ISOLATED AND PARAMETRIZED POINTS ON CURVES

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Abstract. We give a self-contained introduction to isolated points on curves and their counterpoint, parametrized points, that situates these concepts within the study of the arithmetic of curves. In particular, we show how natural geometric constructions of infinitely many degree d points on curves motivate the definitions of $\mathbb{P}^1$- and AV-parametrized points and explain how a result of Faltings implies that there are only finitely many isolated points on any curve. We use parametrized points to deduce properties of the density degree set and review how the minimum density degree relates to the gonality. The paper includes several examples that illustrate the possible behaviors of degree d points.

1. Introduction

Mordell's conjecture, proved by Faltings in 1983, asserts that any curve of genus at least 2 has only finitely many points over any number field. This celebrated theorem is the prototypical example of the guiding philosophy of Diophantine geometry: "geometry controls arithmetic". In this example, the genus, a *geometric* invariant which can be computed over $\mathbb{C}$, controls the arithmetic property of an infinitude of points over some number field.

While the infinitude of points over a number field is a very important arithmetic property, it is not the only one. In some sense, the arithmetic of a curve $C/\mathbb{Q}$ is encoded by all of its algebraic points $C(\overline{\mathbb{Q}})$ equipped with an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. A Galois orbit in $C(\overline{\mathbb{Q}})$ can also be viewed as a (scheme-theoretic) closed point $x \in C$. Through this lens, the degree of the closed point is the size of the orbit (see Section 2 for more details); in particular, the rational points are the same as the degree 1 closed points.

Considering closed points of arbitrary degree gives us more arithmetic information about a curve C. Thus, it is natural to ask: What is the analog of Mordell's Conjecture/Faltings's Theorem for degree d points? In other words, what geometric property of the curve controls the infinitude of degree d points? The Mordell-Lang conjecture, proved by Faltings building on work of Vojta [Fal91, Fal94, Voj91], implies that any infinite collection of degree d points occurs for a geometric reason: namely the infinite collections arise in families parametrized by projective spaces or positive rank abelian varieties. In joint work, the first author with Bourdon, Ejder, Liu, and Odumodu [BELOV19] observed that repeated applications of the Mordell–Lang Conjecture imply the converse, that any such parametrization gives an infinite collection of degree d points. This led to the notion of isolated points, and in op. cit., the authors study the distribution of isolated points in the modular tower $\{X_1(n)\}_{n\in\mathrm{N}}$.

The goal of this paper is to give a self-contained introduction to isolated points and their counterpoint, parametrized points, and situate these concepts in the study of arithmetic of curves (not just modular curves). We propose that this division into parametrized and isolated points gives a fruitful perspective for organizing, describing, and studying the arithmetic of a curve, including characterizing which sets of closed points are Zariski dense.

1.1. **Organization of paper.** We begin with an extended background section which gives an overview of the objects, methods, and results used in the study of rational and degree d points on curves. In Section 2.1, we introduce degree d points on varieties, including related invariants such as the degree set, index, and density degree set, and then discuss their relation to rational points on the symmetric product and the Hilbert scheme (Section 2.2). In Section 2.3, we specialize to the case of curves where we introduce the Abel-Jacobi map and review the results of Faltings on rational points on curves and rational points on subvarieties of abelian varieties. We close the background section in Section 2.4 by reviewing a consequence of Hilbert's Irreducibility Theorem that is particularly useful for producing infinitely many degree d points over number fields.

In Section 3, we describe constructions of infinitely many degree d points, which we use as inspiration for the definitions of $\mathbb{P}^1$-parametrized and AV-parametrized points. We also define the complementary notions of $\mathbb{P}^1$-isolated and AV-isolated points, which were first introduced in [BELOV19], and relate this to the so-called Ueno locus of a subvariety of an abelian variety. In Section 4, we show how a result of Faltings together with the Riemann–Roch Theorem imply that there are only finitely many isolated points on any curve. (Note that isolated and parametrized are properties that apply to *any* closed point on a curve, of which there are always infinitely many.) In Section 5, we use the definition of parametrized points to deduce properties of the density degrees (defined in Section 2.1.4) and relate the minimum density degree to the gonality. We close in Section 6 by highlighting some open questions and further directions.

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2. Background

Notation and conventions. Our primary interest is curves over number fields, so, for simplicity, we assume that our fields have characteristic 0. We use F to denote an arbitrary field of characteristic 0 and we use $\overline{F}$ to denote a fixed algebraic closure. We write G_F for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. We reserve k to denote a number field.

In this paper, we use variety to mean a separated scheme of finite type over a field F. A variety X/F is nice if it is smooth, projective, and geometrically integral over F. For any point $x \in X$, we write $\mathbf{k}(x)$ for the residue field of x; recall that $\mathbf{k}(x)/F$ is an extension with transcendence degree equal to the dimension of the Zariski closure of x. For any extension F'/F, we write $X_{F'}$ for the base change of X to $\operatorname{Spec} F'$ and write $\overline{X} := X_{\overline{F}}$.

A curve is a variety of dimension 1 over a field. Throughout, we use C to denote a nice curve. We write Pic_C for the Picard scheme and $\operatorname{Pic} C$ for the Picard group. In particular, $\operatorname{Pic}_C(F) = (\operatorname{Pic} \overline{C})^{G_F}$ can be larger than $\operatorname{Pic} C$.

2.1. **Degree d points on varieties.**

2.1.1. *Scheme theoretic definition.* Given a nice variety X, a closed point is a scheme theoretic point whose Zariski closure is equal to itself. In an affine open $U = \operatorname{Spec} R \subset X$ the closed points correspond to the maximal ideals $\mathfrak{m} \subset R$. Equivalently, a closed point on X is a reduced and irreducible 0-dimensional subscheme of X.

To any closed point $x \in X$ we may associate its residue field $\mathbf{k}(x)$ (i.e., $R/\mathfrak{m}$ where $x \in \operatorname{Spec} R \leftrightarrow \mathfrak{m} \subset R$), which is a finite extension of F. The degree of a closed point $x \in X$ is the degree of this field extension $[\mathbf{k}(x) : F]$. This agrees with the (constant) Hilbert polynomial of the subscheme $x \subset X$, as described in Section 2.2 below.

2.1.2. *Classical algebrogeometric description.* From a classical viewpoint, we may think of a closed point as a Galois orbit of a geometric point, i.e., a closed point $x \in X$ corresponds to a set $S_x := \{\sigma(y) : \sigma \in G_F\}$ for some $y = y_x \in X(\overline{F})$. From this perspective, the degree of x is the cardinality of the set S_x, and the residue field $\mathbf{k}(x)$ is isomorphic (as an extension of F) to the fixed field of the stabilizer $\operatorname{Stab}_{\sigma(y)} \subset G_F$ of some element $\sigma(y) \in S_x$. Note that the fixed field of the stabilizer $\operatorname{Stab}_{\sigma(y)} \subset G_F$ is an *embedded* field, i.e., a particular subfield of $\overline{F}$ so we have made a choice in how the abstract field $\mathbf{k}(x)$ is embedded in $\overline{F}$. Different choices of this embedding correspond to different choices of elements $\sigma(y) \in S_x$.

2.1.3. *Degree d points under extension.* Given a closed point $x \in C$ and an extension F'/F, the base change $x_{F'}$ can fail to remain a point by becoming reducible. However, this can never occur if F' is linearly disjoint from $\mathbf{k}(x)$.

Lemma 2.1. *Let x be an integral 0-dimensional F-scheme and let F'/F be a finite extension. If $x_{F'}$ is reducible, there exists a minimal subfield $F \subsetneq F_0 \subset F'$ and an integral subscheme $y \subset x_{F'}$ defined over F_0 and $x = \operatorname{Cor}_{F_0/F} y$. In particular,* $\deg(x) = [F_0 : F] \deg(y)$.

Proof. Since F is characteristic 0, $\mathbf{k}(x) = F(\alpha)$ for some $\alpha \in \mathbf{k}(x)$. Thus, the lemma follows from the fact that the minimal polynomial of α over F' divides the minimal polynomial of α over F and that the minimal polynomial over F is the product of the conjugates of the minimal polynomial over F'. □

2.1.4. *Degree sets, index, and density degrees.* There are several important invariants of a variety capturing the behavior of degree d points.

Definition 2.2. Let X be a nice variety defined over a field F.

(1) The degree set $\mathcal{D}(X/F)$ of X is the set of all degrees of closed points on X.
(2) The index of X is the greatest common divisor of the integers in $\mathcal{D}(X/F)$.

(3) The density degree set $\delta(X/F)$ is the set of all positive integers d such that the set of degree d points on X is Zariski dense.
(4) The minimum density degree[1] $\min(\delta(X/F))$ is the smallest positive integer in $\delta(X/F)$.

We always have the containment $\delta(X/F) \subset \mathcal{D}(X/F)$, and consequently, the index divides the minimum density degree. The sets $\mathcal{D}(X/F)$ and $\delta(X/F)$ are sensitive to the ground field F; for example, if $F = \overline{F}$, then they only contain the integer 1! More interestingly, if F is not algebraically closed, integers can be added to $\mathcal{D}(X/F')$ or $\delta(X/F')$ under finite extensions to fields F'/F of definition of closed points. For this reason, we will also be concerned with the following geometric versions of these invariants that is insensitive to finite extensions.

Definition 2.3. Let X be a nice variety defined over a field F.

(1) The potential density degree set $\wp(X/F)$ is the union of $\delta(X/F')$ as F'/F ranges over all possible finite extensions.
(2) The minimum potential density degree is the smallest positive integer in $\wp(X/F)$.

