The Surface Subgroup Theorem

and

The Ehrenpreis Conjecture

by Jeremy Kahn

Joint with Vladimir Markovic

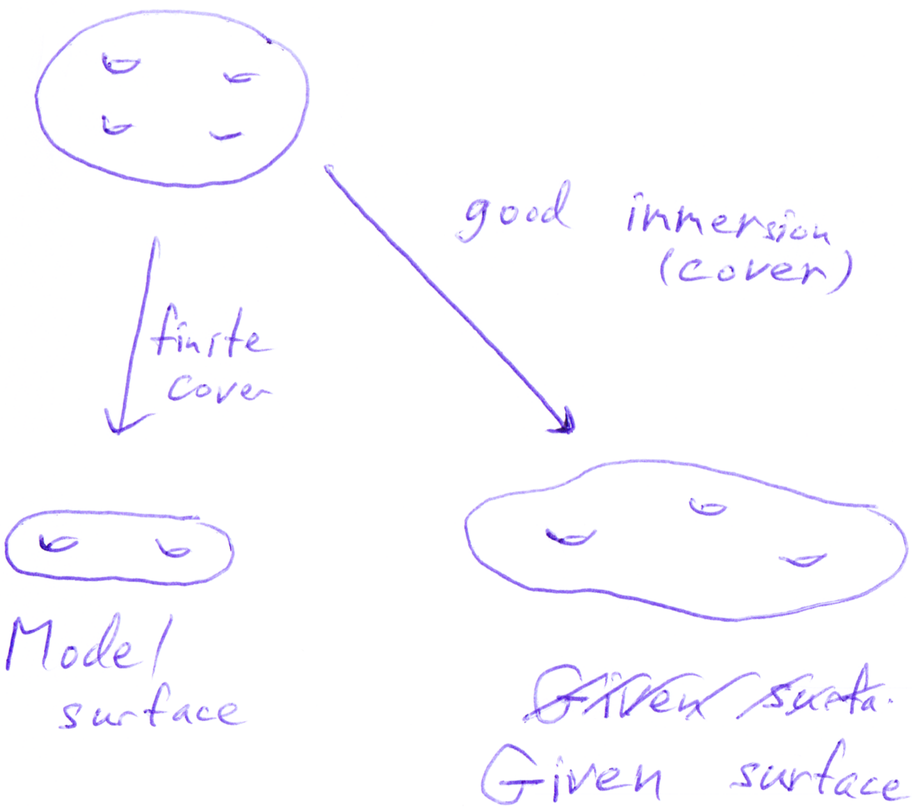
Theorem (KM) (Surface subgroup Conjecture)

Let M be a closed hyperbolic 3-manifold. Then $\pi_1(M)$ has a surface subgroup (the isomorphic image of π_1 of a hyperbolic surface).

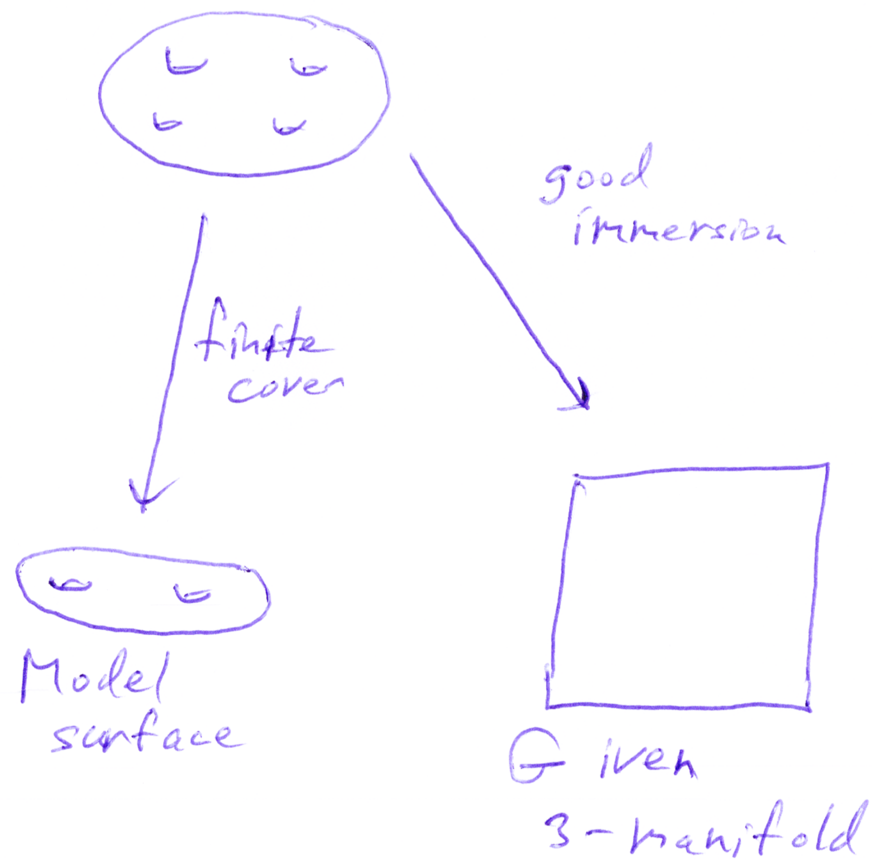
Theorem (KM) (Ehrenpreis conjecture)

Let S and T be closed hyperbolic Riemann surfaces. Then S and T have nearby finite covers (in the Teichmuller metric).

