Critical Sets in Futoshiki Squares

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2012 Joint Mathematics Meetings
January 4, 2012

Sudoku and Latin Squares

A *Latin square* is an n-by-n arrangement of the numbers 1 through n such that no row or column contains a repeated number.

The popular puzzle Sudoku asks the solver to complete a partial Latin square given the additional constraint that certain subregions (usually squares) also may not contain a repeated number.

1			
	2		
		3	
4			

1	4	2	3
3	2	4	1
2	1	3	4
4	3	1	2

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Note that the subsection constraint is necessary for this puzzle to have a unique solution; without it, the given configuration would extend to two different Latin squares.

Futoshiki and Latin Squares

Futoshiki is another logic puzzle type that originated in Japan and appears in British newspapers. The solver must fill in the grid with no repeats in rows or columns, and also obeying “greater than” comparisons between certain squares.

1			
		3	
4		>	

1	2	4	3
3	4	1	2
2	1	3	4
4	3	2 >	1

Futoshiki and Latin Squares

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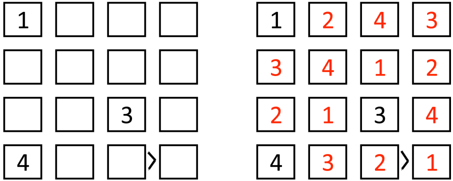

Like Sudoku, solving Futoshiki is completing a Latin square subject to an additional constraint. But in Futoshiki, the additional constraint depends on which comparisons are given (as opposed to the global subsection constraint in Sudoku).

Latin squares and Futoshiki squares

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A *Futoshiki square* is a Latin square together with comparisons for each pair of adjacent cells. (Each comparison specifies which of the two cells is larger.)

A critical set in a Futoshiki square is a collection of numbers and comparisons which uniquely determines that Futoshiki square, and which has no proper subsets which would also do so.

An (h, k)*-critical set* in a Futoshiki square is a critical set consisting of h numbers and k comparisons.

Questions about critical sets in Latin squares

- What is the smallest critical set among all Latin squares of size n?

$$\text{Given } n,\ \min_{L} \min_{S} \{|S| : S \text{ is a critical set for } L\} = ?$$

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$$\text{Given } n, \ \min_L \min_S \{|S| : S \text{ is a critical set for } L\} = ?$$

- What Latin squares of size n have the largest minimal critical sets?

$$\text{Given } n, \ \max_L \min_S \{|S| : S \text{ is a critical set for } L\} = ?$$

Questions about critical sets in Futoshiki squares

- What small (h, k)-critical sets are possible among Futoshiki squares?
  - Given k and n, $\min_L \min_S \{h : \text{there is an } (h,k)\text{-critical set for } L\} = ?$
  - Given h and n, $\min_L \min_S \{k : \text{there is an } (h,k)\text{-critical set for } L\} = ?$

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- What Futoshiki squares require large (h, k)-critical sets?
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An easy general fact

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If S is an (h, k)-critical set for a Futoshiki square of size n,

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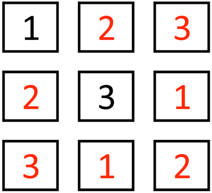
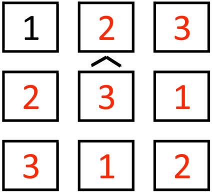
The givens must provide information about at least $(n-1)$ rows and at least $(n-1)$ columns, or else we could permute rows or columns to obtain a non-unique solution.

Each numerical given adds information about at most one row and one column, and each comparison given adds at most two rows and a column (or vice versa).

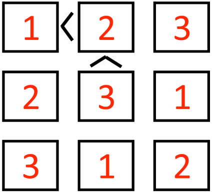
Critical sets in size-3 squares



(2,0)



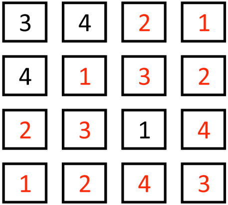
(1,1)



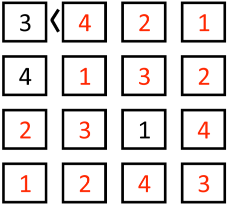
(0,2)

Note that any critical sets of smaller size would violate the inequality given earlier.