2.2. **The Hilbert scheme and symmetric product.**

2.2.1. *The Hilbert scheme of points.* Let X be a reduced projective variety over F. Given an ample line bundle $\mathcal{O}(1)$ on X, for any subscheme $V \subset X$, the sequence of integers $\dim_F H^0(X, \mathcal{O}_V \otimes \mathcal{O}(n))$ agrees with the evaluation of a polynomial $P(n)$ for $n \gg 0$. This polynomial is the Hilbert polynomial of the subscheme V. The Hilbert scheme Hilb^P_X is a projective scheme over F such that

$$\mathrm{Hilb}^P_X(S) = \left\{ \begin{array}{c} \text{subschemes } V \subset X \times S \text{ proper flat over } S\text{, such that} \\ \text{for all points } s \in S,\ V|_s \subset X_{\mathbf{k}(s)} \text{ has Hilbert polynomial } P \end{array} \right\}. \tag{2.1}$$

If V is a zero-dimensional subscheme, then $\mathcal{O}_V \otimes \mathcal{O}(n) \simeq \mathcal{O}_V$ for all n and $P(n)$ is the constant polynomial $\dim_F H^0(X, \mathcal{O}_V)$. This quantity $\dim_F H^0(X, \mathcal{O}_V)$ is called the length of the subscheme. In this case, if $P(n) = d$, the corresponding Hilbert scheme is commonly called the Hilbert scheme of d points on X (although it would be more appropriate to call it the Hilbert scheme of length d subschemes of X) and we denote it Hilb^d_X. The functor of points (2.1) of Hilb^d_X immediately implies a connection with degree d points.

Lemma 2.4. *A degree d point $x \in X$ gives rise to a degree 1 point on Hilb^d_X.*

Proof. By the scheme-theoretic definition of a degree d point as in Section 2.1.1, we have $[\mathbf{k}(x) : F] = d$. Hence x determines a map $\operatorname{Spec} F \to \mathrm{Hilb}^d_X$ by (2.1), which is a degree 1 point on Hilb^d_X. $\square$

On the other hand, applying (2.1) to $S = \mathrm{Hilb}^d_X$, there is a universal degree d subscheme

$$\begin{tikzcd}
\mathcal{U}^d_X \arrow[r, hook] \arrow[d, "\pi_1|_{\mathcal{U}^d_X}"'] & \mathrm{Hilb}^d_X \times X, \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"] & \\
\mathrm{Hilb}^d_X & & X
\end{tikzcd}$$

[1]This invariant was introduced by Smith and the second author [SV22] and was previously referred to as the arithmetic degree of irrationality.

over the Hilbert scheme of d points. The fibers of $\pi_1|_{\mathcal{U}_X^d}$ are precisely the possible degree d subschemes of X. We can therefore view $\pi_2 \circ \pi_1|_{\mathcal{U}_X^d}^{-1}$ as a map from $\mathrm{Hilb}_X^d(F)$ to the set of order d multisets of $X(\overline{F})$.

2.2.2. *The symmetric product.* Given a reduced variety X over a field F, the d^{th} **symmetric product** is the geometric quotient[2]

$$\mathrm{Sym}_X^d := \underbrace{(X \times X \times \cdots \times X)}_{d}/S_d,$$

where the symmetric group acts by permuting the d factors. In general, Sym_X^d can be a singular projective scheme, even if X is smooth. In fact, Sym_X^d is singular whenever $\dim X > 1$.

Given a degree d point $x \in X$, we may view x as a degree 1 point on Sym_X^d. If F is perfect, then we may see this correspondence using the classical algebrogeometric description, i.e., thinking of x as giving a Galois orbit $S_x \subset X(\overline{F})$ of cardinality d. By considering orderings of S_x, we obtain $d!$ points of $(X \times_F X \times_F \cdots \times_F X)(\overline{F})$ that are faithfully permuted by the symmetric group S_d. Thus, these orderings give a single point in the quotient Sym_X^d. Since S_x was a Galois orbit, the image in Sym_X^d is fixed by the absolute Galois group and thus must be a F-rational point of Sym_X^d.

A more precise way to see this correspondence is via Lemma 2.4 and the so-called **Hilbert–Chow** morphism

$$\begin{aligned} \mathrm{Hilb}_X^d &\to \mathrm{Sym}_X^d \\ x &\mapsto \sum_{z \in X} \mathrm{mult}_z(x) \cdot z \ . \end{aligned} \tag{2.2}$$

In the case of a nice curve C over a field F, the symmetric product Sym_C^d is a nice variety for all $d \in \mathbb{Z}_{\geq 1}$. The Hilbert scheme Hilb_C^d is also smooth by a deformation-theoretic argument. Since (2.2) is a bijection on geometric points[3] and F has characteristic 0, we obtain that the Hilbert-Chow morphism gives an isomorphism $\mathrm{Hilb}_C^d \xrightarrow{\sim} \mathrm{Sym}_C^d$. In what follows we will simply refer to this this variety as Sym_C^d to match conventions in the literature.

2.3. **The case of curves.**

2.3.1. *Invariants of curves.* Given a nice curve C/F, we write Pic_C for the Picard scheme of the curve, and we write Pic_C^d for the connected component consisting of degree d line bundles. Recall that Pic_C^0 is a g-dimensional abelian variety where $g := \mathrm{h}^1(C, \mathcal{O}_C)$ is the **genus** of the curve.

The set of degrees of morphisms from C onto $\mathbb{P}_F^1$ has the structure of an additive semigroup called the **Lüroth semigroup** of C/F [MH82, Definition-Proposition 1]. The **gonality** of C/F is the minimal element of the Lüroth semigroup, i.e.,

$$\mathrm{gon}(C) := \min\{\deg \phi : \phi\colon C \twoheadrightarrow \mathbb{P}_F^1\}.$$

[2]This is not the stack quotient, since the action of S_d is not free.

[3]The inverse map on geometric points sends an unordered d-tuple of geometric points to the appropriate powers of the maximal ideals corresponding to the distinct points.

Note that for any extension F'/F, we have $\mathrm{gon}(C) \geq \mathrm{gon}(C_{F'})$. See [Poo07, Theorem 2.5] for further restrictions on the gonality under base extensions.

2.3.2. *Symmetric products of curves and the Abel-Jacobi map.* A fundamental tool in the study of curves is the degree d Abel–Jacobi map $\mathrm{Sym}^d_C \to \mathrm{Pic}^d_C$ which sends an effective degree d 0-cycle D to its divisor class $[D]$. The fiber of the Abel–Jacobi map over $L \in \mathrm{Pic}(C)$ is the complete linear system $|L| \simeq \mathbb{P}^{h^0(L)-1}$. We write $W^d \subset \mathrm{Pic}^d_C$ for the image of the degree d Abel-Jacobi map.

If $g \geq 1$, then by the Riemann–Roch Theorem, the Abel–Jacobi map yields the following isomorphisms: $W^1 \simeq C$ and $W^g \simeq \mathrm{Pic}^g_C$. In general, for $1 \leq d \leq g$, W^d is an irreducible subvariety of dimension d.

If Pic^d_C has no rational points, then neither does W^d nor Sym^d_C. However, if Pic^d_C does have a rational point, then $\mathrm{Pic}^d_C \simeq \mathrm{Pic}^0_C$ and so we may view W^d as a subvariety of an abelian variety. Then we may apply the following theorem of Faltings to obtain a structural description of the rational points of W^d over number fields.

Theorem 2.5 (Faltings's Subvarieties of AV Theorem [Fal91, Fal94])**.**
Let $W \subset P$ be a nice subvariety of an abelian variety over a number field k. Then

$$W(k) = \bigcup_{i=1}^{N} (w_i + A_i(k)),$$

where $w_i \in W(k)$ and A_i are abelian subvarieties $A_i \subset P$ such that $w_i + A_i \subset W$.

In the case of $C \simeq W^1_C \hookrightarrow \mathrm{Pic}^1_C$, this recovers Faltings's earlier result proving the Mordell conjecture.

Theorem 2.6 (Faltings's Curve Theorem [Fal83, Satz 7])**.** *Let C be a nice curve of genus at least 2 defined over a number field k. Then $C(k)$ is finite.*

Remark 2.7. In the literature, Theorems 2.5 and 2.6 are often referred to as the Mordell–Lang conjecture (or Lang Conjecture) and the Mordell Conjecture, respectively. Since we will refer to these results extensively in the paper and using the term 'conjecture' to refer to results that are proved can be confusing to newcomers to the field, we have elected to refer to them as "Faltings's Subvarieties of AV Theorem" and "Faltings's Curve Theorem", even though this is not standard convention.

2.4. **Hilbert's Irreducibility Theorem.** The following corollary of Hilbert's Irreducibility Theorem can be used to produce many degree d points on nice varieties X defined over number fields k that are equipped with dominant degree d maps to projective space.

Corollary 2.8. *Let X be an irreducible variety of dimension $n \geq 1$ defined over a number field k, and suppose that there exists a dominant map $\pi \colon X \to \mathbb{P}^n_k$, which is generically of degree d. Then there exists a Zariski dense subset of points $t \in \mathbb{P}^n(k)$ such that $\pi^{-1}(t)$ is a degree d point on X.*

We give a brief overview of the ingredients going into this result, with references to where the interested reader can find complete details.

Definition 2.9 ([Ser97, Section 9.1]). A subset $\Omega \subset \mathbb{P}^n(F)$ is called thin if it is contained in a finite union of subsets of the following two types:

- (type 1 thin set) $V(F)$ for $V \subset \mathbb{P}^n_F$ a proper closed subvariety,
- (type 2 thin set) $\pi(X(F))$ for X an irreducible variety and $\pi\colon X \to \mathbb{P}^n_F$ a dominant map of degree at least 2.

The relevance of thin sets to Corollary 2.8 is the following result on specialization of Galois groups.

Lemma 2.10 (Specialization of Galois groups)**.** *Let Y/F be an irreducible variety and let $\pi\colon Y \to \mathbb{P}^n$ be a dominant map with $\mathbf{k}(Y)/\mathbf{k}(\mathbb{P}^n)$ a finite Galois extension. There is a thin subset $\Omega \subset \mathbb{P}^n(F)$, such that for all $t \in \mathbb{P}^n(F) \smallsetminus \Omega$, and any closed point $x \in \pi^{-1}(t)$, the extension $\mathbf{k}(x)/F$ is Galois with Galois group $\mathrm{Gal}(\mathbf{k}(Y)/\mathbf{k}(\mathbb{P}^n))$.*

The idea of the proof of this lemma is that the locus in $\mathbb{P}^n(F)$ over which the Galois group is a proper subgroup $H \subsetneq \mathrm{Gal}(\mathbf{k}(Y)/\mathbf{k}(\mathbb{P}^n))$ is contained in the image of the F-points of Y/H under the quotient map $Y/H \to \mathbb{P}^n$.

