Kan Extensions

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Given sets $A \subsetneq B$, a nonempty set C, and a set map $F\colon A \to C$, there are many ways to extend F to a function on the larger domain B, but none of them are "canonical." Contrast this with the following example.

Example 1. The base-2 exponential function $2^{-}\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ has the nice properties that $2^{m+n} = 2^m 2^n$ and $2^0 = 1$. More formally, 2^{-} is a monoid homomorphism from the additive monoid $\mathbb{Z}$ to the multiplicative monoid $\mathbb{R}_{\geq 0}$. We are interested in extending this function to the domain $\mathbb{Z}$ in such a way that those properties are preserved; the natural way to do this is to send $-m \mapsto \frac{1}{2^m}$ for all $m \in \mathbb{N}$. Similarly, to extend further to $\mathbb{Q}$, we define $2^{\frac{1}{q}} = \sqrt[q]{2}$.

The idea is to "categorify" this notion of extending a function canonically. The result is a Kan extension, which turns out to generalize other categorical notions like limits, adjoints, and universal properties. Before we properly define what a Kan extension is, however, let us first go over some preliminary notions.

Definition 2. Given categories A, B, and C and functors S and T—the "source" and "target", respectively—such that

$$A \xrightarrow{S} C \xleftarrow{T} B \tag{1}$$

we define the *comma category* $S \downarrow T$ as follows: the objects consist in ordered triples (a, b, h) such that $a \in A$, $b \in B$, and $h\colon Sa \to Tb$ is a morphism in C; and the morphisms $(a, b, h) \to (a', b', h')$ consist in ordered pairs (f, g) such that $f\colon a \to a'$, $g\colon b \to b'$, and the diagram

$$\begin{tikzcd} Sa \arrow[r, "Sf"] \arrow[d, "h"] & Sa' \arrow[d, "h'"] \\ Tb \arrow[r, "Tg"] & Tb' \end{tikzcd} \tag{2}$$

in C commutes; that is, $h' \circ Sf = Tg \circ h$.

Example 3. Suppose we let $C = A$ and let $S = \mathrm{id}_A$; let $B = \mathbb{1}$, the category with one object, 0, and one morphism, and let a_0 be the unique object in the image of $T\colon B \to A$. Then the comma category $S \downarrow T$ takes on the more familiar form of the slice category $A \downarrow a_0$. We may simplify the commuting diagram as follows:

$$\begin{tikzcd} \mathrm{id}_A a \arrow[r, "\mathrm{id}_A f"] \arrow[d, "h"] & \mathrm{id}_A a' \arrow[d, "h'"] \\ T0 \arrow[r, "T\,\mathrm{id}_0"] & T0 \end{tikzcd} \quad \text{becomes} \quad \begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "h"'] & & a' \arrow[dl, "h'"] \\ & a_0 & \end{tikzcd} \tag{3}$$

In comma categories where one functor has domain $\mathbb{1}$, objects are typically written as ordered pairs (omitting the 0 object), and morphisms are typically written as singletons (omitting the id_0 map).

The next example of a comma category is necessary in stating the definition of a Kan extension.

Example 4. Consider the following diagrams in the 2-category of categories:[1]

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\beta"] & C \\ & B \arrow[ur, "L"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\gamma"] & C \\ & B \arrow[ur, "M"'] & \end{tikzcd} \tag{4}$$

These are not commutative diagrams; the right diagram, for example, says that F and MK are both functors $A \to C$ with a natural transformation $\gamma\colon F \Rightarrow MK$. (Likewise for the left diagram.) Recall that the natural transformation γ consists in a morphism $\gamma a\colon Fa \to MKa$ in C for each object a in A such that, for all $f\colon a \to a'$ in A, the following diagram commutes:

$$\begin{tikzcd} Fa \arrow[r, "\gamma a"] \arrow[d, "Ff"] & MKa \arrow[d, "MKf"] \\ Fa' \arrow[r, "\gamma a'"] & MKa' \end{tikzcd} \tag{5}$$

More notation: C^A is the *functor category* with objects as functors from A to C and morphisms as natural transformations. Then F is in C^A, M is in C^B, and K is in B^A. Then one can verify that pre-composition by K, denoted by $K^*\colon C^B \to C^A$, is in fact a functor of categories. So it makes sense to construct the comma category $F \downarrow K^*$ where F is, by abuse of notation, regarded as the functor $\mathbb{1} \to C^A$ that maps 0 to the object F in C^A; if we have a natural transformation $\alpha : L \Rightarrow M$ such that $\gamma = \alpha K \circ \beta$, then we may construct (and subsequently simplify) the corresponding commutative diagram in C^A:

$$\begin{tikzcd} F0 \arrow[r, Rightarrow, "F\,\mathrm{id}_0"] \arrow[d, Rightarrow, "\beta"] & F0 \arrow[d, Rightarrow, "\gamma"] \\ K^*L \arrow[r, Rightarrow, "K^*\alpha"] & K^*M \end{tikzcd} \quad \text{becomes} \quad \begin{tikzcd} & F \arrow[dl, Rightarrow, "\beta"'] \arrow[dr, Rightarrow, "\gamma"] & \\ LK \arrow[rr, Rightarrow, "\alpha K"] & & MK \end{tikzcd} \tag{6}$$

Since each natural transformation consists in a morphism for each object a in A, we may rewrite the diagram to the right as a diagram in C for each a:

$$\begin{tikzcd} & Fa \arrow[dl, "\beta a"'] \arrow[dr, "\gamma a"] & \\ LKa \arrow[rr, "\alpha K a"] & & MKa \end{tikzcd} \tag{7}$$

Yet another way to write the same data is as a *pasting diagram*:

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\gamma"] & C \\ & B \arrow[ur, "M"'] & \end{tikzcd} = \begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\beta"] & C \\ & B \arrow[ur, "L"] \arrow[ur, bend right=60, "M"'] \arrow[ur, phantom, bend right=30, "\Downarrow\alpha"] & \end{tikzcd} \tag{8}$$

[1] It would be too much of a digression to define here what precisely a 2-category is. Suffice it to say that it is like a category except it has "morphisms between morphisms"; in the present case, where the objects are categories and the morphisms are functors, the *2-morphisms* are natural transformations. Note that this example and the subsequent definition of a Kan extension can be generalized to any 2-category in the natural way.