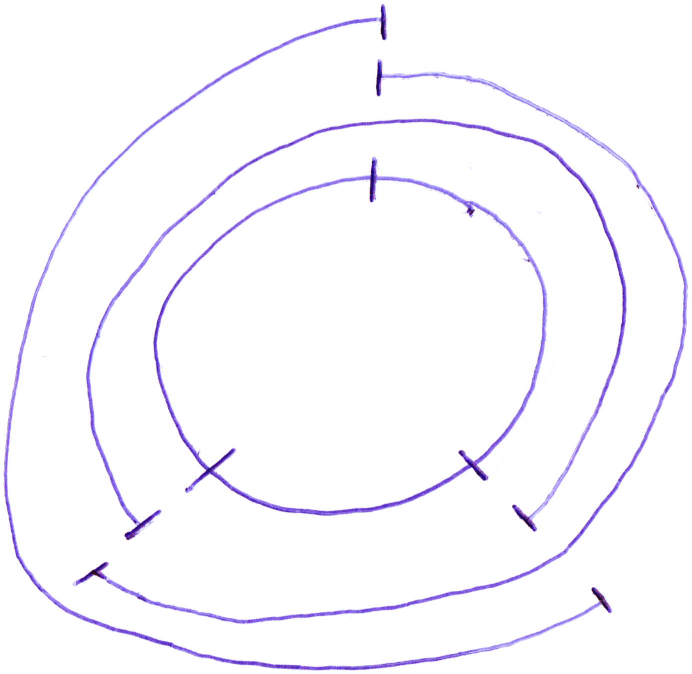
2D


good immersion (cover)
finite cover
Model surface
~~Given surfa.~~
Given surface

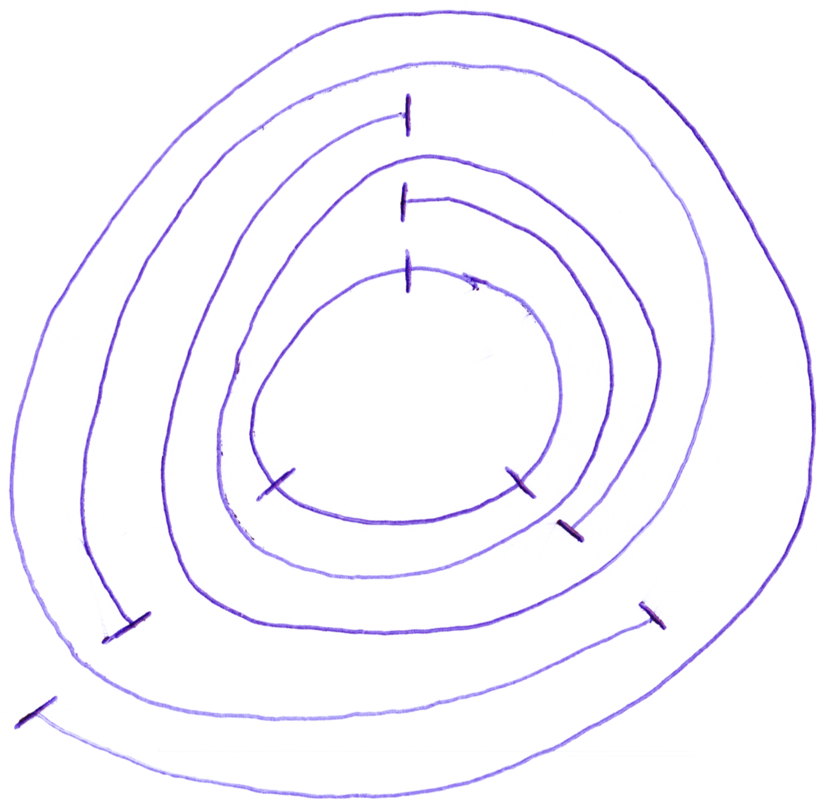
3D


good immersion
finite cover
Model surface
Given 3-manifold

Building Covers of the Circle



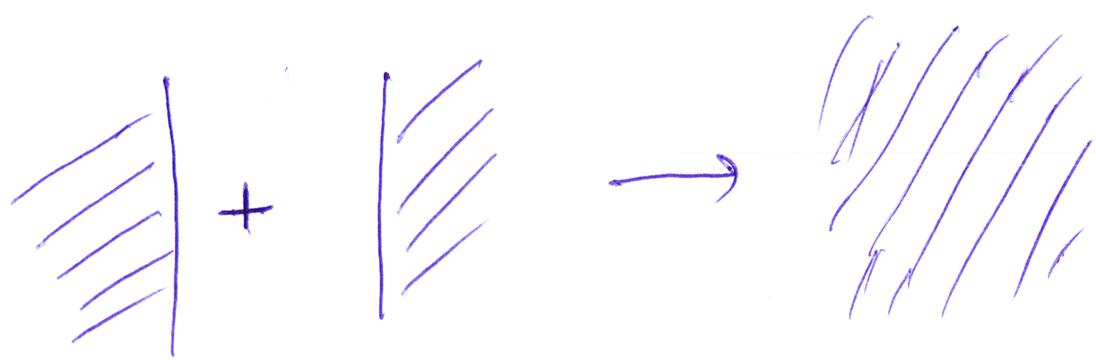
degree 2



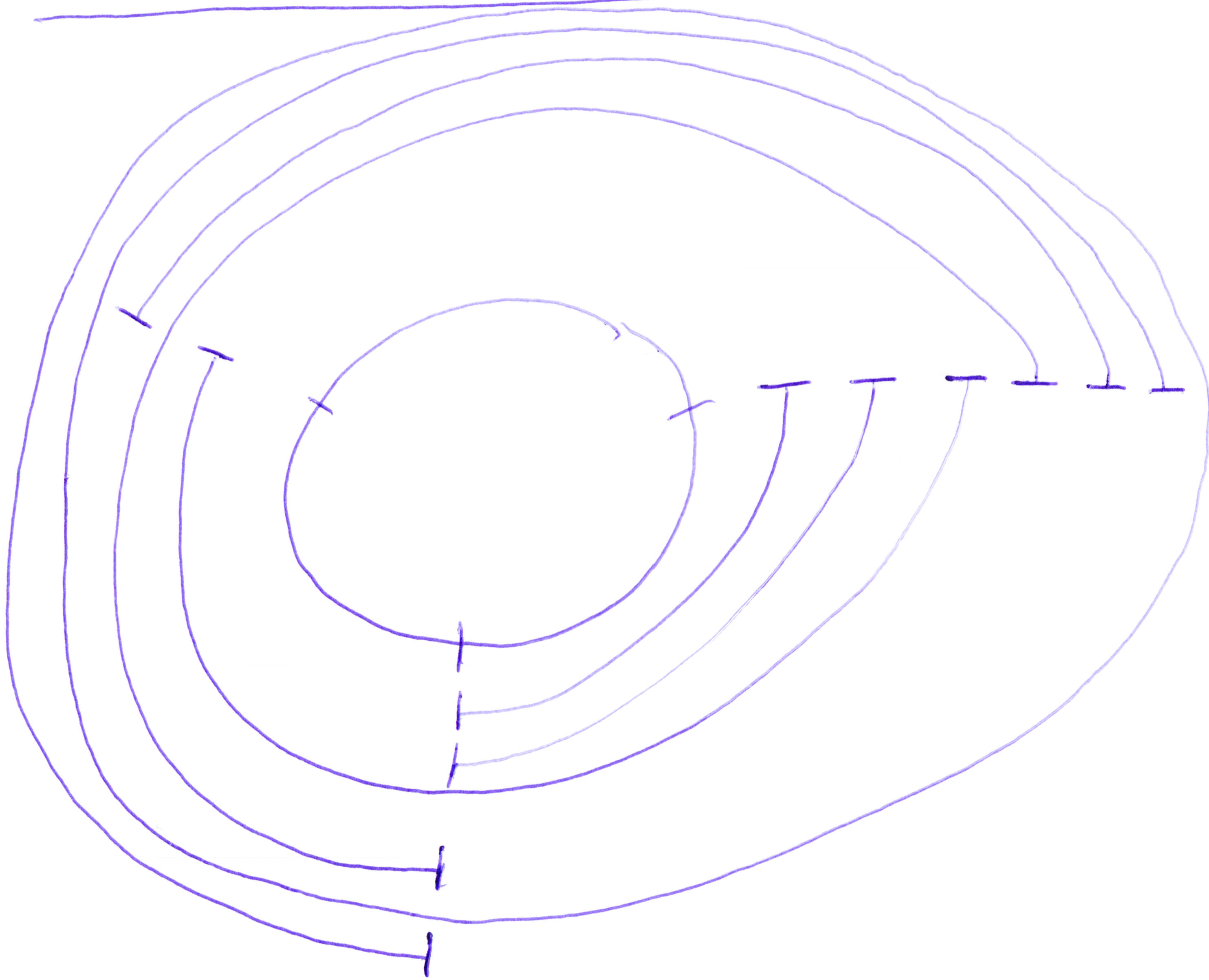
degree 4

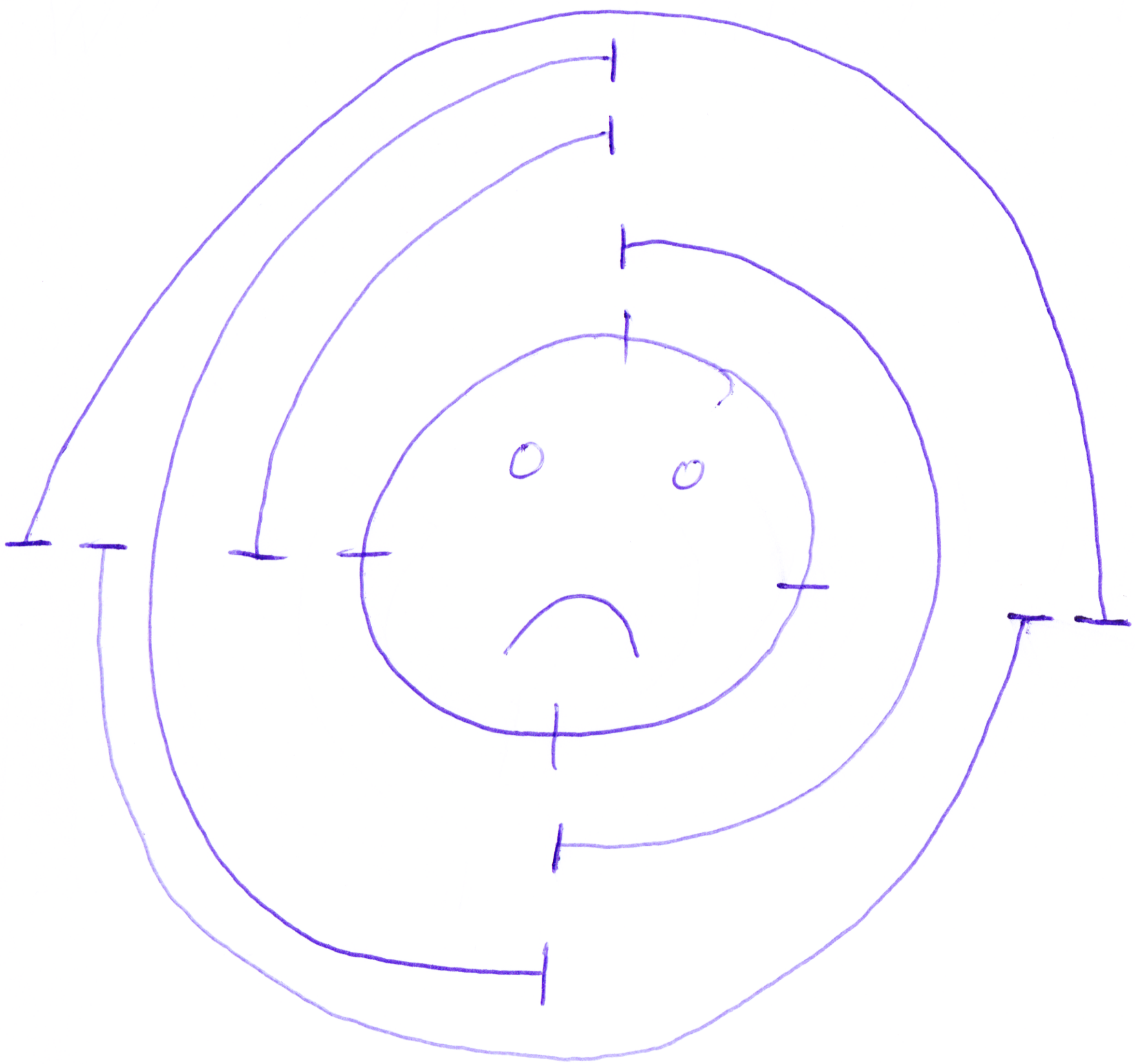
We can join immersed 1-manifolds with boundary to form a cover of a closed 1-manifold (a circle).

We can join immersed orientable n-manifolds with boundary to form a cover of a closed orientable n-manifold.



We can build more than one cover

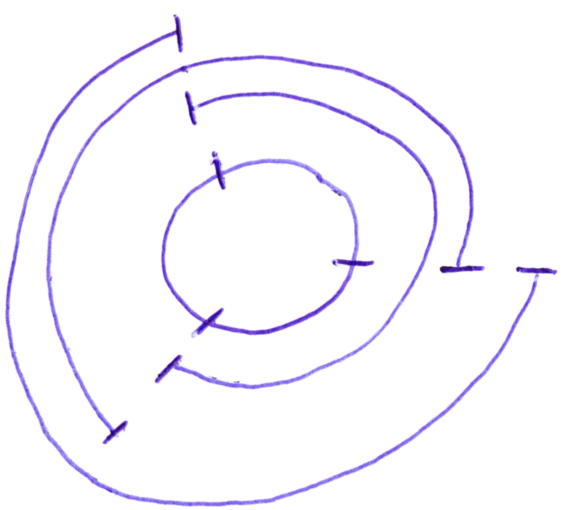




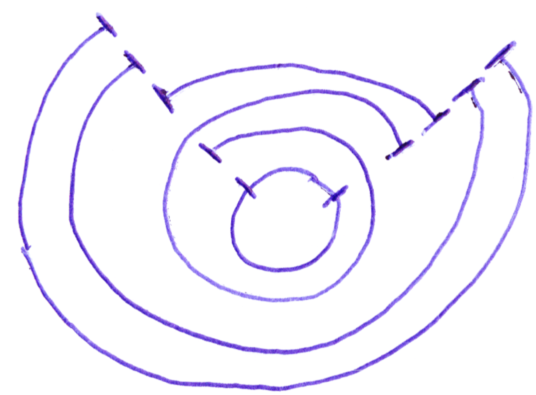
We can fail to build a cover

Given a collection $C = \sum n_i \alpha_i$ of 1-manifolds immersed in a circle, we can assemble C to form a cover if and only if