Definition 2.11. A field F is called Hilbertian if for $n \geq 1$, the set $\mathbb{P}^n(F)$ itself is *not* a thin set. (This is equivalent to requiring only that $\mathbb{P}^1(F)$ is not thin.)

Fields F for which $F^\times/F^{\times 2}$ is finite, like algebraically closed fields, finite fields, or local fields like $\mathbb{Q}_p$ or $\mathbb{R}$, are not Hilbertian, since $\mathbb{P}^1(F)$ is contained in a finite union of type 2 thin sets coming from $\mathbb{P}^1 \xrightarrow{t \mapsto at^2} \mathbb{P}^1$ for $a \in F^\times/F^{\times 2}$.

Lemma 2.12. *Suppose that F is Hilbertian and that $\Omega \subset \mathbb{P}^n(F)$ is a thin set. Then $\mathbb{P}^n(F) \smallsetminus \Omega$ is Zariski dense in $\mathbb{P}^n$.*

Proof. The set $\mathbb{P}^n(F) \smallsetminus \Omega$ cannot be contained in a proper closed subvariety $V \subset \mathbb{P}^n$, since $V(F) \cup \Omega$ is a thin set. □

Theorem 2.13. *A number field k is Hilbertian.*

See [Ser97, Section 9.4] for the reduction to $k = \mathbb{Q}$ and [Ser97, Section 9.6] for a proof of this result when $k = \mathbb{Q}$. For a proof that works in greater generality see [FJ08, Theorem 13.4.2].

Proof of Corollary 2.8. Let G be the Galois group of the Galois closure of $\mathbf{k}(X)/\mathbf{k}(\mathbb{P}^n)$. Since X is irreducible, G is a transitive subgroup of S_d. Spread out the Galois closure of $\mathbf{k}(X)/\mathbf{k}(\mathbb{P}^n)$ to a morphism $Y \to \mathbb{P}^n_k$ that factors $Y \to X \xrightarrow{\pi} \mathbb{P}^n$. By Lemma 2.10, there exists a thin set $\Omega \subset \mathbb{P}^n(k)$ such that for all $t \in \mathbb{P}^n(k) \smallsetminus \Omega$, the fiber Y_t is a single closed point with Galois group over k equal to G. In particular, X_t must also be a single closed point of degree d for all $t \in \mathbb{P}^n(k) \smallsetminus \Omega$. Since k is Hilbertian by Theorem 2.13, it follows from Lemma 2.12 that $\mathbb{P}^n(k) \smallsetminus \Omega$ is Zariski dense in $\mathbb{P}^n$. □

3. Constructing and parametrizing degree d points

3.1. **Constructions of infinitely many degree d points.** Consider a genus 2 curve $C/\mathbb{Q}$ which is given by an equation of the form $y^2 = f(x)$ for $f(x)$ a separable degree 5 or 6 polynomial. Since C has genus 2, Faltings's Curve Theorem tells us that C has only finitely many points over any number field k. *However*, C evidently has infinitely many degree at most 2 points: for any choice of x-value in $\mathbb{Q}$, we need to take at most a quadratic extension to solve for y. Further, by Faltings's Curve Theorem, there are only finitely many rational points (i.e., degree 1 points) on C, so there must be infinitely many points of degree exactly 2!

Geometrically, we can describe the argument as follows. The genus 2 curve C comes equipped with a degree 2 map $\pi\colon C \to \mathbb{P}^1$, $(x, y) \mapsto x$. The base curve $\mathbb{P}^1$ has infinitely many rational points, and, for each $x \in \mathbb{P}^1(\mathbb{Q})$, the fiber $\pi^{-1}(x)$ is a 0-dimensional scheme of degree 2. Furthermore, Corollary 2.8 implies that $\pi^{-1}(x)$ must be a degree 2 point for infinitely many $x \in \mathbb{P}^1(\mathbb{Q})$.

Note that the same construction works for a hyperelliptic curve $C \to \mathbb{P}^1_k$ of arbitrarily large genus: the preimages of the rational points on $\mathbb{P}^1$ again give rise to infinitely many degree 2 points on C.

These examples show that the genus is **not** a fine enough geometric invariant to detect the infinitude/non-infinitude of degree d points for $d \geq 2$.

3.2. **$\mathbb{P}^1$-parametrized points.** The infinitely many degree 2 points that we described in Section 3.1 all "came from $\mathbb{P}^1$". We make this precise with the following definition.

Definition 3.1. Let C be a nice curve over a number field k.

A closed point $x \in C$ is $\mathbb{P}^1$-parametrized if there exists a morphism $\pi\colon C \to \mathbb{P}^1$ with $\deg(\pi) = \deg(x)$ and $\pi(x) \in \mathbb{P}^1(k)$.

A closed point $x \in C$ is $\mathbb{P}^1$-isolated if it is not $\mathbb{P}^1$-parametrized.

Intuitively, a point x is $\mathbb{P}^1$-parametrized if there is a rational parametrization of a subset of degree d points (more precisely, a subvariety of Sym^d_C) that contains x. Unravelling this intuition gives equivalent formulations of the notion of $\mathbb{P}^1$-parametrized.

Lemma 3.2. *Let C be a nice curve over a number field k and let $x \in C$ be a closed point of degree d. The following are equivalent:*

1. *The point x is $\mathbb{P}^1$-parametrized.*
2. *There exists a nonconstant morphism $\mathbb{P}^1 \to \mathrm{Sym}^d C$ whose image contains $x \in (\mathrm{Sym}^d C)(\mathbb{Q})$.*
3. *The dimension $\mathrm{h}^0([x]) \geq 2$, where $[x]$ denotes the divisor class of x.*

Note that condition (3) is intrinsically defined and gives a method of testing whether a given point is $\mathbb{P}^1$-parametrized or $\mathbb{P}^1$-isolated. (In algebraic geometry terminology, condition (3) can be equivalently phrased that the divisor $[x]$ moves.) Furthermore, the Riemann–Roch Theorem and condition (3) imply that all points of sufficiently large degree are $\mathbb{P}^1$-parametrized.

Corollary 3.3. *Let C be a nice curve over a number field k and let $x \in C$ be a closed point of degree d. If $d \geq g + 1$, then x is $\mathbb{P}^1$-parametrized.*

Proof. By the Riemann–Roch Theorem, $\mathrm{h}^0([x]) \geq d + 1 - g$, which by assumption is at least 2. So x is $\mathbb{P}^1$-parametrized by Lemma 3.2. □

Corollary 3.4. *The Abel–Jacobi map gives an bijection between the set of degree d $\mathbb{P}^1$-isolated points and the set of their divisor classes.*

Proof. By Lemma 3.2, any $\mathbb{P}^1$-isolated point x has $\mathrm{h}^0([x]) = 1$, so there is a unique effective representative of $[x]$ or equivalently the fiber above $[x]$ in the Abel-Jacobi map $\mathrm{Sym}^d_C \to \mathrm{Pic}^d_C$ is exactly x. □

Proof of Lemma 3.2. Assume (1). Then we have a degree d morphism $\pi\colon C \to \mathbb{P}^1$ that sends x to a rational point. Thus the morphism $\mathbb{P}^1 \to \operatorname{Sym}^d C, t \in \mathbb{P}^1 \mapsto \pi^* t \in \operatorname{Sym}^d C$ satisfies the conditions from (2).

Now assume (2). Since abelian varieties contain no rational curves, the composition $\mathbb{P}^1 \to \operatorname{Sym}^d C \to \operatorname{Pic}^d_C$ must be constant. In particular, the image of $\mathbb{P}^1 \to \operatorname{Sym}^d C$, which is birational to $\mathbb{P}^1$ and contains x, must map to $[x]$. Thus the fiber of $[x]$ in the Abel-Jacobi map $\operatorname{Sym}^d C \to \operatorname{Pic}^d_C$ is positive dimensional and so $\mathrm{h}^0([x]) \geq 2$, as claimed in (3).

Finally, assume (3). Since $\mathrm{h}^0([x]) \geq 2$, the global sections $\mathrm{H}^0(C, \mathcal{O}(x))$ contains a nonconstant function π whose poles are contained in x. Since x is irreducible and π is nonconstant, the poles of π must be equal to x, a degree d divisor. Thus, f gives a degree d morphism $C \to \mathbb{P}^1$ that sends x to $\infty \in \mathbb{P}^1(k)$, and so x is $\mathbb{P}^1$-parametrized. ☐

Lemma 3.2 assumes the existence of a closed point of degree d and gives criteria for it to be $\mathbb{P}^1$-parametrized. Similar arguments combined with a corollary of Hilbert's Irreducibility Theorem (Corollary 2.8) show that the existence of high-dimensional basepointfree linear systems imply the existence of some $\mathbb{P}^1$-parametrized point.

Lemma 3.5. *Let C be a nice curve over a number field k and fix a positive integer d. Then the following are equivalent.*

(1) *There exists a degree d divisor D with $\mathrm{h}^0(D) \geq 2$ and $|D|$ basepointfree.*
(2) *There exists a degree d morphism $\phi\colon C \to \mathbb{P}^1$.*
(3) *There are infinitely many degree d $\mathbb{P}^1$-parametrized points.*
(4) *There exists a degree d $\mathbb{P}^1$-parametrized point.*

Remark 3.6. Lemma 3.5 implies that the gonality of a curve is the smallest degree of a $\mathbb{P}^1$-parameterized point.

