1. $\vec{F}$ conservative on D (path integrals depend only on beginning / end points)
2. $\oint_\gamma \vec{F} \cdot d\vec{r} = 0$ for any loop γ in D

3. $\nabla \times \vec{F} = \vec{0}$ on D

4. $F = \nabla f$ on D

16.8: Divergence Theorem

Just like in (2D) given a vector field $\vec{F} = (M, N, P)$ we associate the scalar function

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{dM}{dx} + \frac{dN}{dy} + \frac{dP}{dz}$$

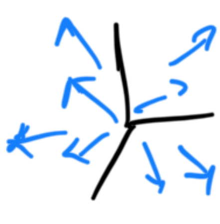
$\nabla \cdot \vec{F}$ is a measure of how much a fluid / gas is expanding / contracting at a point under the vector field $\vec{F}$. If $\operatorname{div} \vec{F} > 0$ expanding

$\operatorname{div} \vec{F} < 0$ contracting

$\operatorname{div} \vec{F} = 0$ no flux.

Measured in flux / unit volume so frequently called 'flux density'.

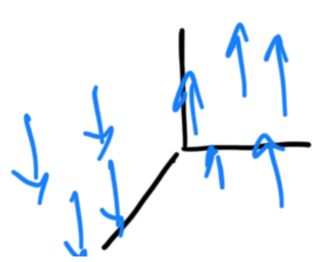
ex: $\vec{F} = (x, y, z)$ $\operatorname{div} \vec{F} = 1 + 1 + 1 > 3$



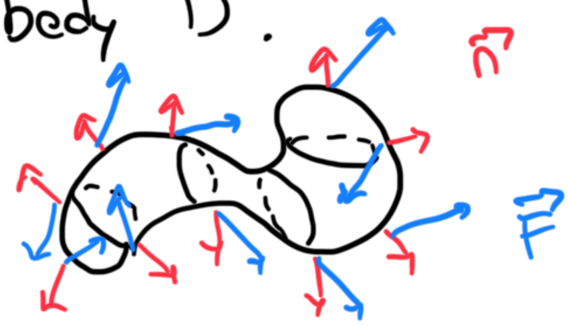
similarly $\vec{F} = (-x, -y, -z)$ then $\operatorname{div} \vec{F} = -3 < 0$

ex: $\vec{F} = (-y, x, 0)$ then $\operatorname{div} \vec{F} = 0$ because fluid just spinning

ex: $\vec{F} = (0, 0, z)$ then $\operatorname{div} \vec{F} > 0$ because stuff moving out



Divergence theorem: let S be a closed oriented surface. Then flux of $\vec{F}$ across S in the direction of ~~outward~~ outward unit normal $\vec{n}$ equals the integral of the divergence over the body D.



Flux $\vec{F}$ over S $= \iint_S (\vec{F} \cdot \vec{n}) \, d\sigma$

$$= \iiint_D (\nabla \cdot \vec{F}) \, dV$$

ex: Verify Divergence Theorem for $\vec{F} = (y^2, x^2, 0)$ over the sphere.

$\iint_S (\vec{F} \cdot \vec{n}) \, d\sigma$: Recall $\iint_S \vec{F} \cdot n \, d\sigma = \iint_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dA$

for a parametrization $\vec{r}$. let $\vec{r}(\phi, \theta) = (\sin\phi\cos\theta,$
$\sin\phi\sin\theta,$
$\cos\phi)$

so that $\vec{r}_\phi \times \vec{r}_\theta = (\cos\theta \sin^2\phi,$
$\sin\theta \sin^2\phi, \cos\phi \sin\phi)$

then $\vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta) = \cos\phi \sin^3\phi \underline{\cos\theta \sin\theta}$

$+ \underline{\cos^2(\theta) \sin(\theta)} \sin^4\phi$

$+ \underline{\cos\theta \sin^2\theta} \sin^4\phi$

Each blue term integrates to 0 on $0 \le \theta \le 2\pi$, so $\iint_S \vec{F}\cdot\vec{n}\, d\sigma = 0$.

