An Interesting Variational Equation

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Summary

Here I provide a proof of a very interesting formula I stumbled upon by chance in Duistermaat and Kolk. The context here is the following. Let V_s be a smooth path of vector fields on a manifold M. Let $\Phi_s^t(x)$ denote the flow of V_s at a point $p \in M$. We then have the following equality

$$\frac{\partial \Phi_{\square}^t(x)}{\partial s} := d\left(\Phi_{\square}^t(x)\right)\left(\frac{\partial}{\partial s}\right) = \int_0^t d\left(\Phi_s^{t-u}\right)_{\Phi_s^u(x)}\left(\frac{\partial V_{\square}}{\partial s}\left(\Phi_s^u(x)\right)\right) du \tag{1}$$

I've introduced squares in an attempt to draw the reader's eye towards what variables are being differentiated in contrast to those that are fixed. This is a weird equation I've never seen in any other book about smooth manifolds, and I think it would be shame for this gem not to get the attention it deserves. These notes largely follow the proof as found in Duistermaat and Kolk, but perhaps with a bit more detail for those not used to taking derivatives on manifolds.

1 Introduction

Before proceeding to the proof of this, I should mention several things. The statement above is purely local, in the sense that s and t are certainly not defined for all time. In fact, we assume that s and t are so small that for all $(s,t) \in (-\epsilon,\epsilon)^2$, $\Phi_s^t(x)$ is defined and contained in a single coordinate chart (U,ϕ) for the sake of ease of calculations. Moreover, $\Phi : (-\epsilon,\epsilon)^2 \times M \longrightarrow M$ is smooth. This is a consequence of the fact that Φ is the flow of the single vector field $X = V_s$ defined on $(-\epsilon,\epsilon) \times M$.

Keeping the above in mind, let us first inspect some of the terms in Equation 1. The left hand side is the derivative of the time t-flows at x the direction transverse to the family of time t-flows. That said, for example, if $t = 0$, we expect the derivative to be zero, as each $\Phi_s^0(x) = \mathrm{id}(x) = x$ for all s, and thus is constant.

Inside the integral on the right hand side of Equation 1, we have the spatial differential of Φ_s^{t-u} evaluated at $\Phi_s^u(x)$. This is a linear map from $T_{\Phi_s^u(x)}M$ to $T_{\Phi_s^t(x)}$, as s is fixed in this equation, so $\Phi_s^a \circ \Phi_s^b = \Phi_s^{a+b}$.

Finally, $\partial V_\square / \partial s$ evaluated at $\Phi_s^u(x)$ which is perhaps more recognizable as

$$\frac{\partial V_\square}{\partial s}\left(\Phi_s^u(x)\right) = d\left(V_\square\right)_{\Phi_s^u(x)}\left(\frac{\partial}{\partial s}\right) \tag{2}$$

Recall, $V(s)$ is a smooth path of vector fields on M, and thus for each point $p \in M$, we have a smooth path in T_xM given by $V(s)|_x$. Because $V(s)|_x \in T_xM$ for all $s \in (-\epsilon,\epsilon)$, its differential as given in Equation 2 is naturally identified as an element of $T_{\Phi_s^u(x)}M$. As per consequence, we may apply the differential $d\left(\Phi_s^{t-u}\right)_{\Phi_s^u(x)}$ to the vector in Equation 2.

After applying the differential, the quantity $d\left(\Phi_s^{t-u}\right)_{\Phi_s^u(x)}\left(\frac{\partial V_\square}{\partial s}\left(\Phi_s^u(x)\right)\right)$ will land in $T_{\Phi_s^t(x)}M$ for all $u \in [0,t]$. As per consequence, we may integrate it, and we claim the integral satisfies Equation 1.

2 Calculations

As a preliminary step, since we assumed $\Phi_\square^\square(x)$ to be contained in a single coordinate chart (U,ϕ) for all $(s,t) \in (-\epsilon,\epsilon)^2$, we may begin by making the identification of $(-\epsilon,\epsilon) \times U$ with the open subset $(-\epsilon,\epsilon) \times \phi(U) \subset \mathbb{R} \times \mathbb{R}^n$.