$$\partial C \equiv \sum n_i \partial \alpha_i = 0$$



$\partial C = 0$



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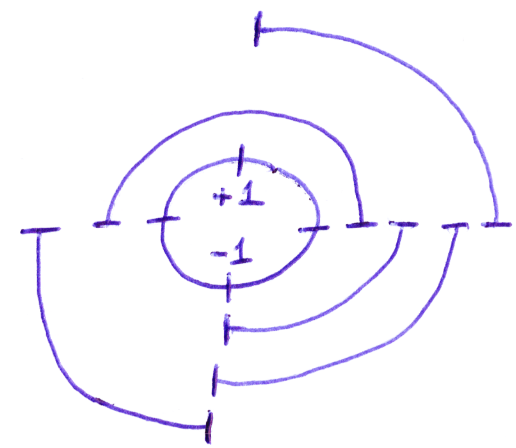
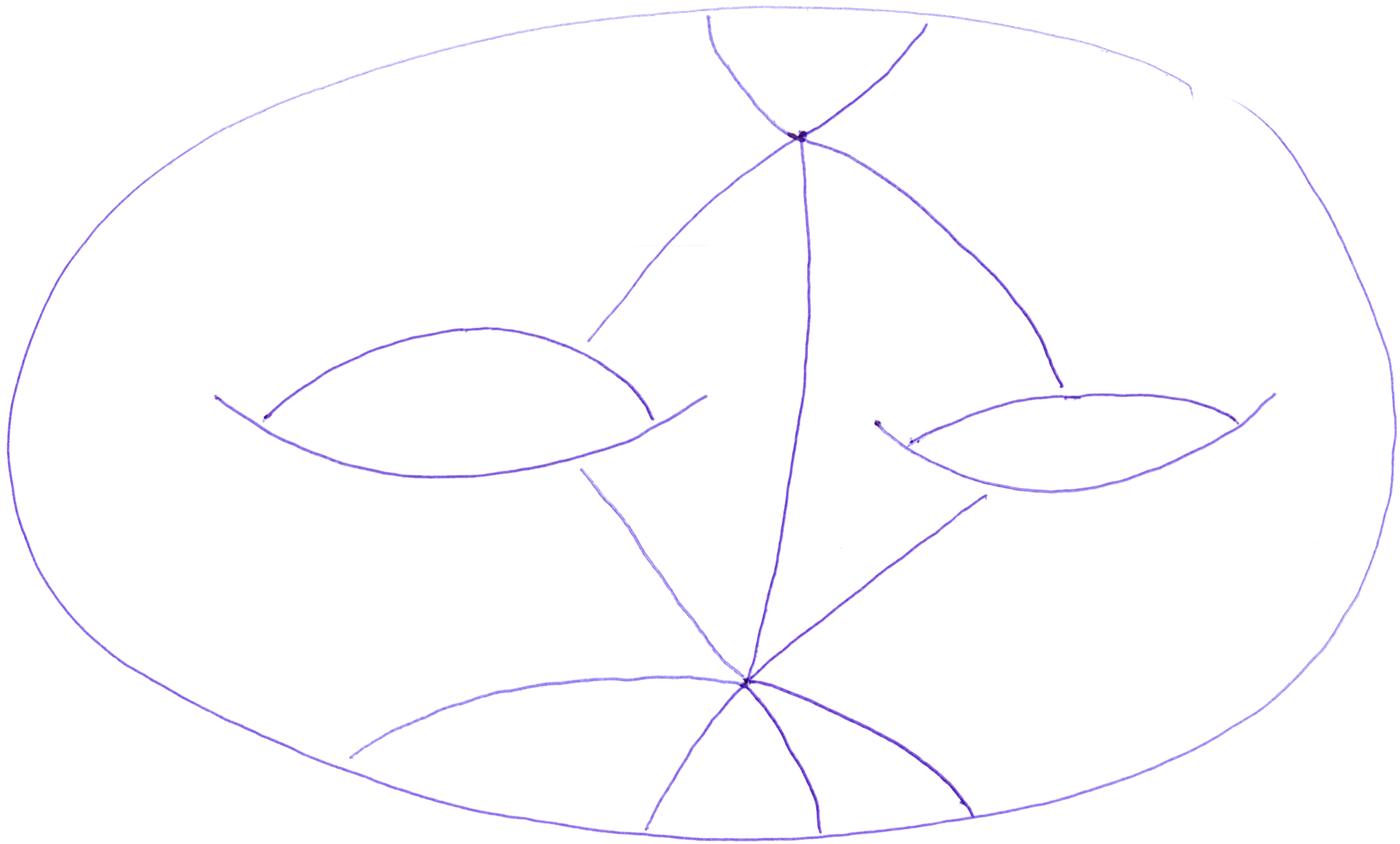
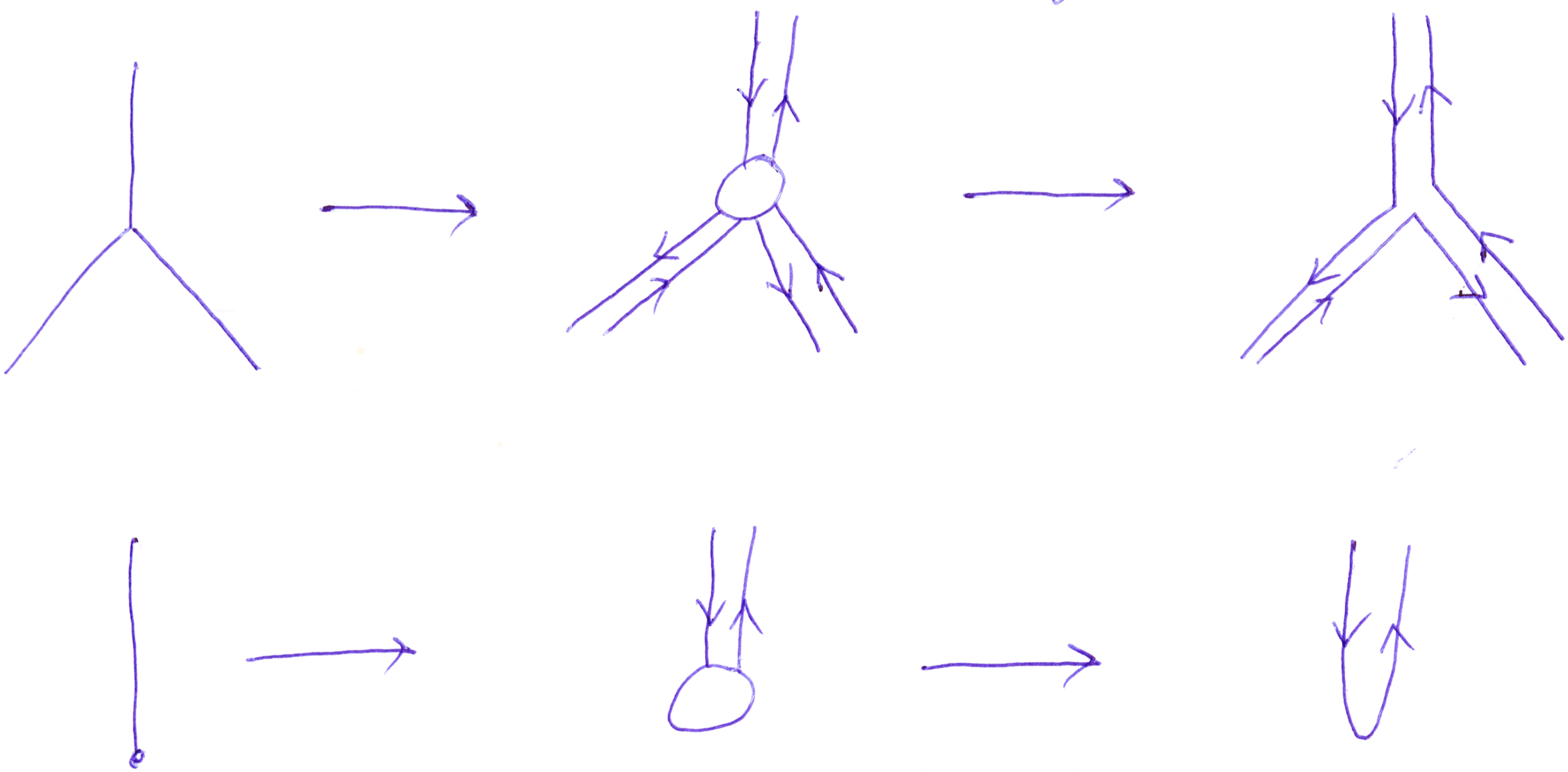
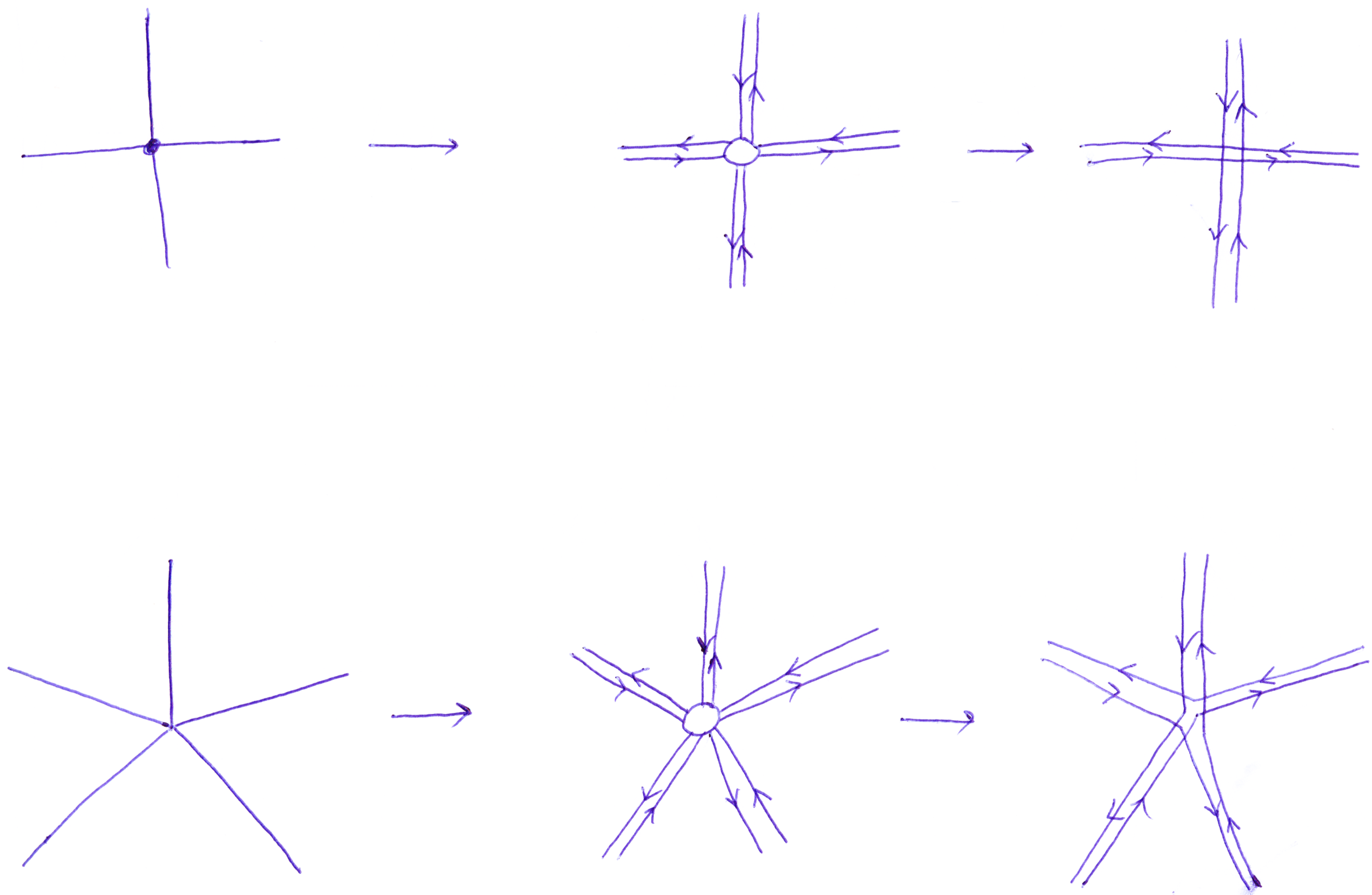


