A_n and E_n Operads

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1 Intuition

We focus on A_n and E_n operads, which are a formalism for discussing various degrees to which associativity and commutativity can fail. Suppose we have a space X with a multiplication $\mu : X \times X \to X$ that is not associative. In particular, $\mu(-, \mu(-,-))$ and $\mu(\mu(-,-),-)$ are two different ternary operations.

The approach we take is to consider a space $\mathcal{O}(3)$ of ternary operations, in which $\mu(-, \mu(-,-))$ and $\mu(\mu(-,-),-)$ are two different points. We can learn about exactly how badly associativity fails in this space by looking at the geometry of this space $\mathcal{O}(3)$. For instance, strict associativity corresponds to this space being one point (i.e. $\mu(-, \mu(-,-)) = \mu(\mu(-,-),-)$), and the next best thing is for the space to be contractible. Similarly for every n, we can also consider a space $\mathcal{O}(n)$ of n-ary operations, where we consider the operations $\mu(\mu(\mu(\cdots),-),-)$, etc. as points.

Do note that there is a slight caveat with this intuition. We are not associating to every space X a collection $\{\mathcal{O}(n)\}_n$ of n-ary operations. Instead, there are some stock collections $\{\mathcal{O}(n)\}$ that we care about, and we can learn something about X if we can interpret $\{\mathcal{O}(n)\}$ as n-ary operations on X in a "nice" way.

2 Definitions and Examples

For simplicity, let $\mathcal{U}$ denote the category of compactly generated Hausdorff spaces and continuous maps (this choice is discussed further in Steenrod's paper [4]). An operad is a collection of certain interrelated spaces $\mathcal{O}(n)$, the points of which are to be thought of as n-ary operations $X^n \to X$. We have two definitions to work with.

2.1 Non-Symmetric Operad

A *non-symmetric operad* is a collection $\{\mathcal{O}(n)\}_{n\geq 0}$ of spaces in $\mathcal{U}$ with some extra structure and properties obtained by thinking of $\mathcal{O}(n)$ as a space of n-ary operations:

1. There are continuous functions (here $k = k_1 + \cdots + k_n$):

$$\gamma : \mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) \to \mathcal{O}(k)$$

   These satisfy the following associativity formula (for all $c \in \mathcal{O}(n)$, $d_i \in \mathcal{O}(k_i)$, and $e_j \in \mathcal{O}(k'_j)$):

$$\gamma(\gamma(c; d_1, \cdots, d_n); e_1, \cdots, e_k) = \gamma(c; f_1, \cdots, f_n)$$

   Here $f_s = \gamma(d_s; e_{k_1+\cdots+k_{s-1}+1}, \cdots, e_{k_1+\cdots+k_s})$, and $f_s = *$ if $k_s = 0$.

2. There is an identity element $1 \in \mathcal{O}(1)$ such that the following hold(for all $d \in C(j)$, $c \in \mathcal{O}(k)$, and here $1^k = (1, \cdots, 1) \in \mathcal{O}(1)^k$):

$$\gamma(1; d) = d$$

$$\gamma(c; 1^k) = c$$

3. $\mathcal{O}(0) = *$

Now given a based space X, we say that "$\mathcal{O}$ acts on X" (or, "X is an $\mathcal{O}$-algebra") if there is a morphism of operads $\mathcal{O} \to \mathrm{End}_X$ (which we will define shortly), where $\mathrm{End}_X(n)$ is the space of maps $X^n \to X$. The simplest example of a non-symmetric operad is the associative operad Ass, defined by setting $\mathrm{Ass}(n) = *$ for all n. An action of Ass on a space X is a map $\mathcal{O}(n) \to \mathrm{End}_X(n)$ for every n. This chooses a single operation $X^n \to X$ for every n. If μ is the chosen binary operation, then the fact that $\mathcal{O} \to \mathrm{End}_X$ preserves operad structure will ensure that the chosen ternary operation is $\mu(-, \mu(-, -)) = \mu(\mu(-, -), -)$, and so on. In particular, $\gamma(\mu; 1, \mu) = \mu(-, \mu(-, -))$ and $\gamma(\mu; \mu, 1) = \mu(\mu(-, -), -)$ are both elements of the image of $\mathrm{Ass}(3) = * \to \{X^3 \to X\}$. This is a point, and so they must be the same. So X is an Ass-algebra iff X has a unital associative multiplication (this works as the spaces X have base-points and the structure maps force the base-point act as the unit with respect to the multiplication).