Proof. We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3)$\Rightarrow$ (4) $\Rightarrow$ (1). Assume (1). Then there exist two effective degree d divisors E_1, E_2 with disjoint support that are both linearly equivalent to D. Thus, there is a function $\phi \in \mathbf{k}(C)^\times$ with $\operatorname{div}(\phi) = E_1 - E_2$ and so ϕ gives a degree d morphism to $\mathbb{P}^1$, as claimed by (2). Assume (2). Then by Corollary 2.8, there exists a Zariski dense (hence infinite) set of $t \in \mathbb{P}^1(k)$ such that C_t is a degree d point, and so Lemma 3.2 implies (3). The implication (3) $\Rightarrow$ (4) is immediate. Lemma 3.2 proves that (4)$\Rightarrow$(1). ☐

3.3. Constructions of infinitely many degree d points that are not $\mathbb{P}^1$-parametrized.

Given the philosophy "Geometry controls arithmetic" and the natural geometric construction of $\mathbb{P}^1$-parametrized points given in Lemma 3.5, it is reasonable to ask whether this is the only way to obtain infinitely many degree d points?

Let us take a moment to review the constructions in Section 3.1. We started with a degree d map $\pi\colon C \to \mathbb{P}^1$ and pulled back the infinitely many rational points to obtain infinitely many degree d effective 0-cycles. Then we used Corollary 2.8 to show that infinitely many of these degree d effective 0-cycles were points.

Notice that the first step requires only that the base curve has infinitely many rational points; it does not use any other properties of $\mathbb{P}^1$. So, we may consider a degree d map $\pi\colon C \to Z$ with $Z(k)$ infinite and $Z \neq \mathbb{P}^1$, and try to determine whether we still obtain infinitely many degree d points on C.

By Faltings's Curve Theorem we know that such a curve Z must have genus 1. For genus 1 curves, we may no longer apply Hilbert's irreducibility theorem; however, Faltings's Curve Theorem allows us to recover a similar result for small d.

Lemma 3.7 (Corollary of Faltings's Curve Theorem)**.** *Let E be a nice genus 1 curve over a number field k with $E(k)$ infinite (i.e., E/k is an elliptic curve of positive rank). Let C be a nice curve of genus at least 2 with a degree d morphism $\pi \colon C \to E$. If $d \leq 3$ then there are infinitely many degree d points on C.*

Proof. Given any $P \in E(k)$, the fiber C_P of C above P is an effective degree d divisor on C. So it is either an irreducible effective degree d divisor, i.e. a degree d point, or, since $d \leq 3$, it is the sum of an effective degree $d-1$ divisor and a rational point. By Faltings's Curve Theorem, $C(k)$ is finite, so the latter possibility only occurs for a finite set of points $P \in E(k)$. Since by assumption $E(k)$ is infinite, there is an infinite set of points $P \in E(k)$ where the fiber C_P is a degree d point. Since all fibers are disjoint, there are infinitely many degree d points on C. □

If $d > 3$, then Faltings's Curve Theorem is not enough to exclude the possibility that the fiber C_P is reducible for all but finitely many points $P \in E(k)$. However, a more involved application of Faltings's (stronger) Subvarieties of AV Theorem does allow us to extend the above lemma to all d (See Theorem 4.4 below).

Thus, it is possible to construct infinitely many degree d points on a curve *without* using a degree d map to $\mathbb{P}^1$; we can instead use a degree d map to a positive rank elliptic curve.

Harris and Silverman proved that if C is a nice curve over a number field k with infinitely many degree 2 points, then (at least up to passing to a finite extension) C must be a double cover of $\mathbb{P}^1$ or an elliptic curve of positive rank [HS91]. Abramovich and Harris extended this result to degree 3 points on all nice curves and degree 4 points on all nice curves of genus different from 7 [AH91]. However, Debarre and Fahlaoui constructed an example showing that, counter to Abramovich and Harris's expection, this is false in general: there exists a nice genus 7 curve with infinitely many degree 4 points and *no* degree 4 map to $\mathbb{P}^1$ or a genus 1 curve (even geometrically) [DF93].

In essence, the reason that such a curve is possible is the fact that Lemma 3.2 has no analog for genus 1 curves. Indeed, the existence of a degree d morphism $f \colon C \to E$ implies the existence of an embedding $E \to \operatorname{Sym}^d C,\ P \mapsto f^*P$. But the converse fails. An embedding $E \to \operatorname{Sym}^d C$ does not necessarily imply the existence of a degree d morphism $C \to E$, as the construction of Debarre and Fahlaoui shows.

3.4. **AV-parametrized points.** Debarre and Fahlaoui's example shows that we need a more expansive interpretation of what it means for a subset of degree d points to be "parametrized by a positive rank elliptic curve". Additionally, Faltings's Subvarieties of AV Theorem suggests that we should also consider parametrizations by positive rank higher dimensional abelian varieties. Taking these as inspiration, we make the following definition (part of which first appeared in [BELOV19]).

Definition 3.8. Let C be a nice curve over a number field k.

A degree d closed point $x \in C$ is AV-parametrized if there exists a positive rank subabelian variety $A \subset \operatorname{Pic}^0_C$ such that $[x] + A \subset W^d = \operatorname{im}(\operatorname{Sym}^d_C \to \operatorname{Pic}^d_C)$ (c.f. Section 2.3.2).

A closed point $x \in C$ is AV-isolated if it is not AV-parametrized.

Corollary 3.9 (Corollary of Faltings's Subvarieties of AV Theorem)**.** *Fix an integer $d \geq 1$. The image of the degree d AV-isolated points under the Abel–Jacobi map is finite.* □

Remark 3.10. Since projective space of any dimension contains $\mathbb{P}^1$, any "$\mathbb{P}^n$-parametrized point" (under the natural generalization) would also be a $\mathbb{P}^1$-parametrized point. However, that is not the case for abelian varieties, since there exist simple abelian varieties of dimension at least 2. Indeed, any nice curve of genus 2 whose Jacobian Pic^0_C is simple with $\mathrm{rk}(\mathrm{Pic}^0_C(k)) \geq 1$ will have $\mathbb{P}^1$-isolated, AV-parameterized points of degree 2 that are *not* parameterized by elliptic curves. This is consistent with the Harris–Silverman result mentioned above because the canonical linear system always gives rise to a degree 2 morphism $C \to \mathbb{P}^1_k$. A similar example can be constructed in genus 3. In other words, the results of Harris–Silverman and Abramovich–Harris yield (after possibly passing to a finite extension) a geometric construction that explains *some* of the infinite set of degree 2 and 3 points, but their results do not give constructions of *all but finitely many* of the degree 2 or 3 points.

3.5. **Comparison with the Ueno locus.** Let A/F be an abelian variety and let $X \subset A$ be a subvariety. The Ueno locus of X is the union of the positive dimensional cosets contained in X, i.e.,

$$\mathrm{Ueno}(X) := \bigcup_{\substack{B<A, x\in X(F)\\ x+B\subset X}} x + B,$$

where $B < A$ denotes a subabelian variety of A. A result of Kawamata [Kaw80, Theorem 4] implies that the Ueno locus is Zariski closed (see [Lan91, Chapter 1, Section 6] for more details on how this implication follows from the statement in Kawamata's work).

Definition 3.11. We say that a closed point $x \in C$ is Ueno if its divisor class $[x]$ lies in the Ueno locus of $W^d \subset \mathrm{Pic}^0_C$, and non-Ueno otherwise.

If X itself is an abelian variety (or a translate) then the Ueno locus of X is all of X. Thus, every closed point of degree at least g is Ueno.

Since any positive rank abelian variety is also positive dimensional, the definitions of AV-parametrized points and the Ueno locus immediately imply the following.

Lemma 3.12. *Let $x \in C$ be a closed point. If x is AV-parametrized then x is Ueno.* □

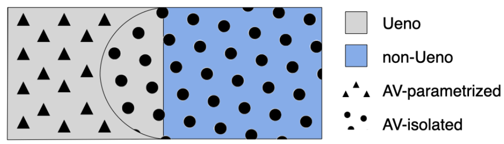
However, the converse does not hold. There can exist Ueno AV-isolated points.

Example 3.13. Consider the genus 2 curve C defined by the equation $y^2 = t_0^6 + 4t_0^4t_1^2 + 4t_0^2t_1^4 + t_1^6$. We have the following morphisms:

$$\begin{tikzcd} & C \arrow[dl, "{[t_0:t_1:y]\mapsto[t_0:t_1]}"'] \arrow[dr, "{[t_0:t_1:y]\mapsto[t_0^2t_1+t_1^3:y:t_1^3]}"] & \\ \mathbb{P}^1 & & E\colon y^2z = x^3 + x^2z - xz^2 \end{tikzcd}$$

The elliptic curve E has rank 0 and six torsion points: $O, (0,0),$ and $(\pm 1, \pm 1)$ [LMFDB, Elliptic Curve 11.a2]. The preimages of $(-1, \pm 1)$ give quadratic points on C that map to quadratic points on $\mathbb{P}^1$. The Jacobian of C has rank 0 [LMFDB, Genus 2 Curve 400.a.409600.1] so every closed point is AV-isolated. In particular, the quadratic points on C that map to $(-1, \pm 1) \in E$ are isolated. However, the map $C \to E$ gives a morphism $E \to \mathrm{Pic}^2_C$, and by definition, the fibers of $C \to E$ are contained in the image of E. Therefore, the quadratic points are Ueno AV-isolated points.

Ueno AV-isolated points should be thought of as fundamentally different than either non-Ueno AV-isolated points or (Ueno) AV-parametrized points, since they are not stable under base extension. For example, Ueno AV-isolated points will become (Ueno) AV-parametrized after a finite extension, whereas non-Ueno AV-isolated points are stable after a finite extension. In the figure below, the grey polka-dot semicircle region represents the AV-isolated points that will become AV-parametrized after a finite extension.



4. Finiteness of isolated points

In the last section, we saw two ways that infinite families of degree d points can arise and used those constructions to motivate the definitions of $\mathbb{P}^1$-parametrized or AV-parametrized points. Despite the rather different nature of these parametrizations (one by a projective space, one by a positive rank abelian variety), both constructions have some similarities, so it will be convenient to think of them in a uniform way.

Definition 4.1. Let C be a nice curve over a number field k.