$\partial C \neq 0$

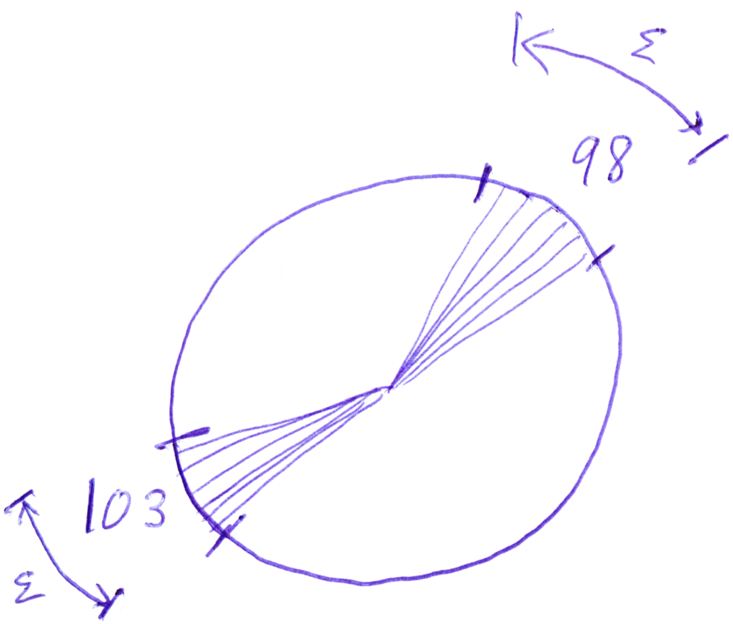


We can always build a piecewise immersed 1-manifold in a closed 2-manifold if we use the "doubling trick"



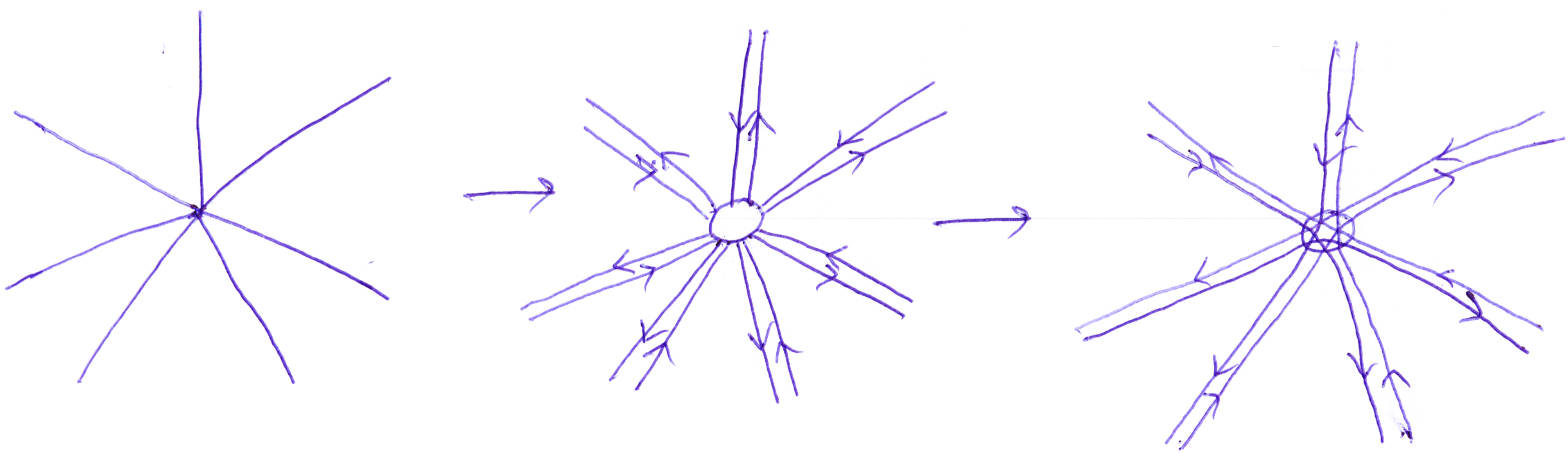


A set of ~~of~~ points on S^1 is "evenly distributed to the ~~scla~~ scale ε" if every ε-interval of S^1 has the same number of points, ~~to~~ up to a factor of $1+\varepsilon$.

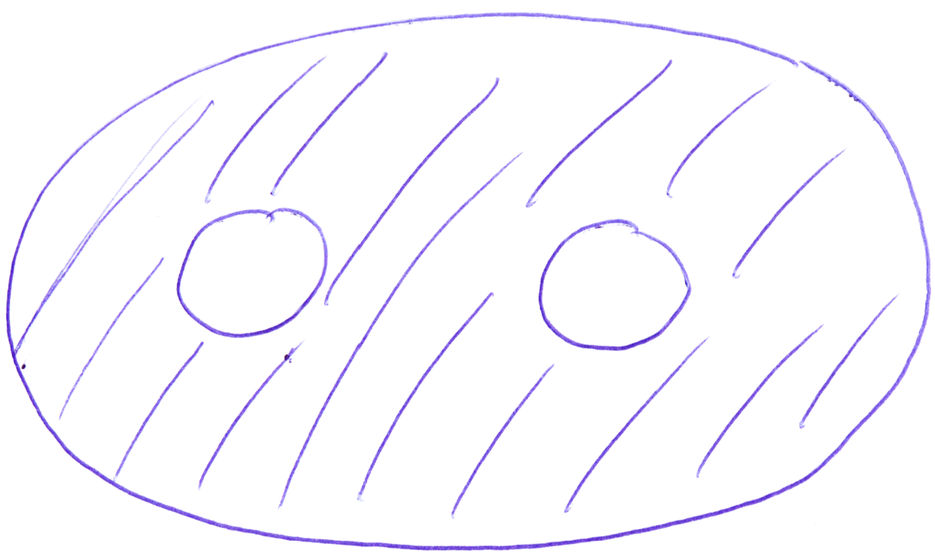


$$\frac{103}{98} < 1+\varepsilon$$

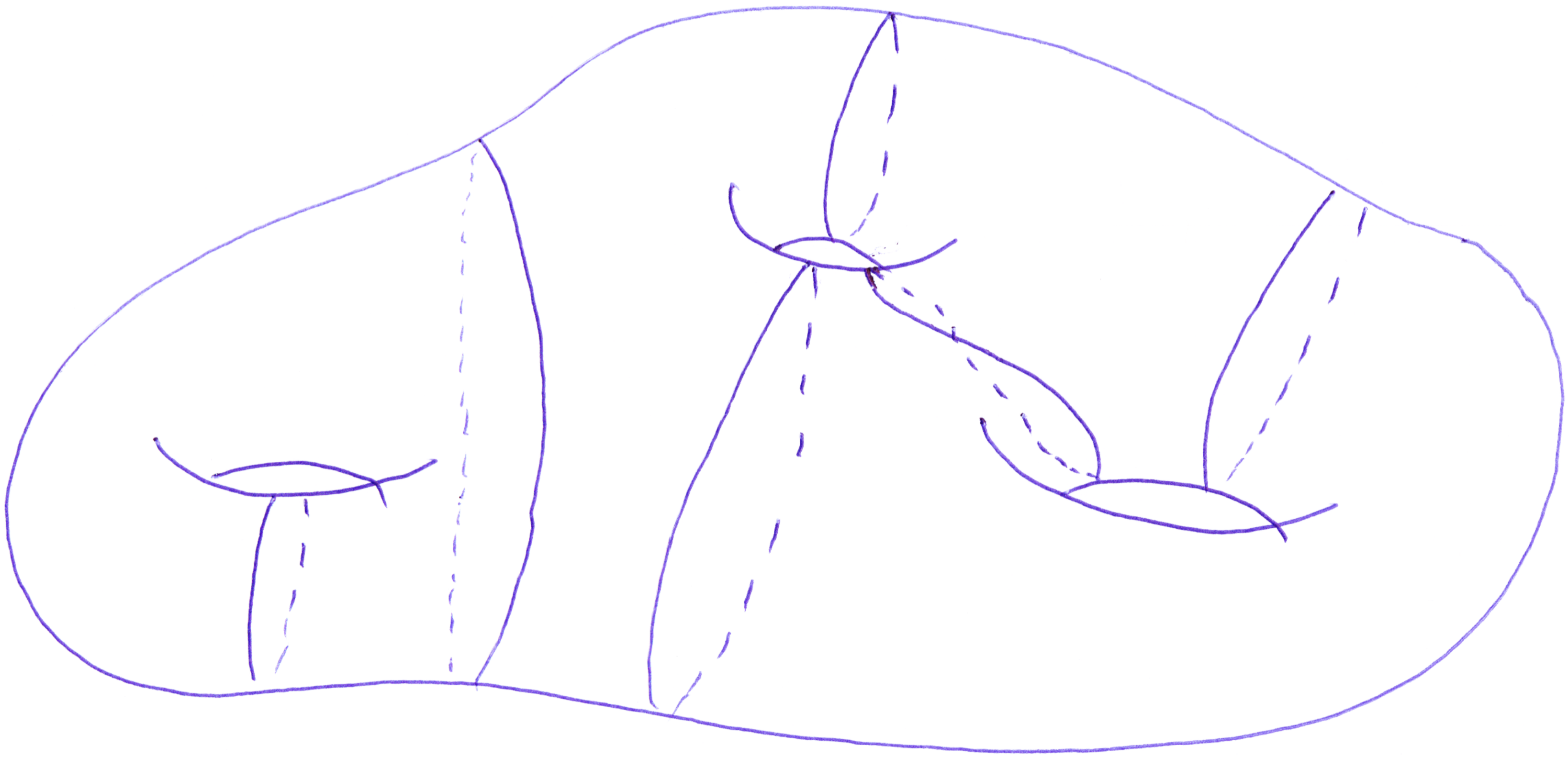
We can make the piecewise immersed 1-manifold nearly geodesic (small bending) if the segments are "evenly distributed" around each meeting ~~point:~~ point:



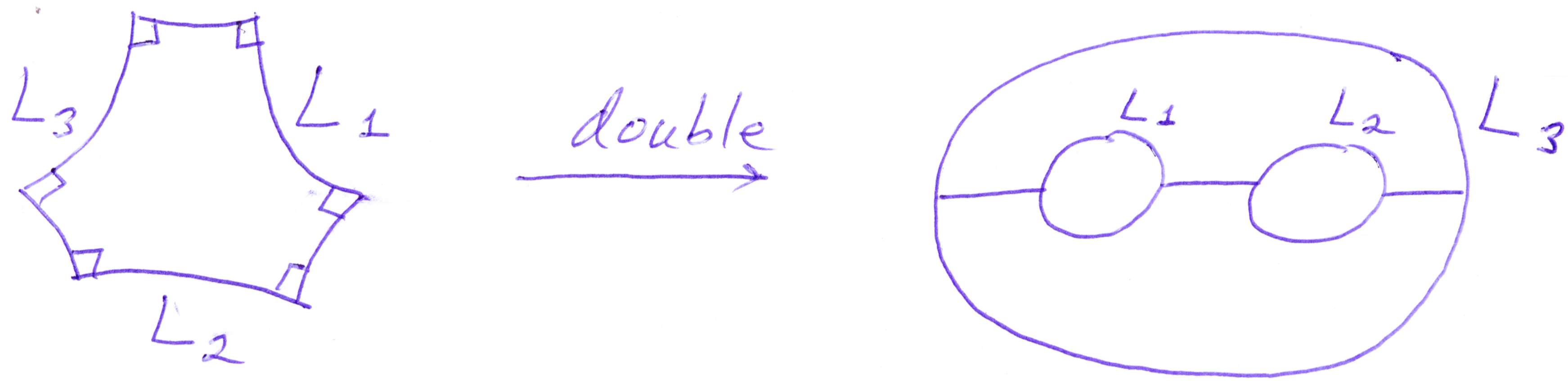
A (hyperbolic) pair of pants is a hyperbolic 2-manifold with geodesic boundary that is diffeomorphic to the following.



We obtain a closed hyperbolic surface when we join pairs of pants

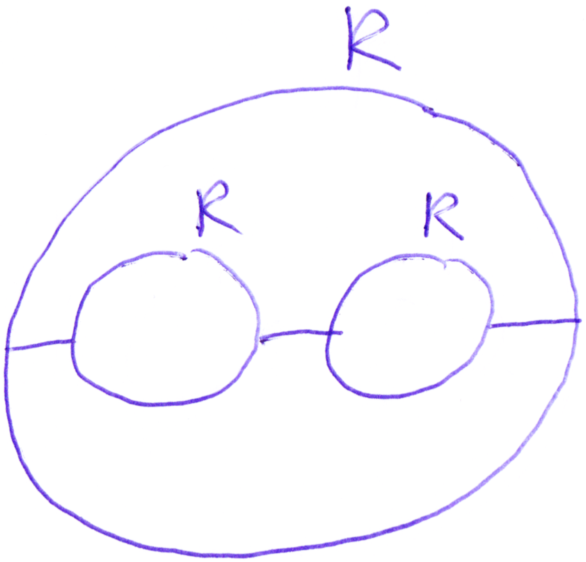


Hyperbolic Pairs of Pants

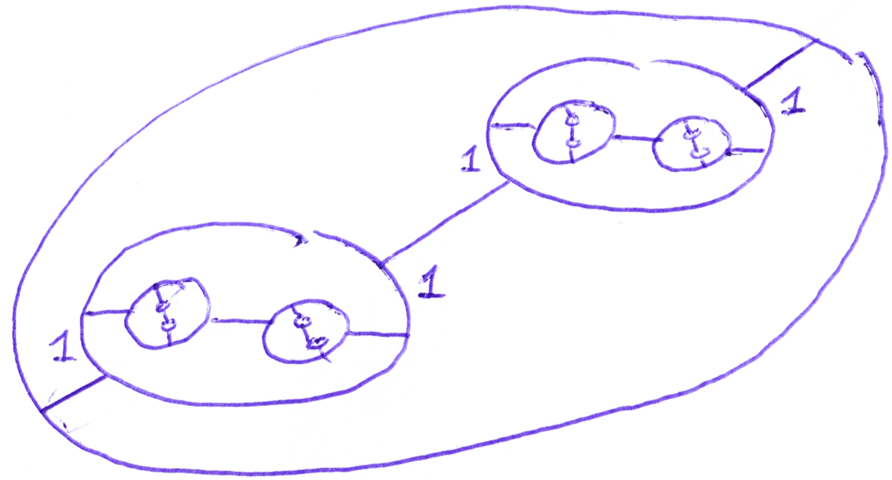


Half-lengths L_1, L_2, L_3

Perfect Pants and Perfect Surfaces



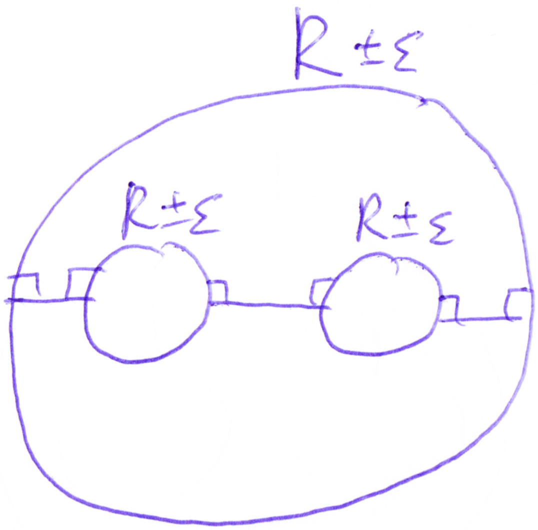
$hl(C) = R$



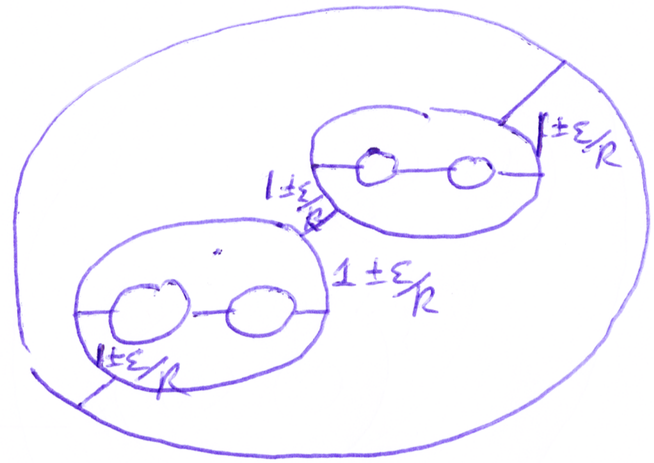
$hl(C) = R$
$s(C) = 1$

Theorem Any two closed ~~surfaces~~ perfect surfaces have a common finite cover.

Good Pants and Good Surfaces



$|\mathbf{hl}(c) - R| < \epsilon$



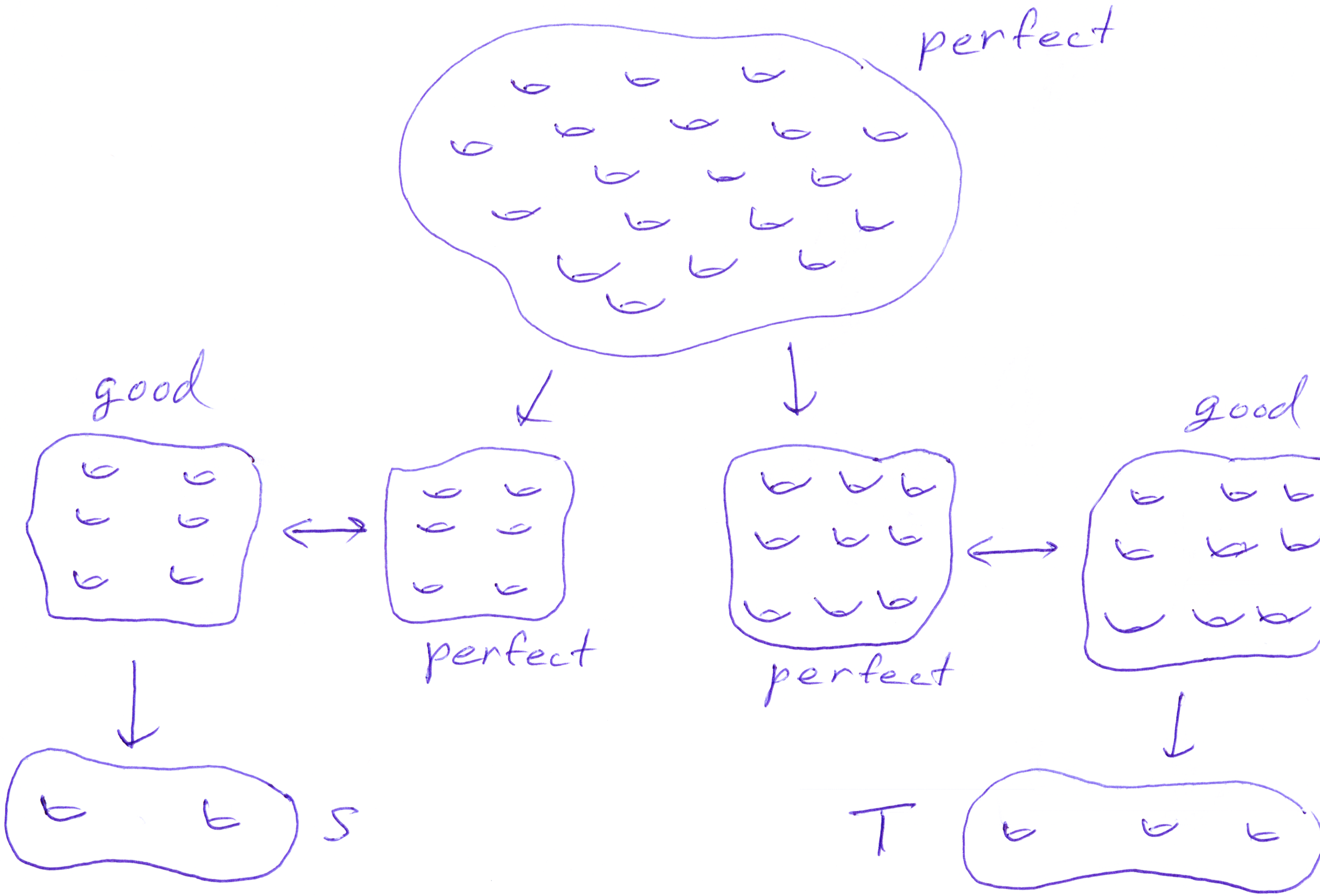
$|s(c) - 1| < \epsilon/R$

<u>Theorem</u> (KM) ~~An~~
A good closed panted surface is close to a perfect surface.