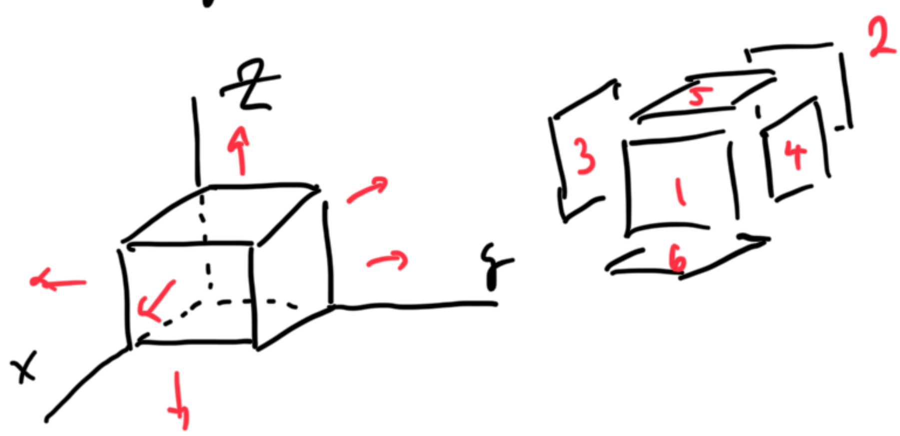
Similarly: $\iiint_D (\nabla\cdot\vec{F})\, dV = \iiint_D 0\, dV = 0$

$\nabla\cdot\vec{F} = (0+0+0) = 0$

ex: Calculate $\iint(\vec{F}\cdot\vec{n})\, d\sigma$ over box $0 \le x, y, z \le 1$

of vector field $\vec{F} = (x^2, y^2, z^2)$ in both ways

using Divergence theorem.



$\vec{F}|_3 = (x^2, y^2, z^2) = (x^2, 0, z^2)$ and $\vec{n}_3 = (0, -1, 0)$

$\vec{F}|_6 = (x^2, y^2, 0^2)$ and $\vec{n}_4 = (0, 0, -1)$

$\vec{F}|_2 = (0^2, y^2, z^2)$ and $\vec{n}_2 = (-1, 0, 0)$

so on $3, 6, 2$ $\quad \vec{F}\cdot\vec{n} = 0$

However, $\vec{F}|_1 = (1^2, y^2, z^2)$ $\quad \vec{n}_1 = (1,0,0)$

$\vec{F}|_4 = (x^2, 1^2, z^2)$ $\quad \vec{n}_4 = (0,1,0)$

$\vec{F}_5 = (x^2, y^2, 1^2)$ $\quad \vec{n}_5 = (0,0,1)$

so $\quad$ Flux $= \iint 1\, dA_1 + \iint 1\, dA + \iint 1\, dA_5$

$\iint_{R_1} \cdots \iint_{R_4} \cdots \iint_{R_5}$

$= 3 \cdot$ Area of unit square $= 3$.

On other hand, $\nabla \cdot \vec{F} = 2x + 2y + 2z$

so $\iiint_D (\nabla \cdot \vec{F})\, dV = 2 \iiint_D (x+y+z)\, dV$

$= 2 \int_{x=0}^{x=1} \int_{y=0}^{y=1} \int_{z=0}^{z=1} (x+y+z)\, dV = 2\left(\frac{3}{2}\right) = 3$

Relation between curl and divergence

theorem: if $\vec{F} = (M, N, P)$ with nice partial derivatives then

$$\operatorname{div}(\nabla \times \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

Proof: Some algebra

ex: Calculate $\iint_S (\vec{F} \cdot \vec{n})\, d\sigma$ over the cube in the previous example

$\vec{F}$ is the vector field $\vec{F} = (2y - 2z, -2x + 2z, 2x - 2y)$

let's say you were having a good math day and noticed that

$\vec{F} = \nabla \times \vec{G}$ where $\vec{G} = (y^2 + z^2, x^2 + z^2, x^2 + y^2)$

$F = \nabla \times G$ where G ...