With this identification, we begin by noting that since $\Phi_s^t(x)$ is the flow at x corresponding to the vector field V_s. As per consequence, it satisfies the following differential equation

$$\frac{\partial \Phi_s^t(x)}{\partial t} = V_s\left(\Phi_s^t(x)\right) \text{ where } \Phi_s^0(x) = x \tag{3}$$

which has the corresponding integral equation

$$\Phi_s^t(x) = x + \int_0^t V_s\left(\Phi_s^u(x)\right) du \tag{4}$$

Differentiating Equation 4 with respect to the transverse parameter s, yields

$$\begin{aligned}\frac{\partial \Phi_s^t(x)}{\partial s} &= \int_0^t \frac{\partial}{\partial s} V_s\left(\Phi_s^u(x)\right) du = \int_0^t \frac{\partial}{\partial s} V\left(s, \Phi_s^u(x)\right) du \\ &= \int_0^t \frac{\partial V}{\partial s}\left(s, \Phi_s^u(x)\right) du + \int_0^t \frac{\partial V}{\partial x}\left(s, \Phi_s^u(x)\right) \circ d\left(\Phi_\square^u(x)\right)\left(\frac{\partial}{\partial s}\right) du \\ &= \int_0^t \frac{\partial V}{\partial s}\left(s, \Phi_s^u(x)\right) du + \int_0^t \frac{\partial V}{\partial x}\left(s, \Phi_s^u(x)\right) \frac{\Phi_s^u(x)}{\partial s} du \end{aligned} \tag{5}$$

It is worth noting that the two integrals in the last line of Equation 5 are, or at least should be, vector quantities as $\frac{\partial \Phi_s^t(x)}{\partial s}$ is. Certainly the expression $\partial V/\partial s$ is a vector, whereas $\partial V/\partial x$ is part of the total derivative of $V : (-\epsilon, \epsilon) \times \phi(U) \longrightarrow \mathbb{R}^n$, and in fact is the spatial derivative of V, and consequently may be identified with a matrix. On the other hand, the differential $d\left(\Phi_\square^u(x)\right)(\partial/\partial s)$ is a vector quantity, so that spatial derivative applied to this vector yields a vector as desired.

Differentiating Equation 5 with respect to t provides the equality

$$\frac{\partial}{\partial t}\frac{\partial \Phi_s^t(x)}{\partial s} = \frac{\partial V}{\partial s}\left(s, \Phi_s^t(x)\right) + \frac{\partial V}{\partial x}\left(s, \Phi_s^t(x)\right)\frac{\Phi_s^t(x)}{\partial s} \text{ where } \frac{\partial \Phi_s^0(x)}{\partial s} = 0 \tag{6}$$

On the other hand, if we differentiate Equation 3 with respect to the spatial component x, we obtain

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial \Phi_s^t(x)}{\partial t} &= \frac{\partial}{\partial x} V\left(s, \Phi_s^t(x)\right) = \frac{\partial V}{\partial x}\left(s, \Phi_s^t(x)\right) \circ d\left(\Phi_s^t(\square)\right) \\ \frac{\partial}{\partial t}\frac{\partial \Phi_s^t(x)}{\partial x} &= \frac{\partial V}{\partial x}\left(s, \Phi_s^t(x)\right)\frac{\partial \Phi_s^t(x)}{\partial x}\end{aligned} \tag{7}$$

where the second line of equality follows from the fact that the partial derivatives of t and x commute. Note that $\partial \Phi_s^t/\partial x$ satisfies the homogenous linear differential equation

$$\frac{\partial}{\partial t}\heartsuit = \frac{\partial V}{\partial x}\left(s, \Phi_s^t(x)\right)\heartsuit \tag{8}$$

where Equation 8 is in fact the homogenous part of Equation 6. From here we apply the method of variation of parameters to $\partial \Phi_s^t(x)/\partial s$ and equate the result to $\partial \Phi_s^t(x)/\partial s$.

To begin let $v(t)$ denote a smooth vector quantity. If we want $\left(\partial \Phi_s^t(x)/\partial x\right) v(t)$ to be the solution to Equation 6, then

$$\frac{\partial \Phi_s^0(x)}{\partial x} v(0) = I_n v(0) = 0 \implies v(0) = 0 \tag{9}$$

where the second equality follows as $\Phi_s^0(x) = x$ for all $s \in (-\epsilon, \epsilon)$. Now differentiating the product and subjecting it to the constraints as defined by Equation 6,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial \Phi_s^t(x)}{\partial x} v(t)\right) &= \left(\frac{\partial}{\partial t}\frac{\partial \Phi_s^t(x)}{\partial x}\right) v(t) + \frac{\partial \Phi_s^t(x)}{\partial x}\frac{dv}{dt}(t) \\ &= \frac{\partial V}{\partial s}\left(s, \Phi_s^t(x)\right) + \frac{\partial V}{\partial x}\left(s, \Phi_s^t(x)\right)\left(\frac{\partial \Phi_s^t(x)}{\partial x} v(t)\right)\end{aligned} \tag{10}$$

