From last time:

- The <u>FI category</u> is the one with objects finite sets and morphisms injective maps.
- The full subcategory of FI of finite sets of the form $[n]$ is equivalent to FI and is a skeleton of FI.
- Let R comm. ring. An FI-mod V over R is a functor from $FI \longrightarrow R\text{-mods}$.

Last time, we also saw that

- $\mathrm{End}_{FI}([n]) \simeq S_n$.

Def. $\iota_{n,n+1} : [n] \hookrightarrow [n+1]$ is the canonical inclusion that sends i to i.

Prop. The morphisms in this category are generated by the endomorphisms S_n and the canonical inclusions $\iota_{n,n+1}$

Proof. Let $f : [m] \hookrightarrow [n]$, then

$$\begin{tikzcd}
{[m]} \arrow[r, hook, "\iota_{n,n+1}"] \arrow[drr, "f"'] & {[m+1]} \arrow[r, hook] & \cdots \arrow[r, hook] & {[n]} \arrow[dl, dashed, "\sigma"] \\
 & & {[n]} &
\end{tikzcd}$$

Now, a theorem about the set $\mathrm{Hom}_{FI}([m],[n])$.

Thm. $\mathrm{Hom}_{FI}([m],[n]) \simeq S_n / S_{n-m}$ as $\underbrace{S_n\text{- set}}_{\text{set with an } S_n\text{-action}}$.

Def. Two G-sets X and Y are called isomorphic if there is a bijection $\phi : X \to Y$ such that $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X$.

Lemma. A transitive G-set is isomorphic to G/A where A is the stabilizer of any set element.

Proof. Let $A = \mathrm{Stab}(x)$. Let $\phi : G/A \to X$ that sends the coset $gA \mapsto g(x)$. One can check this is well-defined and is indeed an isomorphism of G-sets. ▨

Proof (of the thm) We have a natural S_n-action on $\mathrm{Hom}_{FI}([m],[n])$ on the left by postcomposition:

$$\mathrm{End}_{FI}([n]) \cong S_n \times \mathrm{Hom}_{FI}([m],[n]) \longrightarrow \mathrm{Hom}_{FI}([m],[n])$$
$$(\sigma, \alpha) \mapsto \sigma\circ\alpha$$

When σ is the identity, $\alpha \mapsto \alpha$; and for $\sigma, \delta \in S_n$, $(\sigma\delta)\alpha = \sigma(\delta\alpha)$.
This action is transitive. Let $\alpha, \beta \in \mathrm{Hom}_{FI}([m],[n])$, construct $\sigma: S_n \to S_n$ so that $\alpha(i) = \sigma(\beta(i))$, $i \in \{1, \dots, n\}$. Define $\sigma: [n]\setminus \mathrm{im}\beta \to [n]\setminus \mathrm{im}\alpha$ to be injective. σ is well-defined b/c β is injective. Since α is also inj, $|\mathrm{im}\alpha| = |\mathrm{im}\beta|$, so σ is a permutation

Consider the point $\iota_{m,n} \in \mathrm{Hom}([m],[n])$. $\mathrm{Stab}(\iota_{m,n}) =$
$$\{\sigma \in S_n \mid \sigma\circ\iota_{m,n} = \iota_{m,n}\}$$
So for $\sigma \in \mathrm{Stab}(\iota_{m,n})$, $\sigma(i) = \begin{cases} i & \text{if } i \le m \\ j \text{ for some } j > m & \text{if } i > m \end{cases}$

So $\mathrm{Stab}(\iota_{m,n}) \cong S_{n-m}$.

Therefore, by lemma, $\mathrm{Hom}([m],[n]) \cong S_n / S_{n-m}$. ▨

Notation: Let V be an FI-module, denote the object $V([n])$ by V_n.

Let's now consider a theorem associating an FI-mod to S_n-representations.

Thm. Let V be an FI-module over a ring R.

(1) For each $n \in \mathbb{Z}_{n\ge 0}$, the action of S_n on the R-mod V_n gives V_n the structure of an S_n-representation;

(2) The map $(\iota_{m,n})_* = V(\iota_{m,n}): V_m \to V_n$ is S_m-equivalent w.r.t.

the action of S_m on V_m, and the action of the subgroup S_m defined by $I_{m,n}$ in S_n, on V_n.

Proof. (1) It suffices to show that V_n is an $R[S_n]$-module. An endomorphism $\sigma \in S_n$ on $[n]$ is mapped to an endomorphism $V(\sigma): V_n \to V_n$. For some $\sum_{i=1}^{n} r_i\sigma_i \in R[S_n]$ and $u \in V_n$, $(\sum r_i\sigma_i)\cdot u = \sum(r_i \cdot V(\sigma_i)(u))$.