then by divergence theorem,

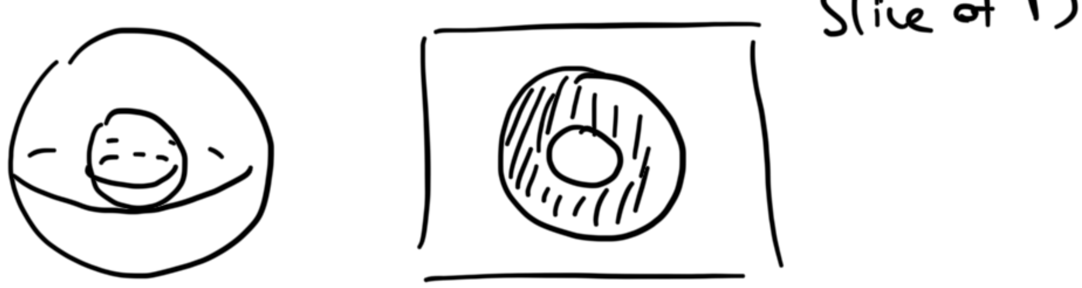
$$\iint_S (\vec F \cdot \vec n)\, d\sigma = \iiint_D (\operatorname{div} \vec F)\, dV = \iiint_D \operatorname{div} (\nabla \times \vec G)\, dV$$

$$= \iiint_D 0\, dV = 0$$

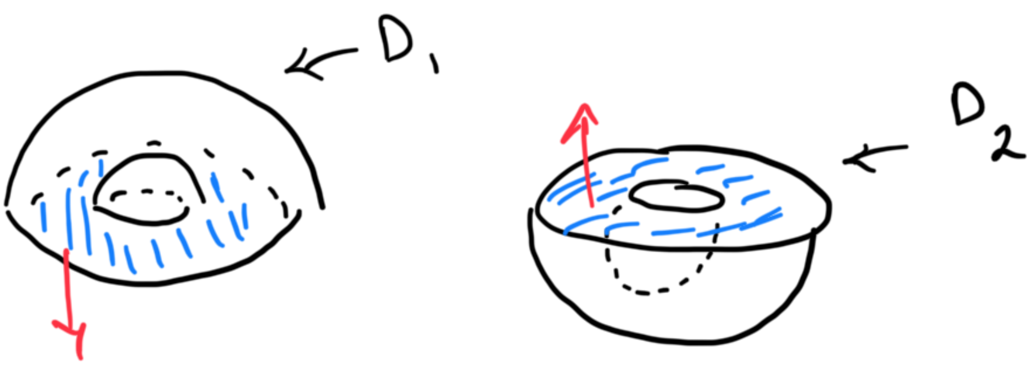
Also it's a neat way to check if $\vec F = \nabla \times \vec G$ for some $\vec G$. ex: $\vec F = (x, y, z)$ has $\operatorname{div} \vec F \neq 0$ so no solution to $\vec F = \nabla \times \vec G$.

Divergence theorem holds for more general regions.

e.g. let D be region between two concentric spheres.



Can break D up into



S_1 = semi-outer sphere
semi-inner sphere
flat annulus

S_2 = semi-outer sphere
semi-inner sphere
flat annulus

By Divergence theorem $\iiint_{D_1} (\nabla \cdot \vec{F})\, dV = \iint_{S_1} \vec{F} \cdot \vec{n}\, d\sigma$

and $\iiint_{D_2} (\nabla \cdot \vec{F})\, dV = \iint_{S_2} \vec{F} \cdot \vec{n}\, d\sigma$. so

$$\iiint_{D} (\nabla \cdot \vec{F})\, dV = \iint_{S_1} (\vec{F} \cdot \vec{n})\, d\sigma + \iint_{S_2} (\vec{F} \cdot \vec{n})\, d\sigma$$

$$= \iint_{\text{outer sphere}} (\vec{F} \cdot \vec{n})\, d\sigma + \iint_{\text{inner sphere}} \vec{F} \cdot \vec{n}\, d\sigma$$

$$+ \underbrace{\iint_{\text{Annulus 1}} (F \cdot \vec{n})\, d\sigma + \iint_{\text{Annulus 2}} (\vec{F} \cdot \vec{n})\, d\sigma}_{\text{two cncl b/c opposite orientation!}}$$

$$\iint_{\partial D} (\vec{F} \cdot \vec{n})\, d\sigma = \iiint_{D} (\nabla \cdot \vec{F})\, dV$$ where ∂D is boundary of D with possibly many pieces.