A closed point $x \in C$ is **parametrized** if it is $\mathbb{P}^1$-parametrized *or* AV-parametrized.

A closed point $x \in C$ is **isolated** if it is both $\mathbb{P}^1$-isolated *and* AV-isolated.

Remark 4.2. Note that parametrized and isolated are properties of *algebraic* points, and curves of any genus always have infinitely many algebraic points.

The definition of parametrized points provides us with additional structure (albeit not necessarily in an explicit way) that we can use to study these points. Indeed, a $\mathbb{P}^1$-parametrized point of degree d is witnessed by a degree d morphism to $\mathbb{P}^1$. An AV-parametrized degree d point x is witnessed by a subabelian variety $A \subset \mathrm{Pic}^0_C$ of positive rank such that the translate $x + A \subset \mathrm{im}(\mathrm{Sym}^d_C \to \mathrm{Pic}^d_C)$. Hence, the parametrized points should be thought of as the ones with a *geometric* reason for existence. The isolated points are mysterious – there is no good geometric reason for their existence. Furthermore, Corollaries 3.3, 3.4, and 3.9 imply that any curve has only finitely many isolated points (out of the infinitely many algebraic points), which is perhaps another reason to think of them as mysterious!

Corollary 4.3 ([BELOV19, Theorem 4.2]; relies on [Fal94])**.** *Let C be a nice curve over a number field. There are finitely many isolated points on C (regardless of degree).* □

We used constructions of curves with infinitely many degree d points to motivate the definition of $\mathbb{P}^1$- and AV-parametrized points. Faltings's Subvarieties of AV Theorem allows us to *characterize* when we have infinitely many degree d points in terms of the existence of degree d parametrized points.

Theorem 4.4 ([Fal91]+ε; [BELOV19, Theorem 4.2])**.** *Let C be a nice curve over a number field. There are infinitely many degree d points if and only if there exists a degree d parametrized point.*

In this section, we review the proof of Theorem 4.4 and in particular, highlight a result about AV-parametrized $\mathbb{P}^1$-isolated points that appears within [BELOV19, Proof of Theorem 4.2] (see Proposition 4.5 below). From this proposition, Theorem 4.4 follows from straightforward applications of Hilbert's Irreducibility Theorem and Faltings's Subvarieties of AV Theorem.

4.1. Points that are AV-parametrized and $\mathbb{P}^1$-isolated.

Proposition 4.5 ([BELOV19, Proof of Theorem 4.2]; relies on [Fal94])**.** *Let C be a nice curve over a number field k. Let $x \in C$ be a degree d point that is* AV*-parametrized and $\mathbb{P}^1$-isolated. Let $A \subset \operatorname{Pic}^0_C$ be a positive rank abelian variety such that $x + A \subset W^d$. Then there exists a finite index subgroup $H \subset A(k)$ such that every element of $[x] + H$ is represented by a degree d point. In particular, there are infinitely many degree d points on C.*

The bulk of the argument is to show that every element of $[x] + H$ is represented by an *irreducible* effective divisor, and it is in this step that we require Faltings's Subvarieties of AV Theorem. The existence of a finite index subgroup $B \subset A(k)$ such that all elements of $[x] + H'$ are *effective* is a much more straightforward argument.

Lemma 4.6. *Let C be a nice curve over a number field k, let E be an effective degree d divisor on C, and let $A \subset \operatorname{Pic}^0_C$ be a positive rank abelian variety such that $E + A \subset W^d$. Then there is a finite index subgroup $B \subset A(k)$ such that $E + B \subset \operatorname{im}(\operatorname{Sym}^d_C(k) \to W^d(k))$.*

Proof. Since $E + A(k) \subset W^d$, every divisor *class* in $E + A(k)$ is represented by a $\overline{k}$-effective divisor. The divisor classes of $E + A(k)$ that are contained in the image of $\operatorname{Sym}^d_C(k) \to W^d(k)$ are exactly the divisor classes that have k-rational representatives. In other words,

$$\begin{aligned}(E + A(k)) \cap \operatorname{im}(\operatorname{Sym}^d_C(k) \to W^d(k)) &= (E + A(k)) \cap \operatorname{im}(\operatorname{Pic}(C) \to \operatorname{Pic}_C(k)) \\ &= E + (A(k) \cap \operatorname{im}(\operatorname{Pic}(C) \to \operatorname{Pic}_C(k))\,.\end{aligned}$$

The exact sequence of low degree terms from the Hochschild-Serre spectral sequence (see [Poo17, Section 6.7, Corollary 6.7.8]) shows that

$$A(k) \cap \operatorname{im}(\operatorname{Pic}(C) \to \operatorname{Pic}_C(k)) = A(k) \cap \ker(\operatorname{Pic}_C(k) \to \operatorname{Br} k) = \ker(A(k) \to \operatorname{Br} k).$$

Since $A(k)$ is finitely generated and $\operatorname{Br} k$ is torsion, we obtain the desired result. ☐

Lemma 4.7. *If $x \in C$ is a point of degree d that is $\mathbb{P}^1$-isolated, then $[x]$ is not in the image of $W^e \times W^{d-e}$ for any $e < d$.*

Proof. Any divisor y in the image of $W^e \times W^{d-e}$ is geometrically represented by the sum of an effective degree e divisor E and an effective degree $d - e$ divisor D. By assumption, x is a degree d point hence an irreducible divisor and so it is not equal to a sum of divisors of lower degree. Further, since $x \in C$ is $\mathbb{P}^1$-isolated, Lemma 3.2 implies that $\mathrm{h}^0([x]) = 1$. Thus, there is a unique effective representative for $[x]$, even over $\overline{k}$, so $[x]$ is not in the image of $W^e \times W^{d-e}$ for any $e < d$. ☐

Proof of Proposition 4.5. By Lemma 4.6, there is a finite index subgroup $B \subset A(k)$ such that every point in $x + B$ is represented by an effective k-rational divisor. We will need to pass to a finite index subgroup of B to ensure that the points are represented by *irreducible* effective k-rational divisors, i.e., degree d points.

Fix a positive integer $e < d$. By Faltings's Subvarieties of AV Theorem, there exist finitely many degree e divisors $D_1^{(e)}, \dots, D_{r_e}^{(e)}$ and finitely many subabelian varieties $A_1^{(e)}, \dots, A_{r_e}^{(e)} \subset \mathrm{Pic}^0_C$ such that

$$W^e_C(k) = \bigcup_{i=1}^{r_e} (D_i^{(e)} + A_i^{(e)}(k)).$$

Thus, the image of $W^e_C(k) \times W^{d-e}_C(k)$ in $W^d_C(k)$ is the union

$$\bigcup_{i=1}^{r_e} \bigcup_{j=1}^{r_{d-e}} \left(D_i^{(e)} + D_j^{(d-e)} + A_i^{(e)}(k) + A_j^{(d-e)}(k) \right).$$

Taking the union also over all $1 \leq e \leq d/2$, we see that the points of W^d_C that are represented by reducible divisors are contained in a finite union of translates $D_i + A_i(k)$ where the divisors D_i are reducible degree d divisors and A_i are subabelian varieties of Pic^0_C.

To prove the Proposition, it suffices to show that there exists a finite index subgroup $H \subset B$ such that $(x+H) \cap (D_i + A_i(k)) = \emptyset$ for all i. Since there are finitely many translates $D_i + A_i(k)$, if there exists a finite index subgroup $H_i \subset B$ such that $(x+H_i) \cap (D_i + A_i(k)) = \emptyset$, then $H = \cap_i H_i$ has the desired properties. Thus, we may reduce to considering a single translate $D + A'(k)$, and we may assume that $(x+B) \cap (D + A'(k)) \neq \emptyset$.

Let $z \in (x+B) \cap (D+A'(k))$; then $(x+B) \cap (D+A'(k)) = z + B \cap A'(k)$. By Lemma 4.7, $x \notin D + A'(k)$, so $z + B \cap A'(k) = (x+B) \cap (D+A'(k)) \subsetneq x + B$, hence $B \cap A'(k) \subsetneq B$ and $x \notin z + B \cap A'(k)$. Since B is a finitely generated abelian group, and $z - x \notin B \cap A'(k)$, by the structure theorem for finitely generated abelian groups and the isomorphism theorems, there exists a group H with the following properties

$$z - x \notin H, \quad B \cap A'(k) \subset H \subset B, \quad \text{, and} \quad [B : H] < \infty.$$

Then, $(x+H) \cap (D+A'(k)) \subset (x+B) \cap (D+A'(k)) = z + B \cap A'(k) \subset z + H$, but also $(x+H) \cap (D + A'(k)) \subset x + H$. Since $z - x \notin H$, $z + H$ and $x + H$ are disjoint cosets and so $(x+H) \cap (D+A'(k)) = \emptyset$ as desired. □

4.2. **Proof of Theorem 4.4.** The ideas to prove the forward direction have appeared many times; we repeat them here for the convenience of the reader. Assume that C has infinitely many degree d points. Then Sym^d_C has infinitely many k points. First assume that there is some degree d point x such that there are infinitely many k-points in the fiber of the Abel-Jacobi map $\mathrm{Sym}^d_C \to \mathrm{Pic}^d_C$ over $[x]$. Then the fiber above $[x]$ has positive dimension so by the definition of the Abel-Jacobi map $\mathrm{h}^0([x]) \geq 2$. Thus, Lemma 3.2 implies that x is $\mathbb{P}^1$-parametrized.

If there is no such degree d point x, then the infinitely many k-points of Sym^d_C must map to infinitely many k points of W^d. Fixing a degree d-point $x \in C$, we may translate by x and embed $W^d \hookrightarrow \mathrm{Pic}^0_C$. After translating back the description in Faltings's Subvarieties of AV Theorem, we see that

$$W^d(k) = \bigcup_{i=1}^{r} D_i + A_i(k),$$

where each of the A_i are subabelian varieties of Pic^0_C and $D_i + A_i \subset W^d$. Since $W^d(k)$ contains infinitely many images of degree d points, there must be some i such that A_i has positive rank and $D_i + A_i$ contains the image of a degree d point x. Thus, $D_i + A_i = x + A_i \subset W^d$ and so x is an AV-parametrized point.