Example 5. A description of the comma category $K^* \downarrow F$ (cf. $F \downarrow K^*$ above) is obtained by reversing the direction of all natural transformations (and all arrows in (7)) in the previous example.

We are now ready to define the two (dual) notions of the Kan extension. The previous examples allow us to state them succinctly:

Definition 6. Given a diagram of categories

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "K"'] & & C \\ & B & \end{tikzcd} \tag{9}$$

the *left Kan extension* of F along K, denoted by $\mathrm{Lan}_K F$, is the initial object in the comma category $F \downarrow K^*$. The *right Kan extension* of F along K, denoted by $\mathrm{Ran}_K F$, is the final object in the comma category $K^* \downarrow F$.

To be more explicit, we may regard a left Kan extension as an ordered pair $(L\colon B \to C, \beta\colon F \Rightarrow LK)$ such that for any pair $(M\colon B \to C, \gamma\colon F \Rightarrow MK)$, γ factors uniquely through β; i.e., there exists a unique natural transformation $\alpha\colon L \Rightarrow M$ such that $\gamma = \alpha K \circ \beta$ (represented by the pasting diagram (8)). The dual of this statement applies to right Kan extensions.

Note that neither extension is guaranteed to exist. The following theorem outlines a sufficient condition to guarantee existence of a left Kan extension. (Again, a dual theorem can be stated.) Recall that a category A is *small* if the collection of all objects and morphisms of A is a set. Moreover, a category C is *cocomplete* if it contains all small colimits (i.e., all small diagrams in C have a colimit in C).

Theorem 7. *Given the diagram of categories in (9), if A is small and C is cocomplete, then* $\mathrm{Lan}_K F$ *exists and is defined by* $b \mapsto \operatorname{colim} FU_b$, *where* $U_b\colon (K \downarrow b) \to A$ *sends objects* $(a, h) \mapsto a$ *and morphisms* $(a \to a') \mapsto (a \to a')$.

We omit the proof. It is pertinent to note, however, that C need not be cocomplete for the same idea to work: indeed, it is sufficient to contain the colimit of every diagram FU_b for all objects b in B. This the key to realizing that the extension of the base-2 exponential function explored earlier is in fact a Kan extension.

Example 8. We will focus our attention on the case of extending $F \doteq 2^-\colon \mathbb{Q} \to \mathbb{R}_{>0}$ to $\mathbb{R}$. Note that each of these sets is totally ordered by the familiar ordering $\leq$. Thus, each set can be regarded as a *toset category*, where $r \to r'$ if and only if $r \leq r'$ (and any Mor-set has cardinality at most 1). We note that F is monotonic over $\mathbb{Q}$ and is thus a functor of tosets. Note also that we are taking K to be the inclusion functor $\mathbb{Q} \to \mathbb{R}$. What is $K \downarrow b$ for any b in $\mathbb{R}$? It is precisely the full subcategory of $\mathbb{Q}$ containing all rational numbers a such that $a \leq b$. In other words, it is a set in $\mathbb{Q}$ bounded above. Again, F is monotonic, so any upper bound on a set in $\mathbb{Q}$ is sent by F to an upper bound on the image of that set in $\mathbb{R}_{<0}$; thus, the image of the set in question is bounded above. We now appeal to analysis and use the fact that any such set in $\mathbb{R}$ (and so $\mathbb{R}_{>0}$) has a least upper bound, in this case: a positive real number c such that $Fa \leq c$ for all $a \leq b$, and for all c' such that $Fa \leq c'$ for all $a \leq b$, it is the case that $c \leq c'$. Observe that this is exactly what it means for c to be a colimit of the corresponding diagram FU_b. So the theorem (or at least the spirit of it) gives us the left Kan extension defined by $b \mapsto \operatorname{colim} FU_b$. This, of course, is exactly the extension of 2^- that we are used to: any other choice would fail to preserve monotonicity.

There is much more to Kan extensions than we can say here. Needless to say, the adage "all concepts are Kan extensions" is not hyperbole. We conclude with just a flavour of this idea.

Theorem 9. *Given any functor $F\colon A \to C$, if F has a colimit, then it is the left Kan extension of the diagram*

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr] & \arrow[d, phantom, "\Downarrow\beta"] & C \\ & \mathbb{1} \arrow[ur, dotted, "\operatorname{colim} F"'] & \end{tikzcd} \tag{10}$$

where $\beta a\colon Fa \to \operatorname{colim} F$ is the corresponding colimit morphism.

Theorem 10. *Given any functor $F\colon A \to B$, F has a left adjoint if and only if*

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "F"'] & & A \\ & B & \end{tikzcd} \tag{11}$$

has a right Kan extension preserved by F; that is, $F \operatorname{Ran}_F \mathrm{id}_A = \operatorname{Ran}_F F$. In the case that F has a left adjoint, then it is given by $\operatorname{Ran}_F \mathrm{id}_A$.