From which we substitute Equation 7 to yield

$$\frac{\partial V}{\partial x}\left(s,\Phi_s^t(x)\right)\frac{\partial \Phi_s^t(x)}{\partial x}v(t)+\frac{\partial \Phi_s^t(x)}{\partial x}\frac{dv}{dt}(t)=\frac{\partial V}{\partial s}\left(s,\Phi_s^t(x)\right)+\frac{\partial V}{\partial x}\left(s,\Phi_s^t(x)\right)\left(\frac{\partial \Phi_s^t(x)}{\partial x}v(t)\right)$$

This necessitates that

$$\frac{\partial \Phi_s^t(x)}{\partial x}\frac{dv}{dt}(t)=\frac{\partial V}{\partial s}\left(s,\Phi_s^t(x)\right) \tag{11}$$

Since

$$\Phi_s\left(-t,\Phi_s\left(t,x\right)\right)=x \implies \frac{\partial \Phi_s}{\partial x}\left(-t,\Phi_s(t,x)\right)\circ\frac{\partial \Phi_s}{\partial x}(t,x)=\text{ id}$$

Right inverting yields

$$\frac{\partial \Phi_s^t(x)}{\partial x}^{-1}=\frac{\partial \Phi_s}{\partial x}(t,x)^{-1}=\frac{\partial \Phi_s^{-t}}{\partial x}\left(\Phi_s(t,x)\right) \tag{12}$$

Combining the results of Equation 11 and Equation 12 implies that

$$\frac{dv}{dt}(t)=\frac{\partial \Phi_s^t(x)}{\partial x}^{-1}\frac{\partial V}{\partial s}\left(s,\Phi_s^t(x)\right)=\frac{\partial \Phi_s^{-t}}{\partial x}\left(\Phi_s(t,x)\right)\frac{\partial V}{\partial s}\left(s,\Phi_s^t(x)\right)$$

Integrating and recalling the condition as imposed in Equation 9

$$v(t)=\int_0^t\frac{\partial \Phi_s^{-u}}{\partial x}\left(\Phi_s(u,x)\right)\frac{\partial V}{\partial s}\left(s,\Phi_s^u(x)\right)du$$

Returning to the original form of the solution,

$$\begin{aligned}\frac{\partial \Phi_s^t(x)}{\partial x}v(t)&=\frac{\partial \Phi_s^t(x)}{\partial x}\int_0^t\frac{\partial \Phi_s^{-u}}{\partial x}\left(\Phi_s(u,x)\right)\frac{\partial V}{\partial s}\left(s,\Phi_s^u(x)\right)du\\&=\int_0^t\frac{\partial \Phi_s^{t-u}}{\partial x}\left(\Phi_s(u,x)\right)\frac{\partial V}{\partial s}\left(s,\Phi_s^u(x)\right)du\end{aligned}$$

By uniqueness of solutions, we have the desired equality

$$\frac{\partial \Phi_s^t(x)}{\partial s}=\int_0^t\frac{\partial \Phi_s^{t-u}}{\partial x}\left(\Phi_s(u,x)\right)\frac{\partial V}{\partial s}\left(s,\Phi_s^u(x)\right)du$$

which the coordinate expression of

$$\frac{\partial \Phi_s^t(x)}{\partial s}=\int_0^t d\left(\Phi_s^{t-u}\right)_{\Phi_s^u(x)}\left(\frac{\partial V_s}{\partial s}\left(\Phi_s^u(x)\right)\right)du$$

which is Equation 1 where the variables have not been suppressed by squares.

3 Conclusion

Equation 1 finds it use in the theory of differential equations, in particular, if one has a family of time dependent vector fields, Equation 1 can be used to calculate derivatives transverse to the spatial direction. For example, as done in Duistermaat and Kolk [?], Equation 1 is used to calculate the derivative of the exponential map of a lie group. In particular, if one fixes two elements of the lie algebra $X, Y \in \mathfrak{g}$, one can consider the family of vector fields, $X + sY$ to show the derivative of $\exp : \mathfrak{g} \to G$ is given by

$$d_X \exp = d(L_{\exp X})_e \left(\int_0^1 \exp^{-s\,\mathrm{ad}_X} \, ds \right)$$

References

[1] Hans Duistermaat and Johan Kolk. Lie Groups. pages 333–336, 2000.