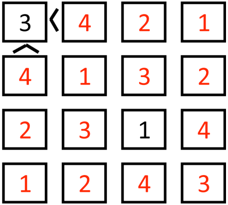
Critical sets in size-4 squares



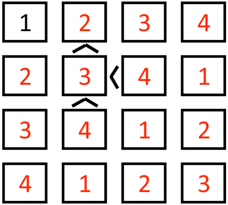
(4,0)



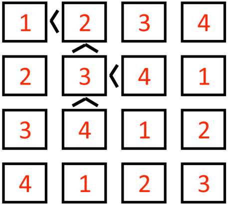
(3,1)



(2,2)

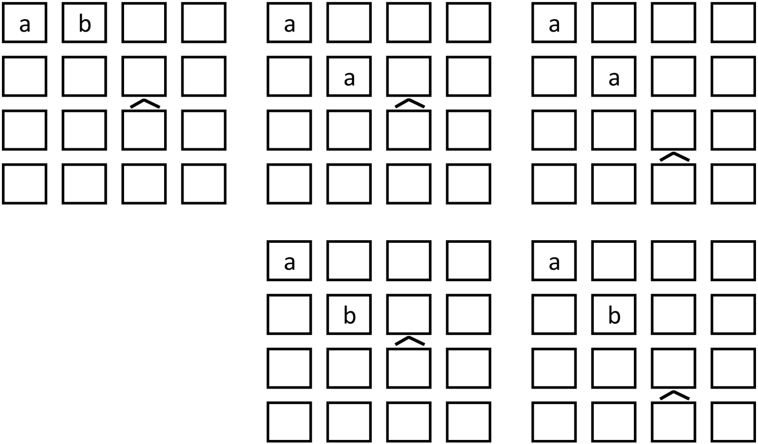


(1,3)



(0,4)

Showing there is no size-4 (2,1)-critical set



Showing there is no size-4 (2,1)-critical set

a	b		
c	d	a	b
d	c	b	a
b	a		

a		b	
d	a	c	b
c	b	d	a
b		a	

a	b	c	d
b	a	d	c
		a	b
		b	a

a		b	
d	b	c	a
c	a	d	b
b		a	

a	c	b	d
d	b	c	a
	d	a	
	a	d	

Critical sets in size-5 squares

2	1	5	4	3
1	5	4	3	2
5	4	3	2	1
4	3	2	1	5
3	2	1	5	4

(6,0)

1	< 2	3	4	5
2	3	< 4	< 5	1
3	4	5	1	2
4	5	1	2	3
5	1	< 2	3	4

(0,6)

[Can be modified to (5,1), (4,2), (3,3), (2,4)]

[Can be modified to (1,5), (2,4), (3,3), (4,2)]

Potential patterns

For size-4 Futoshiki squares, there are $(h, 4-h)$-critical sets for $h = 0, 1, \ldots, 4$, achievable using one of two structures.

For size-5 Futoshiki squares, there are $(h, 6-h)$-critical sets for $h = 0, 1, \ldots, 6$, achievable using one of two structures.

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For size-5 Futoshiki squares, there are $(h, 6-h)$-critical sets for $h = 0, 1, \ldots, 6$, achievable using one of two structures.

Question 1: Given n, let H be the minimal value for which there exists an $(H, 0)$-critical set.
Does there always exist an $(h, H-h)$-critical set for $h = 0, 1, \ldots, H$?

Question 2: How many structures are necessary to achieve the smallest possible critical sets?

The value of H for general n is unknown for $n > 7$ (as far as I know). It would be interesting to know if these questions can be addressed without finding this value.

A Futoshiki square that requires large h

Define the n-by-n Latin square A_n by letting

$$a_{ij} \equiv i + j - 1 \pmod{n}.$$

A_n is referred to as a back-circulant Latin square, and is a standard example used when studying critical sets in Latin squares.

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Let $E_n = (2a_{ij})$ and $O_n = (2a_{ij} - 1)$, and let

$$F_{2n} = \begin{bmatrix} O_n & E_n \\ E_n & O_n \end{bmatrix}.$$

Proposition

For any (h, k)-critical set in F_{2n}, $h \geq n^2$.

A Futoshiki square that requires large h

1	3	5	7	2	4	6	8
3	5	7	1	4	6	8	2
5	7	1	3	6	8	2	4
7	1	3	5	8	2	4	6
2	4	6	8	1	3	5	7
4	6	8	2	3	5	7	1
6	8	2	4	5	7	1	3
8	2	4	6	7	1	3	5

We can partition the square into n^2 sets of four cells.
Any of these sets can be permuted to form a valid Latin square, for which none of the comparisons change.

Thus, any critical set in this Futoshiki square must include at least n^2 numerical givens (one from each set), and no comparison givens can provide this information.

Some variants to consider

- How do these problems change if we allow global comparisons (between non-adjacent squares)?

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- In general, what additional Latin square constraints yield interesting mathematical structures and/or puzzles that are satisfying to solve?

Thanks, and enjoy the Meetings!