ex: let $\vec{F} = (x, y, z)/\rho^3$ where $\rho = \sqrt{x^2 + y^2 + z^2}$ and calculate the flux of $\vec{F}$ across the boundary of D as above. One can calculate for $\rho \neq 0$ that $\operatorname{div} \vec{F} = 0$ (a little weird, looks like it wouldn't)

so $$\iiint_{D} \operatorname{div} \vec{F}\, dV = \iint_{S_1} (\vec{F} \cdot n)\, d\sigma + \iint_{S_2} (\vec{F} \cdot n)\, d\sigma = 0$$

So $\iint_{\text{sphere}} (\vec{F}\cdot\vec{n})\, d\sigma$ independent of radius of sphere.

In fact if we chose sphere to be radius a

then $\vec{F}\cdot\vec{n} = \dfrac{(x,y,z)}{a^3} \cdot \dfrac{(x,y,z)}{a} = \dfrac{x^2+y^2+z^2}{a^4}$

$= 1/a^2$

So $\text{Flux} = \iint_{\text{sphere}} (\vec{F}\cdot\vec{n})\, d\sigma = \iint_{\text{sphere}} (1/a^2)\, d\sigma = (4\pi a^2)/a^2$

$= 4\pi$

Gauss' Law: let $\vec{E} = \dfrac{1}{4\pi \varepsilon_0}\, q\, \dfrac{\vec{r}}{|\vec{r}|^3}$

where ε_0 is a physical constant (electric permittivity of free space $\approx 10^{-11} \dfrac{\text{Colomb}}{N\, m^2}$)

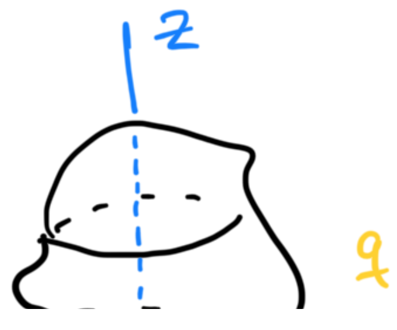
q is point charge located at origin, $\vec{r}$ is position vector.

Note by previous calculations, $\nabla\cdot\vec{E} = 0$.

So if S is any surface containing the origin point charge

Flux of $\vec{E}$ across S $= \iint_S \vec{E}\cdot\vec{n}\, d\sigma = 4\pi \cdot \dfrac{q}{4\pi\varepsilon_0}$

$= q/\varepsilon_0$



Note that Green's theorem from before was

$$\oint_\gamma \vec{F}\cdot\vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

So if we consider $\vec{F} = (M, N, 0)$ then $\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

so $\oint (\vec{F}\cdot\vec{n})\, ds = \iint_R (\nabla\cdot\vec{F})\, dA$

Similarly $(\nabla\times\vec{F})\cdot\hat{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ so

$$\oint_\gamma (\vec{F}\cdot\vec{T})\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (\nabla\times\vec{F})\cdot\hat{k}\, dA$$

Comparing:

Tangential Green in plane: $\oint (\vec{F}\cdot\vec{T})\, ds = \iint_R (\nabla\times\vec{F})\cdot\hat{k}\, dA$

Stokes: $\oint_\gamma (\vec{F}\cdot\vec{T})\, ds = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma$

Normal Green in plane: $\oint (\vec{F}\cdot\vec{n})\, ds = \iint (\nabla\cdot\vec{F})\, dA$

Divergence: $\iint_S (\vec{F} \cdot \vec{n})\, d\sigma = \iiint_D (\nabla \cdot \vec{F})\, dV$

All of these can be unified by a more general Stokes's Theorem. Very loosely speaking

$$\int_D \text{derivative of something} = \int_{\partial D} \text{something}$$

something is a <u>differential form</u>.