(2) Consider the following commutative diagram:

$$\begin{tikzcd} {[m]} \arrow[r, "I_{m,n}"] \arrow[d, "\sigma"'] & {[n]} \arrow[d, "\sigma \in S_m \leq S_n"] \\ {[m]} \arrow[r, "I_{m,n}"'] & {[n]} \end{tikzcd} \quad \overset{\text{Apply } V}{\Longrightarrow} \quad \begin{tikzcd} V_m \arrow[r, "(I_{m,n})_*"] \arrow[d, "V(\sigma)"'] & V_n \arrow[d, "V(\sigma)"] \\ V_m \arrow[r, "(I_{m,n})_*"'] & V_n \end{tikzcd}$$

We get another comm. diag. So for $\sigma \in S_m$ and $u \in V_m$,

$$\sigma \cdot (I_{m,n})_*(u) = (I_{m,n})_*(\sigma \cdot u).$$

This is exactly saying $(I_{m,n})_*$ is S_m-equivariant. ▨

Every FI-module V determines a sequence of S_n-rep V_n along with S_n-equivariant maps $V_n \to V_{n+1}$. The converse, however, is not quite true. Now we introduce a theorem that determines when such a sequence arises from an FI-module.

<u>Thm</u> (The FI-module criterion)

Suppose that $\{W_n\}$ is a sequence of S_n-rep with S_n-equivariant maps $\phi_n: W_n \to W_{n+1}$. Denote $G \cong S_{n-m}$ the stabilizer of $I_{n,m}$ under the action of S_n by postcomposition. Then $\{W_n\}$ can be promoted to an FI-mod with $(I_{n,n+1})_* = \phi_n$ iff for all $m<n$, $\sigma \cdot v = v$ for all $\sigma \in G$ and $v \in \operatorname{im}((\iota_{m,n})_*)$.

Proof.

$$\begin{tikzcd} u & v = (\iota_{m,n})_*(u) \\ V_m \arrow[r, "(\iota_{m,n})_*"] \arrow[d, "V(\sigma)"'] & V_n \arrow[d, "V(\sigma)"] \\ V_m \arrow[r, "(\iota_{m,n})_*"'] & V_n \quad v \end{tikzcd}$$

$\sigma \in G = \operatorname{stab}(\iota_{m,n})$

$\sigma \circ \iota_{m,n} = \iota_{m,n}$

$V(\sigma) \circ (\iota_{m,n})_* = (\iota_{m,n})_*$

($\Rightarrow$) Suppose $\{W_n\}$ can be promoted to an FI-mod. Since $\sigma \in G$, $\sigma \circ \iota_{m,n} = \iota_{m,n}$, so $W(\sigma) \circ (\iota_{m,n})_* = (\iota_{m,n})_*$. Let $v = (\iota_{m,n})_*(u)$ for some $u \in W_m$, then $W(\sigma) \circ (\iota_{m,n})_*(u) = ((\iota_{m,n})_*(u)$, So $\sigma \cdot v = W(\sigma)(v) = v$.

$\phi_{m,n} = \phi_{n-1} \circ \cdots \circ \phi_m$

($\Leftarrow$) For all $m<n$, $\sigma \cdot v = v$ for all $\sigma \in G \simeq S_{n-m}$ and $v \in \operatorname{im}(\phi_{m,n})$.
Define W to be a map $FI \to R\text{-mod}$ where $W([n]) = W_n$ and $W(\iota_{m,n}) = \phi_{m,n}$. $W(\sigma) = \sigma$ for $\sigma \in \operatorname{End}([n]) \simeq S_n$ and define composition in a way that its preserved.

Want to show the def is self-consistent. Let $\sigma \in G$.

$$W(\underbrace{\sigma \circ \iota_{m,n}}_{= \iota_{m,n}}) := W(\sigma) \circ W(\iota_{m,n}) = \sigma \cdot \phi_{m,n} = \phi_{m,n}.$$

$\parallel$

$\phi_{m,n}$

This is enough since every mor is gen. by σ and $\iota_{m,n}$. $\square$

Last time we saw that $V_n = \mathbb{Z}[S_n]$ with action of S_n by conjugation. $\phi_n : V_n \to V_{n+1}$ natural inclusion. is an example of an FI-mod, but $V_n = \mathbb{Z}[S_n]$ with action by left multiplication, ϕ_n nat. inclusion is a non-example.

$$\mathbb{Z}[S_2] \xrightarrow{\phi_{2,4}} \mathbb{Z}[S_4]$$

$$\underline{(1)(2)} \mapsto \underline{(1)(2)(3)(4)}$$

$(34) \in G$, but $(34)\,\underline{(1)(2)(3)(4)} = \underline{(1)(2)(34)}$.