The backwards direction follows from Lemma 3.5 and Proposition 4.5. $\square$

5. The density degree set and the Lüroth semigroup for curves over number fields

Since a proper Zariski closed subset of a curve consists of a *finite* union of closed points, Theorem 4.4 connects the general notion of the density degree set of a curve C defined over a number field with degrees of parameterized points.

Corollary 5.1 (Corollary of Theorem 4.4)**.** *Let C be a nice curve over a number field k. An integer d is in $\delta(C/k)$ if and only if there exists a parameterized point of degree d.*

Thus, Corollary 5.1 implies that $\delta(C/k)$ is the union of the following two sets.

Definition 5.2.

1. $\delta_{\mathbb{P}^1}(C/k) := \{\deg(x) : x \text{ is a } \mathbb{P}^1\text{-parameterized point}\}$
2. $\delta_{\mathrm{AV}}(C/k) := \{\deg(x) : x \text{ is a AV-parameterized point}\}$

By Lemma 3.2, $\delta_{\mathbb{P}^1}(C/k)$ is the same as the Lüroth semigroup of C/k.

5.1. Asymptotics of the density degree set.

When the degree is large enough, the behavior of the degrees of $\mathbb{P}^1$-parametrized points (and thus the integers in the density degree set) is completely determined by the index of the curve.

Proposition 5.3. *Let C be a nice curve over a number field k. If $d \geq \max(2g, 1)$, then the following are equivalent:*

1. *d is a multiple of the index of C*
2. *$d \in \delta_{\mathbb{P}^1}(C/k)$*
3. *$d \in \delta(C/k)$*
4. *$d \in \mathcal{D}(C/k)$*

Proof. We will prove (1) $\Rightarrow$(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Assume that d is a multiple of the index. Since the index is the greatest common divisor of the degrees of closed points on C, there exists a linear combination of closed points, i.e., a k-rational divisor D, of degree d. Since $d \geq 2g$, the Riemann–Roch Theorem implies that $h^0(D) \geq \max(g + 1, 2) \geq 2$ and that the complete linear system of D is basepoint free. Hence, there exists a map $C \to \mathbb{P}^1_k$ of degree d, and C has a parameterized point of degree d by Lemma 3.5. The implication (2) $\Rightarrow$ (3) is Corollary 5.1. The implication (3) $\Rightarrow$ (4) is immediate. Similarly the implication (4) $\Rightarrow$ (1) follows from the definition of the index as the greatest common divisor of the integers in $\mathcal{D}(C/k)$. $\square$

Remark 5.4. The proof of Proposition 5.3 crucially uses Hilbert's irreducibility theorem, and hence that the ground field is Hilbertian. If, instead, the ground field were Henselian, counterexamples to the analogous statement are given in [CV].

The assumption $d \geq \max(2g, 1)$ in Proposition 5.3 cannot be weakened to $d \geq g + 1$, which merely guarantees that a linear system has dimension at least 1, without basepoint freeness. Compare this to Corollary 3.3, where the fact that x is a closed point guarantees basepoint freeness. The following example illustrates this, as well as the fact that $\mathrm{Sym}^d_C(k)$ being infinite is not sufficient to guarantee that $d \in \delta(C/k)$.

Example 5.5. Let C be a curve of genus 2 for which $C(k) = \{p\}$ and $\mathrm{Pic}^0_C(k) = \{[\mathcal{O}_C]\}$ (in particular, Pic^0_C has rank 0 over k). Since $C(k) \neq \emptyset$, we have $1 \in \mathcal{D}(C/k)$; on the other hand, $1 \notin \delta(C/k)$ by Faltings's Curve Theorem. The complete linear system of K_C gives a degree 2 map $C \to \mathbb{P}^1_k$, and hence $2 \in \delta(C/k)$. Since $\mathrm{Pic}^0_C(k) = \{[\mathcal{O}_C]\}$, the only k-point on Pic^3_C is $[K_C(p)]$. Even though $h^0(K_C(p)) = 2$, there is no degree 3 map $C \to \mathbb{P}^1_k$: since $h^0(K_C) = 2$, the point p is a basepoint of this linear system. Hence $3 \notin \mathcal{D}(C/k)$ (even though $3 \geq 2+1$), and consequently $3 \notin \delta(C/k)$. Combining this with Proposition 5.3, we see that $\mathcal{D}(C/k) = \mathbb{Z}_{>0} \setminus \{3\}$ and $\delta(C/k) = \mathbb{Z}_{>0} \setminus \{1, 3\}$. Examples of such curves over $\mathbb{Q}$ can be found by the following search [LMFDB].

The degrees of AV-parametrized points also have a uniform behavior when d is large.

Lemma 5.6. *Let C be a nice curve over a number field k. If $d \geq g \geq 1$, then the following are equivalent:*

1. $d \in \delta_{\mathrm{AV}}(C/k)$
2. $d \in \mathcal{D}(C/k)$ *and* $\mathrm{rk}(\mathrm{Pic}^0_C(k)) > 0$

Proof. Since $W^d = \mathrm{Pic}^d_C$ for all $d \geq g$ by the Riemann–Roch theorem, once there exists a closed point x of degree $d \geq g$, we have $x + \mathrm{Pic}^0_C \subset W^d$. ☐

Remark 5.7. If Pic^0_C is a simple abelian variety, then there are no AV-parameterized points of degree $1 \leq d \leq g-1$, since $\dim W^d = d < g$ in this range. Hence the existence of AV-parameterized points is controlled exclusively by $\mathcal{D}(C/k)$ and $\mathrm{rk}(\mathrm{Pic}^0_C(k))$ by Lemma 5.6.

Remark 5.8. If $\mathrm{rk}(\mathrm{Pic}^0_C(k)) = 0$, then every $x \in C$ is AV-isolated and $\delta_{AV}(C/k) = \emptyset$.

5.2. **Multiplicative structure of the density degree set.** The set $\delta_{\mathbb{P}^1}(C/k)$ is closed under multiplication by positive integers, since a degree d morphism $\phi\colon C \to \mathbb{P}^1$ can be composed with any degree n morphism $\psi_n\colon \mathbb{P}^1 \to \mathbb{P}^1$ to yield the degree dn morphism $\psi_n \circ \phi\colon C \to \mathbb{P}^1$. The set of degrees of AV-parametrized points also has this structure.

Proposition 5.9. *Let C be a nice curve defined over a number field k. If $d \in \delta_{\mathrm{AV}}(C/k)$, then for all $n \geq 2$, we have*

$$nd \in \delta_{\mathrm{AV}}(C/k) \cap \delta_{\mathbb{P}^1}(C/k).$$

In particular, $\delta_{\mathrm{AV}}(C/k)$, and hence $\delta(C/k)$, is closed under multiplication by $\mathbb{Z}_{\geq 1}$.

Proof. Let x be an AV-parameterized point of degree d. Then there exists a positive rank abelian variety $A \subset \mathrm{Pic}^0_C$ such that $x + A \subset W^d$. Since all positive multiples of an effective divisor are also effective, we have $[nx] + [n]A \subset W^{nd}$, where $[n]$ is the multiplication by n map on Pic^0_C. Since $[n]A = A$, if we can show that there exists a closed point $z \in |nx|$, then $z + A \subset W^{nd}$ and $nd \in \delta_{\mathrm{AV}}(C/k)$.

By Lemma 4.6, there exists a finite index subgroup $H \subset A(k)$ such that every divisor in $x + H$ is in the image of $\mathrm{Sym}^d_C(k)$. Let $D_0 \in H \subset A(k)$ be a point of infinite order. Then for all integers i, the divisor class $x + iD_0$ is represented by some degree d k-rational effective divisor y_i (we will take $y_0 = x$). Note that since D_0 has infinite order $y_i \not\sim y_j$ if $i \neq j$. Then for any integer n and any integer i, we have

$$(n-1)y_i + y_{i-in} \;\sim\; (n-1)(x + iD_0) + (x + (i - in)D_0) \;\sim\; nx = ny_0.$$

Thus, for $n \geq 2$, $[nx]$ has infinitely many effective representatives with disjoint support and so $\mathrm{h}^0([nx]) \geq 2$ and $[nx]$ is basepointfree. Hence, by Lemma 3.5, there exists a closed point of degree nd in $|nx|$ and we conclude $nd \in \delta_{\mathbb{P}^1}(C/k) \cap \delta_{AV}(C/k)$. ☐

The $n = 2$ case of Proposition 5.9 implies that geometric properties, such as the gonality, can exclude the existence of a parametrized point, which is another example of geometry controlling arithmetic.

Corollary 5.10 ([AH91, Fre94])**.** *Let C be a nice curve over a number field k. If $d \in \delta_{\mathrm{AV}}(C/k)$, then $\operatorname{gon}(C) \leq 2d$.*

This multiplicative structure of the density degree set of a curve over a number field can be applied to understand the behavior under finite extensions of the ground field. As is true in general, integers can be added to $\delta(C/k)$ under finite extensions k'/k. However, they can not be *removed* from $\delta(C/k)$.

Corollary 5.11. *Let C be a nice curve defined over a number field k, and let k'/k be a finite extension. Then*

$$\delta(C/k) \subseteq \delta(C/k').$$

Remark 5.12. Corollary 5.11 implies that there exists a single finite extension k_0/k such that $\wp(C/k) = \delta(C/k_0)$.

Proof. Let $d \in \delta(C/k)$. Then by Corollary 5.1, there exists a degree d parametrized point $x \in C_k$. If $x_{k'}$ is irreducible, then it is a degree d point on $C_{k'}$ and Definitions 3.1 and 3.8 imply that $x_{k'}$ is parametrized.