Theorem (KM)

For all $\varepsilon < \varepsilon_0$, and $R > R_0$, any (ε, R) good surface is $10^{12}\varepsilon$-close to an R-perfect surface in the Teichmuller metric.

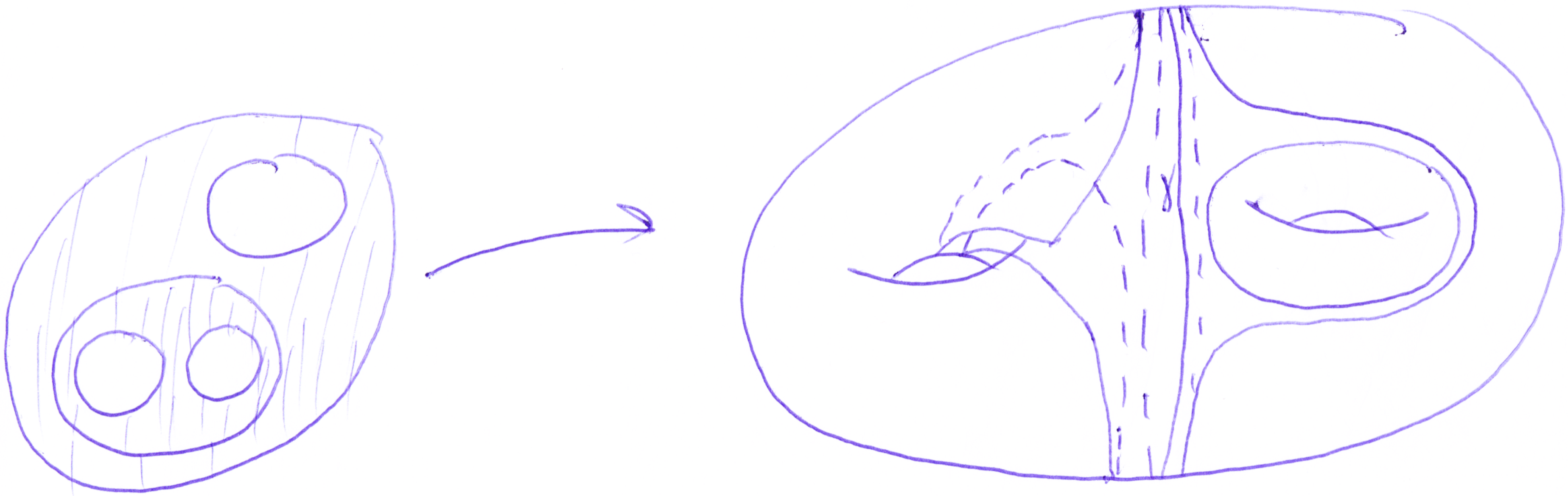
Good Covers $\Rightarrow$ Ehrenpreis



An Immersed Pair of Pants

We can think of the pants as being isometrically immersed, and speak of the goodness of pants.

Gluing Together Immersed Pants



We can ~~ind~~ ~~in~~ identify cuffs of the parametrizing pants whenever the cuffs map to the same ~~go~~ geodisic γ, and the pants are on opposite sides of γ.

Easy Theorem

Let Q be a finite set of immersed pants, such that for every closed geodesic γ on S we have the same number of pants in Q on both sides of γ. (That number can be zero). Then we can assemble Q to form a finite cover of S.

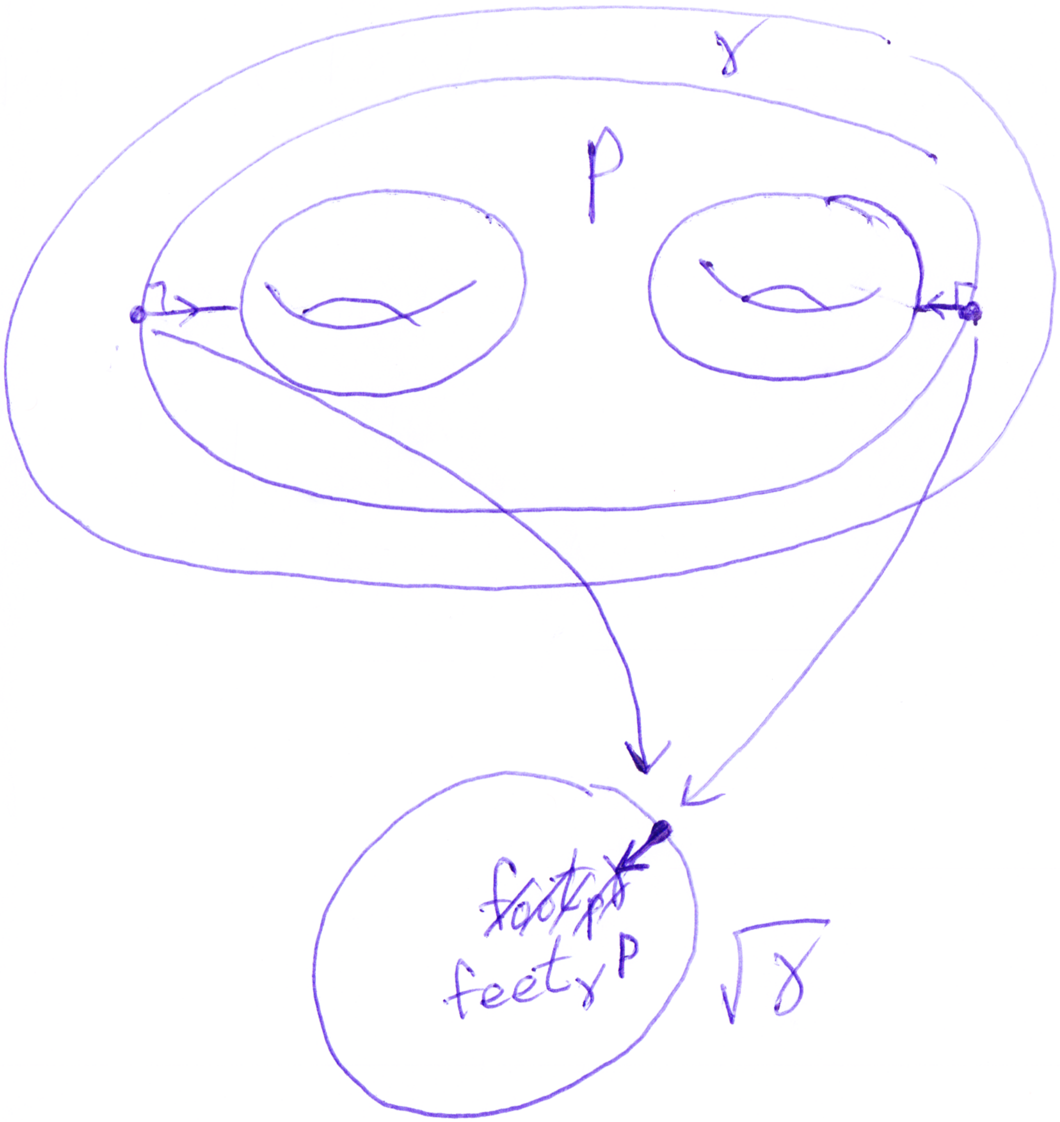
Good Pants and Good Curves

We let

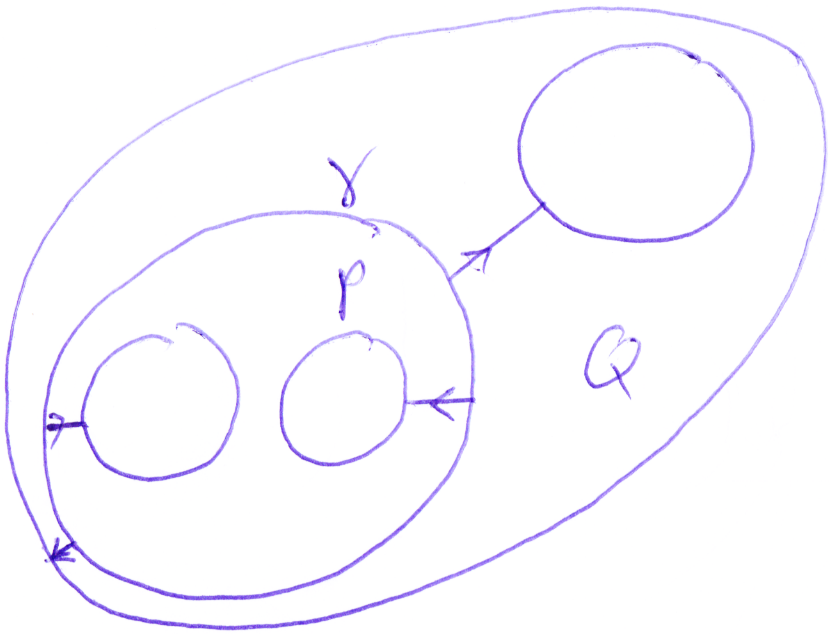
$$\Gamma_{\varepsilon,R} = \left\{ \gamma \;\middle|\; \begin{array}{l} \gamma \text{ is a closed geodesic on } S \\ |\mathbf{hl}(\gamma) - R| < \varepsilon \end{array} \right\}$$