We can now discuss A_n operads. The second best kind of associativity after strict associativity is when all the $\mathcal{O}(n)$'s are contractible. In this case, we say the multiplication is A_∞ (or that $\mathcal{O}$ is an A_∞ operad). The weaker notion of A_n operads parametrizes multiplication associative upto certain levels of homotopies (i.e. $\mathcal{O}(n)$ is contractible up to a certain n). We say that X is an A_∞-space (similarly A_n-space) if it has an action of an A_∞ (similarly A_n) non-symmetric operad. We can further describe some A_n-spaces as follows:

1. A_1-spaces are pointed spaces.
2. A_2-spaces are H-spaces (topological unital magmas) with no associativity conditions.
3. A_3-spaces are homotopy associative H-spaces.

2.2 Symmetric Operads

A *symmetric operad* is a non-symmetric operad with a right action of the symmetric group S_n on each $\mathcal{O}(n)$ such that the following equivariance formulas are satisfied for all $c \in \mathcal{O}(n)$, $d_i \in \mathcal{O}(k_i)$, $\sigma \in S_k$, and $\tau_j \in S_{k_j}$:

$$\gamma(c\sigma; d_1, \cdots, d_n) = \gamma(c; d_{\sigma^{-1}(1)}, \cdots, d_{\sigma^{-1}(n)})\sigma(k_1, \cdots, k_n)$$

$$\gamma(c; d_1\tau_1, \cdots, d_n\tau_n) = \gamma(c; d_1, \cdots, d_n)(\tau_1 \oplus \cdots \oplus \tau_n)$$

Here $\sigma(k_1, \cdots, k_n)$ denotes the permutation of k letters which permutes the n blocks of letters determined by the given partition of k as σ permutes n letters, and $\tau_1 \oplus \cdots \oplus \tau_n$ denotes the image of $(\tau_1, \cdots, \tau_n)$ under the natural inclusion of $S_{k_1} \times \cdots \times S_{k_n}$ in S_k.

We can further define *operad morphisms* $\psi : \mathcal{O} \to \mathcal{O}'$ as a sequence of S_n-equivariant maps $\psi_n : \mathcal{O}(n) \to \mathcal{O}'(n)$ such that $\psi_1(1) = 1$ and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) & \xrightarrow{\gamma} & \mathcal{O}(k) \\ \downarrow{\scriptstyle \psi_n \times \psi_{k_1} \times \cdots \times \psi_{k_n}} & & \downarrow{\scriptstyle \psi_k} \\ \mathcal{O}'(n) \times \mathcal{O}'(k_1) \times \cdots \times \mathcal{O}'(k_n) & \xrightarrow{\gamma'} & \mathcal{O}'(k) \end{array}$$

Again, the simplest example is the commutative operad Comm, defined to have $\mathrm{Comm}(n) = *$. We have that Comm-algebras are the same as unital, associative, commutative monoids. The idea here is that $\mathrm{Comm}(2) = *$ forces $(a, b) \mapsto \mu(a, b)$ and $(a, b) \mapsto \mu(b, a)$ to be the same point. Furthermore, given a non-symmetric operad $\mathcal{O}$, we can convert it into a symmetric operad $\mathcal{O}^S$ (non-standard notation) by allowing S_n to acts freely on the right, thereby defining:

$$\mathcal{O}^S := \mathcal{O} \times S_n$$

This allows us to discuss E_n operads. If the space fails to have strict commutative multiplication, the next best thing to have the action of a symmetric operad, all of whose spaces are contractible. A symmetric operad $\mathcal{E}$ is an E_∞ operad if $\mathcal{E}(n)$ is contractible for all n, and the S_n action is free. (The free action is a technical condition and more abstractly, an E_∞ operad is any cofibrant resolution of Comm [6]). The weaker notion of E_n operads parametrizes multiplication commutative upto certain levels of homotopies (i.e. $\mathcal{O}(n)$ is contractible up to a certain n). We say that X is an E_∞-space (similarly E_n-space) if it has an action of an E_∞ (similarly E_n) symmetric operad. We can further describe some E_n-spaces as follows:

1. E_1-spaces are A_∞-spaces.
2. E_2-spaces are homotopy commutative A_∞-spaces.

References

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