Thus, we have reduced to the case that $x_{k'}$ is reducible, and further, we may assume that the above basechange construction results in reducible divisors for all (infinitely many) degree d parametrized points $x \in C_k$. Then by Lemma 2.1, each $x_{k'}$ is the corestriction of a degree e point for some $e|d$ (that depends on $x_{k'}$). By the pigeonhole principle, there must be infinitely many $x_{k'}$ that are the corestriction of a degree e point for the same fixed $e|d$. In particular $C_{k'}$ has infinitely many degree e points. Thus $e \in \delta(C/k')$, so Proposition 5.9 implies that $d \in \delta(C/k')$. ☐

5.3. **Additive structure of the density degree set.** Since $\delta_{\mathbb{P}^1}(C/k)$ is equal to the Lüroth semigroup, it is closed under addition. Given Proposition 5.9, one may ask whether $\delta(C/k)$ *also* has the structure of an additive semigroup. The following example shows that this is not necessarily the case.

Example 5.13. Let $E/\mathbb{Q}$ be the elliptic curve defined by $y^2 = x^3 - x + 1$ whose Mordell-Weil group is free of rank 1 [LMFDB, 92.a1]. Let $\psi \colon \mathbb{P}^1 \to \mathbb{P}^1$ be the degree 3 map given by $[t_0 : t_1] \mapsto [t_0^3 + 7t_0^2 t_1 - 9t_0 t_1^2 + 7t_1^3 : t_0 t_1 (t_0 - t_1)]$, and let C be the fiber product $E \times_{\mathbb{P}^1} \mathbb{P}^1$. Then C is a smooth hyperelliptic curve of genus 5 that is a degree 3 cover of E. In particular, C has a degree 2 $\mathbb{P}^1$-parametrized point and a degree 3 AV-parametrized point. We claim that C has no degree 5 parametrized points. The key features for this proof are that C cannot admit a degree 3 or 5 map to $\mathbb{P}^1$ for genus reasons and that $\operatorname{Pic}_C(\mathbb{Q}) / \operatorname{Pic}_E(\mathbb{Q})$ is generated by rational Weierstrass points of C.

If $5 \in \delta_{\mathbb{P}^1}(C/k)$, then by Lemma 3.5 there is a degree 5 map $\phi \colon C \to \mathbb{P}^1$. Considering the product of the hyperelliptic map and ϕ gives a birational model of C as a $(2, 5)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$. However, the arithmetic genus of $(2, 5)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is $(2-1)(5-1) = 4 < g = 5$. So there is no such morphism ϕ.

Thus it remains to prove that $5 \notin \delta_{AV}(C/k)$. Since $g = 5$, we have $W^5 = \operatorname{Pic}_C^5$, and since E has rank 1, the Jacobian of C has positive rank. Thus any point in $\operatorname{Pic}_C^5(\mathbb{Q})$ that is represented by an *irreducible* divisor will be an AV-parametrized point. We must therefore prove that $5 \notin \mathcal{D}(C/\mathbb{Q})$. In other words, we must show that every point in $\operatorname{Pic}^5_C(\mathbb{Q})$ is represented by a *reducible* divisor. (Since there are no $\mathbb{P}^1$-parametrized points of degree 5, every degree 5 linear system has a base point. In particular, if there is a reducible representative of a degree 5 divisor class, then every effective representative of this class will be reducible.)

The degree 3 morphism $C \to E$ induces an embedding $E \hookrightarrow \operatorname{Sym}^3_C$ where all the points in the image have disjoint support. In addition, C has no degree 3 map to $\mathbb{P}^1$ because its genus is too large, so composing with the degree 3 Abel-Jacobi map and translating gives an embedding $\operatorname{Pic}^0_E \hookrightarrow \operatorname{Pic}^0_C$. A `Magma` computation shows that the rank of $\operatorname{Pic}^0_C(\mathbb{Q})$ is at most 1, so $\operatorname{Pic}^0_C(\mathbb{Q})/\operatorname{Pic}^0_E(\mathbb{Q})$ must be torsion. This finite torsion group must embed into $\operatorname{Pic}^0_C(\mathbb{F}_p)/\operatorname{Pic}^0_E(\mathbb{F}_p)$ for any prime p of good reduction. For $p = 5$ and $p = 13$, this group has order $2^3 \cdot 3 \cdot 29$ and $2^2 \cdot 17 \cdot 311$ respectively, so $\operatorname{Pic}^0_C(\mathbb{Q})/\operatorname{Pic}^0_E(\mathbb{Q})$ has order at most 4. By construction, C has 3 rational Weierstrass points w_0, w_1, w_2 (contained in $\psi^{-1}(\infty)$), which generate an order 4 torsion subgroup. Since E has a free Mordell-Weil group, these Weierstrass points must generate $\operatorname{Pic}^0_C(\mathbb{Q})/\operatorname{Pic}^0_E(\mathbb{Q})$. Thus, $W^5(\mathbb{Q}) = \operatorname{Pic}^5_C(\mathbb{Q})$ consists of four translates of $\psi^* \operatorname{Pic}^1_E$ by the following four degree 2 divisors:

$$[2w_0],\ [2w_0] + [w_1 - w_0],\ [2w_0] + [w_2 - w_0],\ \text{and } [2w_0] + [w_1 + w_2 - 2w_0].$$

We see, by inspection, that every divisor in these translates has a reducible effective representative.

5.4. **The minimum density degree of a curve over a number field.** The minimum density degree of a curve over a number field is related to the gonality, but also accounts for the additional possibility of AV-parameterized points of small degree.

Proposition 5.14. *Let C be a nice curve over a number field k.*

(1) *We have* $\operatorname{gon}(C)/2 \leq \min(\delta(C/k)) \leq \operatorname{gon}(C)$. *In particular, for $g \neq 1$, we have* $\min(\delta(C/k)) \leq |2g-2|$.
(2) *If all closed points on C of degree strictly less than* $\operatorname{gon}(C)$ *are AV-isolated, for example, if* $\operatorname{rk}\operatorname{Pic}^0_C(k) = 0$, *then* $\min(\delta(C/k)) = \operatorname{gon}(C)$.

Remark 5.15. The bounds in (1) are sharp [SV22, Theorem 1.1].

Proof.

(1) The inequality $\min(\delta(C/k)) \leq \operatorname{gon}(C)$ follows from Lemma 3.5. If $g \neq 1$, either $|K_C|$ or $|-K_C|$ is a basepoint free linear system of degree $|2g-2|$, and hence gives an upper bound on the gonality. By Corollary 5.1, there exists a parameterized point of degree $\min(\delta(C/k))$. The inequality $\operatorname{gon}(C)/2 \leq \min(\delta(C/k))$ thus follows from Corollary 5.10.
(2) By definition, all points on C of degree strictly less than $\operatorname{gon}(C)$ are $\mathbb{P}^1$-isolated. Hence, all points on C of degree strictly less than $\operatorname{gon}(C)$ are isolated if and only if they are AV-isolated. ∎

If Pic^0_C is a simple abelian variety, there is only one case where $\min(\delta(C/k))$ can take a different value than $\operatorname{gon}(C)$.

Corollary 5.16. *If* Pic^0_C *is a simple abelian variety, then*

$$\min(\delta(C/k)) = \begin{cases} g & \text{if } g \in \mathcal{D}(C/k), \mathrm{rk}(\mathrm{Pic}^0_C(k)) > 0, \textit{ and } \mathrm{gon}(C) \geq g, \\ \mathrm{gon}(C) & \textit{otherwise.} \end{cases}$$

Proof. It suffices to consider the case $\min(\delta(C/k)) = \min(\delta_{\mathrm{AV}}(C/k))$. By Remark 5.7, we have $\min(\delta_{\mathrm{AV}}(C/k)) \geq g$. If $\min(\delta(C/k)) \geq g+1$, any closed point (which may or may not be AV-parameterized) of degree $\min(\delta(C/k))$ is $\mathbb{P}^1$-parameterized by Corollary 3.3 and so $\min(\delta(C/k)) = \mathrm{gon}(C)$. The only remaining case is $\min(\delta(C/k)) = g$, which requires $\mathrm{gon}(C) = \min(\delta_{\mathbb{P}^1}(C/k)) \geq g$ by Remark 3.6, and $g \in \mathcal{D}(C/k)$ and $\mathrm{rk}(\mathrm{Pic}^0_C(k)) > 0$ by Lemma 5.6. □

When g is small, the minimum density degree depends only on the arithmetic of rational points and the gonality.

Proposition 5.17. *If* $g \leq 2$*, then*

$$\min(\delta(C/k)) = \begin{cases} 1 & : \textit{if } C(k) \textit{ is infinite} \\ \mathrm{gon}(C) & : \textit{otherwise.} \end{cases}$$

Proof. If $C(k)$ is infinite (hence $g \leq 1$ by Faltings's Curve Theorem), then $1 \in \min(\delta(C/k))$ and so $\min(\delta(C/k)) = 1$. We may assume for the remainder that $C(k)$ is finite so that $\mathrm{gon}(C) \geq \min(\delta(C/k)) \geq 2$. If $g \leq 1$, then $2 \geq \max(2g, 1)$, so by Proposition 5.3, the minimum density degree is the smallest degree of a $\mathbb{P}^1$-parameterized point. Hence, we must have equality $\min(\delta(C/k)) = \mathrm{gon}(C)$ by Remark 3.6. For $g = 2$, the canonical linear system exhibits that $\mathrm{gon}(C) = 2$. Hence every inequality in $2 \leq \min(\delta(C/k)) \leq \mathrm{gon}(C) = 2$ is an equality. □

On the other hand, once the genus of C is at least 3, the minimum density degree becomes more subtle, and genuinely reflects new information coming from *other* geometric sources of degree d points.

Example 5.18. Consider the plane quartic $C : y^4 = x^3z - xz^3 + z^4 \subset \mathbb{P}^2$, which is a smooth curve of genus 3. By Faltings's Curve Theorem, we have $\min(\delta(C/k)) > 1$. The morphism $\mathbb{P}^2 \to \mathbb{P}(1,2,1)$, $[x : y : z] \mapsto [x : y^2 : z]$ induces a degree 2 morphism from C to $E : Y^2 = X^3Z - XZ^3 + Z^4$. The elliptic curve E has rank 1 [LMFDB, 92.a1], and thus, by Lemma 3.7, C has degree 2 AV-parametrized points and $\min(\delta(C/k)) = 2$. On the other hand, smooth plane quartics are canonically embedded, so C is not hyperelliptic. Projection from the point $[0 : 1 : 1]$ gives a degree 3 map to $\mathbb{P}^1$. Hence, $3 = \mathrm{gon}(C) > 2 = \min(\delta(C/k))$.