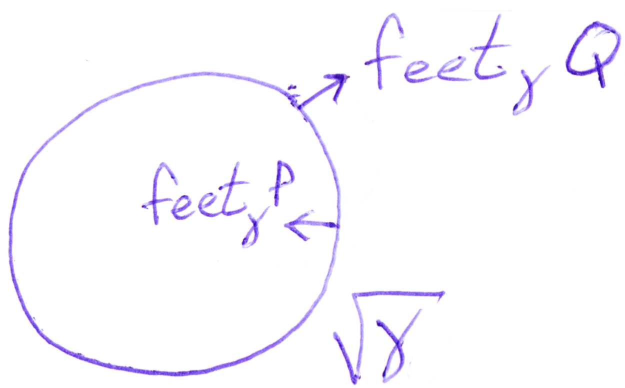
$$\Pi_{\varepsilon,R} = \left\{ P \;\middle|\; \begin{array}{l} P \text{ is an immersed pair} \\ \text{of pants in } S, \text{ and} \\ \partial P \subseteq \Gamma_{\varepsilon,R} \end{array} \right\}$$

The square root of a geodesic



Shear Coordinates and $N^4(\sqrt{\gamma})$





The Equidistribution Theorem

For every good curve γ, ~~and large R,~~

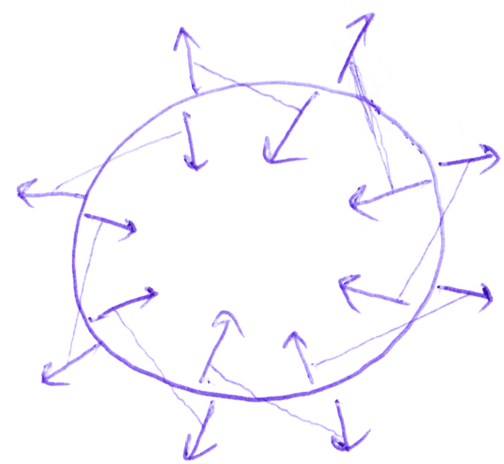
$$\{\text{feet}_\gamma P \mid P \in \Pi_{\varepsilon,R} \text{ and } \gamma \in \partial P\} \subseteq N^1(\sqrt{\gamma})$$

is evenly distributed on $N^1(\sqrt{\gamma})$ to the scale e^{-qR}, for $q = q(S, \varepsilon)$.

(We assume that $R > R_0(S, \varepsilon)$).

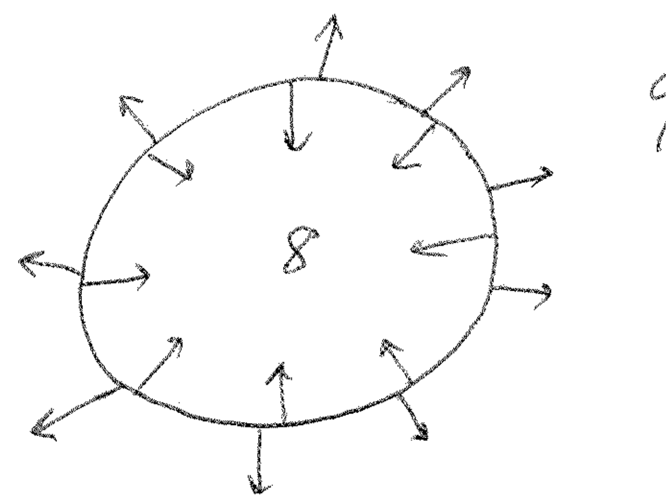
Equidistributed & Balanced $\Rightarrow$ Good Cover

If there were exactly the same number of pants on both sides of each geodesic γ, then we would be ~~all~~ able to assemble the pants in $\Pi_{\epsilon,R}$ to form a good cover of S:



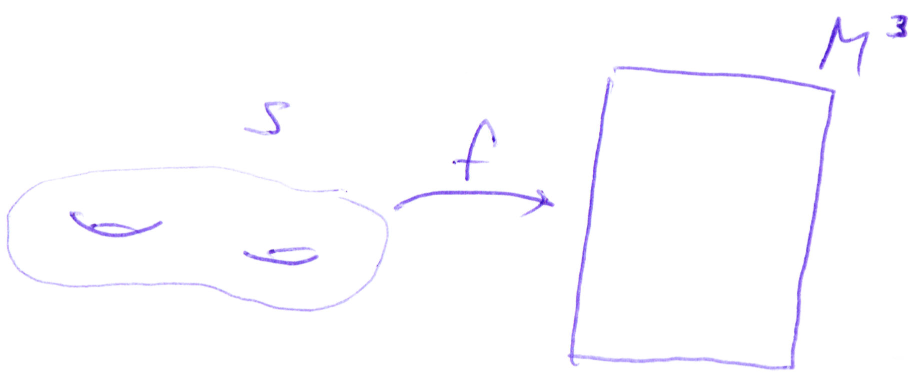
Equidistributed $\Rightarrow$ Nearly balanced

It follows from equidistribution that there are nearly the same number of pants in $\Pi_{\varepsilon,R}$ on either side of a good geodesic γ



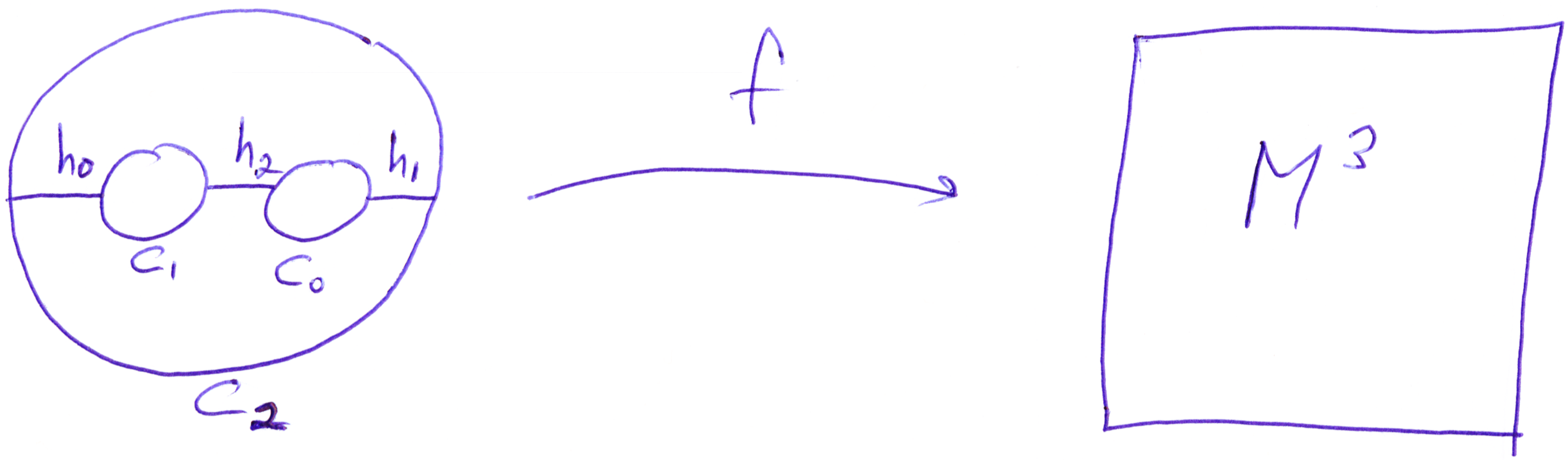
An Interlude in Three Dimensions

In a hyperbolic 3-manifold, $N^1(\gamma)$ (and $N^1(\sqrt{\gamma})$) is connected, so we can use the "doubling trick" to build a nearly geodesic immersed surface (which is therefore essential).



$\pi_1(f): \pi_1(S) \to \pi_1(M^3)$
injective

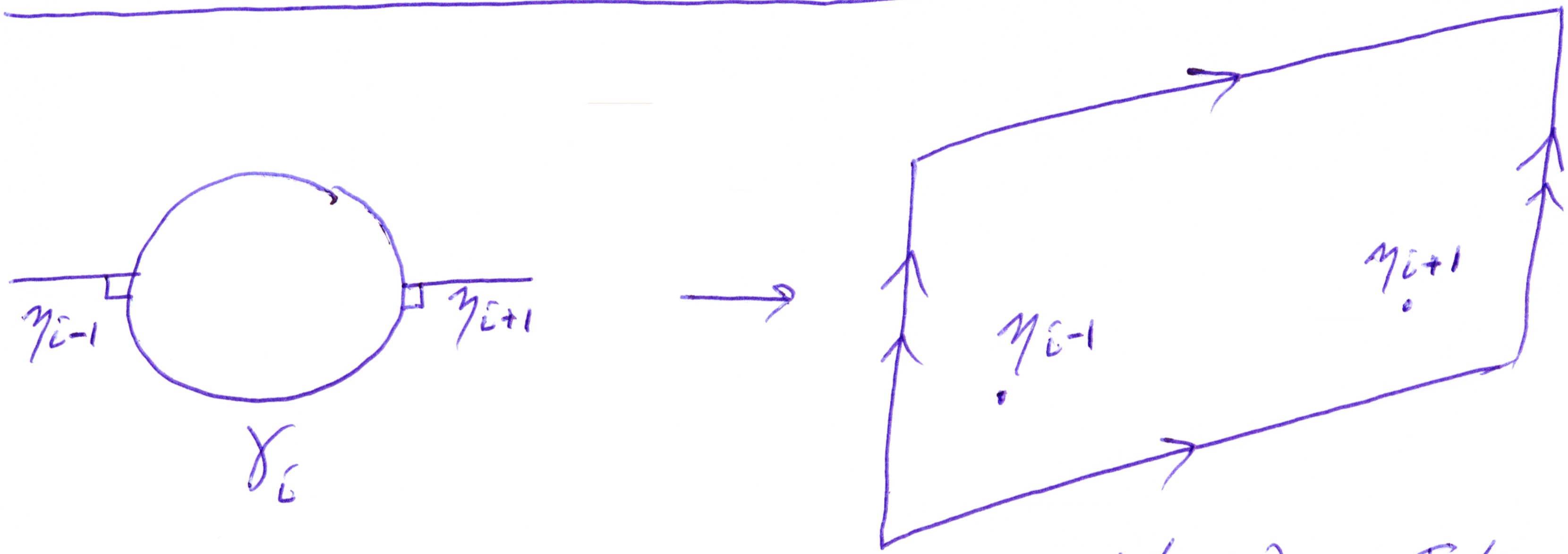
A skew pair of pants



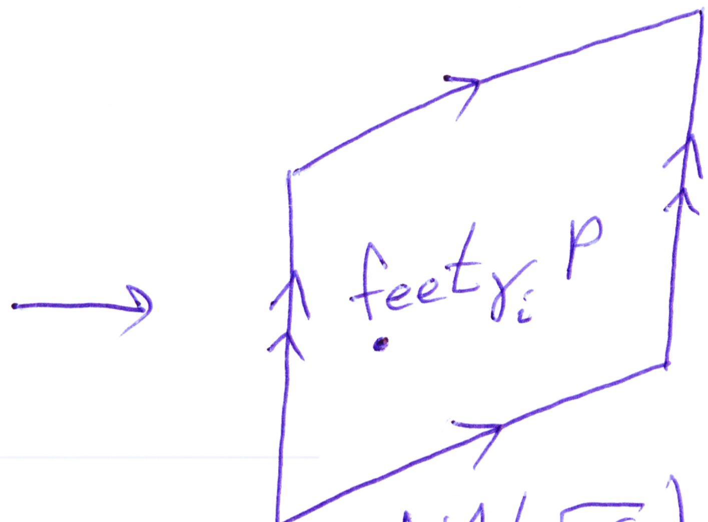
$\gamma_i \equiv f(c_i)$ is a closed geodesic

$\eta_i \equiv f(h_i)$ is a geodesic segment orthogonal to $f(\gamma_{i\pm 1})$.

The feet of a skew pair of pants



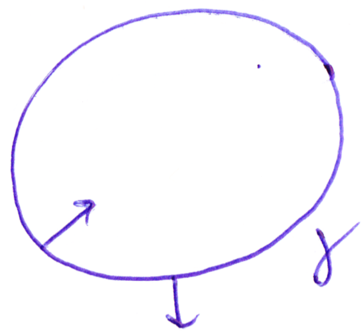
$$N^1(\gamma_i) \cong \mathbb{C}/2\pi i\mathbb{Z} + hl(\gamma)\mathbb{Z}$$



$N^1(\sqrt{\gamma_i})$

Two and Three Dimensions

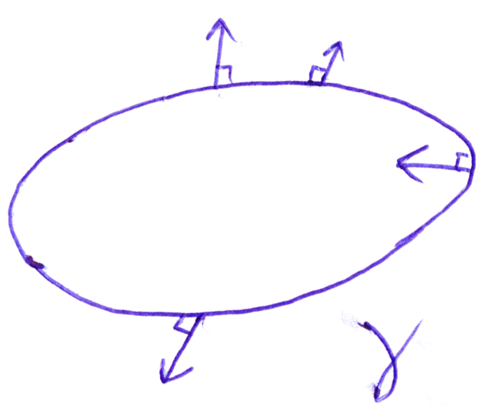
2D



$$N^1(\gamma) \cong \mathbb{R}/\ell(\gamma)\cdot\mathbb{Z} \times \{-1,+1\}$$

$$\cong \mathbb{R}^*/\langle x \mapsto e^{\ell(\gamma)}x \rangle$$

3D



$$N^1(\gamma) \cong \mathbb{C}/\ell(\gamma)\cdot\mathbb{Z} + 2\pi i\,\mathbb{Z}$$

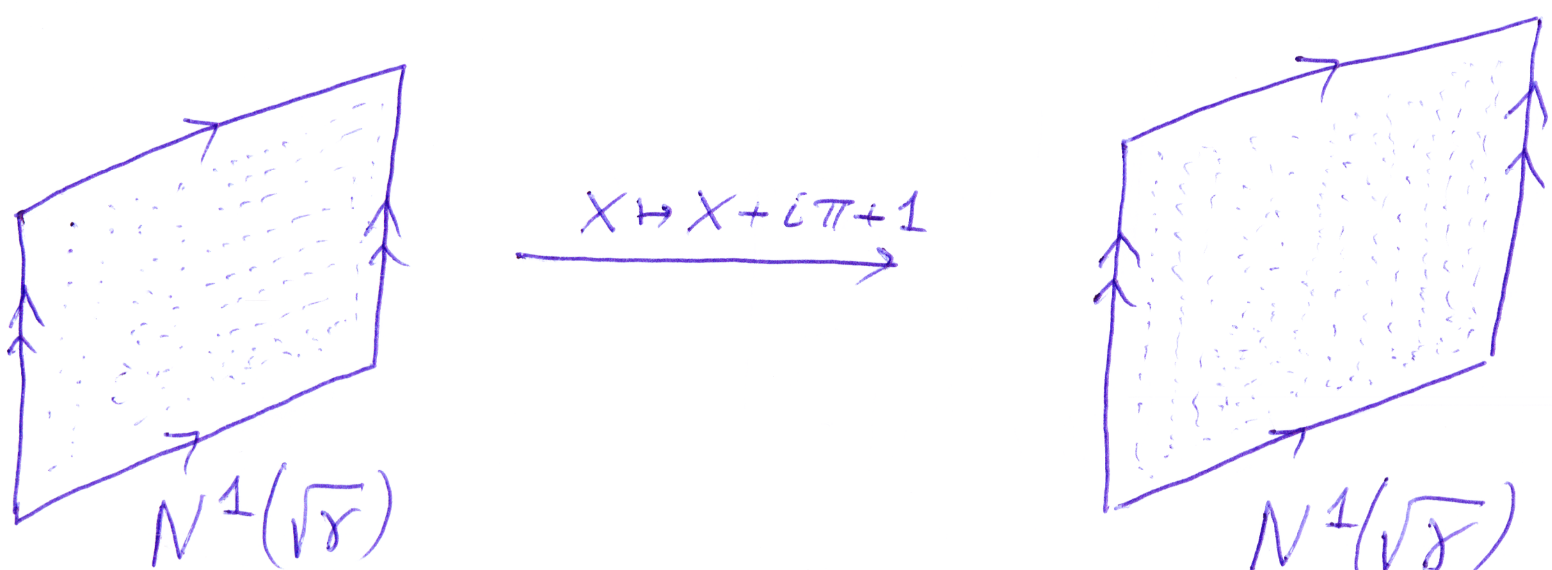
$$\cong \mathbb{C}^*/\langle z \mapsto e^{\ell(\gamma)}z \rangle$$

The meaning of equidistribution

The feet $\left\{ \text{feet}_\gamma P : \begin{array}{l} \gamma \in \partial P \\ P \in \Pi_{\epsilon,R} \end{array} \right\}$

are e^{-qR} - evenly distributed

as points on $N^1(\sqrt{\gamma})$



The "doubling trick" revisited

We take

$$A_\gamma = \{ \mathrm{foot}_\gamma P : \gamma \in \partial P, \ P \in \Pi_{\varepsilon, R} \}$$

We can find

$\sigma : A_\gamma \to A_\gamma$ a permutation

such that $|\sigma(x) - x - i\pi - 1| < \varepsilon/R$
(for all $x \in A_\gamma$)

and then define

$$\begin{tikzcd}
\tau : A_\gamma^+ \arrow[rrrrrr, bend right, "\sigma"'] & \sqcup & A_\gamma^- \arrow[rr, bend right, "\sigma^{-1}"'] & \longrightarrow & A_\gamma^+ & \sqcup & A_\gamma^-
\end{tikzcd}$$

The Idea of Self-Correction

We find $g : \mathbb{Q}\Gamma_{\varepsilon,R} \to \mathbb{Q}\Pi_{\varepsilon,R}$ such that

1. $\partial g(\alpha) = \alpha$ whenever $\alpha = \partial \pi$

2. $\|g(\alpha)\|_\infty \leq P(R) e^{-R} \|\alpha\|_\infty$ for all α.

(where $P(R) = CR^N$ is a polynomial in R)

Counting Good Pants and Good Curves

$$\# \Pi_{\varepsilon, R} \asymp e^{3R}$$

$$\# \Gamma_{\varepsilon, R} \asymp e^{2R}/R$$

$$\# \{ P \in \Pi_{\varepsilon, R} \mid \gamma \in \partial P \} \asymp e^{R}/R$$

for each $\gamma \in \Gamma_{\varepsilon, R}$

Then let $\Pi = \sum_{P \in \Pi_{\varepsilon, R}} P$, and $\alpha = \partial \Pi$.

Then $\|\alpha\|_\infty \leq e^{(1-q)R}$ by equidistribution

So $\|g(\alpha)\|_\infty \leq P(R) e^{-qR} << 1$.

So $\Pi - g(\alpha)$ is positive and equidistributed

And $\partial(\Pi - g(\alpha)) = \partial \Pi - \partial g(\partial \Pi) = \partial \Pi - \partial \Pi = 0$.

After clearing denominators, we can assemble the pants of $\Pi - g(\alpha)$ to form a good cover!

The Relative Surface Subgroup Theorem

Let M be a closed hyperbolic 3-manifold, and let γ be a closed geodesic in M of length $l(\gamma) \geq R_0(M, \varepsilon)$. (and suppose γ bounds a singular chain).
Then we can find a compact hyperbolic surface S with boundary and $f: S \to M$ that is ε-close to being an isometric immersion, and for which $f(\eta) = \gamma$ for every component η of ∂S.

The Good Pants Homology

We let $H_1^{\varepsilon,R}(S;\mathbb{Q}) = \mathbb{Q}\Gamma_{\varepsilon,R} / \partial\mathbb{Q}\Pi_{\varepsilon,R}$.

We find a series of identities leading to $H_1^{\varepsilon,R}(S;\mathbb{Q}) = H_1(S;\mathbb{Q})$.

The Good Pants Homology

We let $H_1^{\varepsilon,R}(S;\mathbb{Q}) = \mathbb{Q}\Gamma_{\varepsilon,R} / \partial\mathbb{Q}\Pi_{\varepsilon,R}$.

We find a series of identities leading to $H_1^{\varepsilon,R}(S;\mathbb{Q}) = H_1(S;\mathbb{Q})$.

Then we observe that $g: \mathbb{Q}\Gamma_{\varepsilon,R} \to \mathbb{Q}\Pi_{\varepsilon,R}$ has been implicitly defined, such that $\partial g(\alpha) = \alpha$ when $\alpha = 0$ in H_1.

The Algebraic Square Lemma

Under conditions of reasonable geometry,

$$\sum_{i,j=0,1} (-1)^{i+j} [A_i U B_j V] \equiv 0$$

in $H_1^{\varepsilon,R}$

(where $[X]$ denotes the closed geodesic freely homotopic to X, for $X \in \pi_1(S,*)$).

We then define

$$A_T = \frac{1}{2}\left([T A \bar{T} B] - [T \bar{A} T B]\right)$$

(which is independent of the choice of B in $H_1^{\varepsilon,R}$)

and prove

$$(XY)_T = X_T + Y_T$$

in $H_1^{\varepsilon,R}$.