The phenomena of Example 5.18 is not the only way an AV-parameterized point on a genus 3 curve can witness the minimum density degree. Indeed, any smooth plane quartic that has index 1 and whose Jacobian is simple and has positive rank will have no $\mathbb{P}^1$-parametrized points of degree at most 2, and $\min(\delta_{\mathrm{AV}}(C/k)) = 3$.

5.5. **Potential density degrees.** The density degree set is sensitive to arithmetic information about the curve, in particular, the index. Since the potential density degree set is the union of the density degree sets over finite extensions, it has no dependence on the index of the curve. In addition, the minimum potential density degree depends only on $C_{\overline{k}}$.

Proposition 5.19. *Let C/k be a nice curve. Then*
$$\min(\wp(C/k)) = \min\left(\{\mathrm{gon}(C_{\overline{k}})\} \cup \left\{d : \mathrm{Ueno}\left(W^d_{\overline{k}}\right) \neq \emptyset\right\}\right).$$

Proof. Let k'/k be a finite extension. Then $d \in \delta_{\mathbb{P}^1}(C_{k'})$ implies $\mathrm{gon}(C_{\overline{k}}) \leq d$. Similarly, $d \in \delta_{\mathrm{AV}}(C_{k'})$ implies the existence of a degree d point x on $C_{k'}$ and an a positive dimensional abelian subvariety $A \subset \mathrm{Pic}^d_{C_{k'}}$ such that $x + A \subset W^d_{k'}$. Therefore $\min(\wp(C/k))$ is an upper bound for quantity on the right.

On the other hand, if d is the minimum value of the quantity on the right, we want to show that there exists a finite extension k'/k for which $d \in \delta(C_{k'})$. By the previous paragraph, for all finite extensions k'/k, we have $\min(\delta(C_{k'})) \geq d$. If $d = \mathrm{gon}(C_{\overline{k}})$, then $d \in \delta_{\mathbb{P}^1}(C_{k'})$ for the field of definition k' of this map. We may therefore assume that d is the smallest positive integer for which $\mathrm{Ueno}\left(W^d_{\overline{k}}\right) \neq \emptyset$, i.e., the smallest positive integer for which there exists a positive-dimensional abelian translate A in $W^d_{\overline{k}}$. Let k'/k be a finite extension such that A is defined over k' and $\mathrm{rk}(A(k')) > 0$, so we have $\mathrm{Sym}^d_C(k')$ is infinite. Since $\min(\delta(C_{k'})) \geq d$, we must have that for all $d' < d$, $\mathrm{Sym}^{d'}_C(k')$ is finite. Thus there are infinitely many closed points of degree exactly d, and $d \in \delta(C_{k'})$ as desired. □

6. Further directions

6.1. **Uniform bounds on the number of isolated points.** Given that the set of isolated points is finite, it is natural to ask how large this set can be. By interpolation, we can always obtain the existence of a curve of genus at least 2 with arbitrarily many rational points, which, by Faltings's Curve Theorem, are necessarily isolated. However, interpolation yields a curve whose genus grows with the number of rational points. Thus, a better question is how large the set of isolated points can be on curves *of fixed genus.*

Recent work of Dimitrov–Gao–Habegger [DGH21] combined with work of Kühne [Kü] gives a bound on the number of rational points of a curve of genus g that depends only on g and the rank of the Jacobian (see [Gao] for a survey of how the works come together). Further work of Gao–Ge–Kühne gives a uniform version of the Mordell-Lang conjecture, which gives us the following bound on non-Ueno isolated points

Theorem 6.1 (Corollary of [GGK, Theorem 1.1])**.** *Let C be a smooth genus g curve over a number field k. Then the number of non-Ueno isolated points is bounded by a constant that depends only on g and the rank of* $\mathrm{Pic}^0_C(k)$.

Proof. By Corollary 3.3, any isolated point has degree at most g. Further, every point of degree g is Ueno (Definition 3.11). Thus, it suffices to show that for each $d < g$, the number of non-Ueno isolated points of degree d is bounded by a constant that depends only on g, d and the rank of $\mathrm{Pic}^0_C(k)$.

Recall that W^d denotes the image of the Abel-Jacobi map $\mathrm{Sym}^d_C \to \mathrm{Pic}^d_C$. Let $\theta \in \mathrm{Pic}^0_C$ denote the (ample) theta divisor.

The rational points on Sym^d_C corresponding to isolated degree d points on C inject into the rational points of W^d, so it suffices to show that the images of the non-Ueno isolated points in $W^d(k)$ are bounded by a constant that depends only on g, d and the rank of $\mathrm{Pic}^0_C(k)$. By Faltings's Subvarieties of AV Theorem and [GGK, Theorem 1.1], there exists

(1) a constant $c = c(g, \deg_\theta W^d)$,
(2) a positive integer $N \leq c(g, \deg_\theta W^d)^{1+\mathrm{rank}\,\mathrm{Pic}^0_C(k)}$, and

(3) for each $1 \le i \le N$, a point $x_i \in \mathrm{Pic}^d_C(k)$ and a subabelian varieties $B_i \subset \mathrm{Pic}^0_C$ with $x_i + B_i \subset W^d$,

such that

$$W^d(k) = \bigcup_{i=1}^{N} (x_i + B_i(k)).$$

Since, by definition, the image of the non-Ueno points must be contained in the union of $x_i + B_i(k)$ where B_i is 0-dimensional (and hence B_i equals the identity), the number of images of the non-Ueno points is bounded by $N \le c(g, \deg W^d)^{1+\mathrm{rank}\,\mathrm{Pic}^0_C(k)}$. Finally, [ACGH85, Poincaré's Formula, Chapter I, Section 5] gives the following formula for $\deg_\theta W^d$ that depends only on g and d:

$$\deg_\theta W^d = \deg_\theta \left(\frac{\theta^{g-d}}{(g-d)!} \right) = \frac{\theta^g}{(g-d)!} = \frac{g!}{(g-d)!}. \qquad \square$$

Any Ueno isolated point x of degree d lies on a translate of a positive dimensional rank 0 abelian variety $A \subset \mathrm{Pic}^0_C$ such that $x + A \subset W^d$. The theorem of Gao–Ge–Kühne bounds the number of such abelian varieties by a constant that depends on g, d, and $\mathrm{rank}\,\mathrm{Jac}_C(k)$. Since any isolated point has $d \le g$, this implies that the number of Ueno isolated points is bounded depending only on g and $\mathrm{rank}\,\mathrm{Jac}_C(k)$ if the size of k-rational torsion on abelian varieties of dimension at most g is bounded by a constant that depends only on g and the rank of the Mordell–Weil group. The existence of such a uniform bound remains open.

6.2. **Algebraic points on rank 0 curves.** Let C be a curve over a number field k whose Jacobian has rank 0. Then C has no AV-parametrized points, so, by Corollary 4.3 all but finitely many closed points of C are $\mathbb{P}^1$-parametrized. The definition of $\mathbb{P}^1$-parametrized points then gives the following result:

Corollary 6.2 (Specialization of Corollary 4.3)**.** *Let C be a curve over a number field k whose Jacobian has rank 0. Then there exists a proper closed subscheme $Z \subset C$ such that all closed points $x \in C \setminus Z$ appear as a fiber of some morphism to $\mathbb{P}^1$.*

In other words, all but finitely many points appear as divisors in a positive-dimensional Riemann–Roch space. In addition, since the finitely many exceptions are isolated points, by definition the exceptions are all 0-dimensional Riemann–Roch spaces. Thus, given a fixed divisor D on C whose degree is equal to the index, every point of degree d on C lies in one of the finitely many Riemann–Roch spaces

$$|P + dD|, \quad \text{for } P \in \mathrm{Pic}^0_C(k) = \mathrm{Pic}^0_C(k)_{\mathrm{tors}}.$$

(Note that the collection of these Riemann–Roch spaces doesn't depend on D, since for any other choice D' and any integer d, $d(D - D') \in \mathrm{Pic}^0_C(k)$.) Moreover, for any $P, P' \in \mathrm{Pic}^0_C(k)$ and any positive integers d, d', addition of divisors gives morphisms

$$\Sigma_{P,P';d,d'} \colon |P + dD| \times |P' + d'D| \longrightarrow |(P + P') + (d + d')D|.$$

Thus, the closed points of C in some sense can be encoded as a limit of unions of projective spaces. Since the geometry and arithmetic of projective space is completely understood, we could speculate whether this perspective gives a way to "understand" all closed points on a curve.

6.3. **Classification of curves C with small minimum potential density degree.** Faltings's Curve Theorem gives a geometric classification of curves C over number field k with $\min(\wp(C/k)) = 1$: they are precisely the curves of genus 0 or 1. This raises the question of classifying curves for which $\min(\wp)$ takes other (small) values.

The results of Harris–Silverman [HS91] and Abramovich–Harris [AH91] quoted in Section 3.3 classify the curves C with $\min(\wp(C/k)) = 2$ or 3: over $\overline{k}$ they are always degree 2 or 3 covers of curves of genus 0 or 1. On the other hand, Debarre–Fahlaoui give examples that show that $\min(\wp(C/k))$ can equal 4 even for curves C that are not degree 4 covers of a curve of genus at most 1 [DF93]. In [KV], Kadets and the second author extend Harris–Silverman's and Abramovich–Harris's classification results to $d \in \{4, 5\}$: the only curves with $\min(\wp(C/k)) = d$ besides the curves with a degree d cover of a curve of genus at most 1 are the examples that Debarre–Fahlaoui constructed! It is an intriguing problem to extend this $d > 5$, where, presumably, examples beyond those constructed by Debarre–Fahlaoui can be found.

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