MATH 2520 ALGEBRA

SHIYUE LI

FI-MODULES

FI-modules naturally arise when we analyze sequence of algebraic objects indexed by the non-negative integers. This presentation follows the treatment of [2]. Interested audience may refer to the seminal paper on FI-modules by Church, Ellenberg and Farb [1].

Definition. The category FI consists of the following data:

- Objects: **F**inite sets;
- Morphisms: **I**njective (set) maps.

Example. The sets $[n] := \{1, \dots, n\}$ for $n \in \mathbb{Z}_{\geq 0}$ are objects in FI.

Example. The injective set maps $[n] \hookrightarrow [m]$ for $n \leq m$ are morphisms in FI.

Let $\mathcal{F}$ be the category where objects are $[n] := \{1, \dots, n\}$ for any $n \in \mathbb{Z}_{\geq 0}$ and morphisms are injective maps $[n] \hookrightarrow [m]$ for $n \leq m$. It is a subcategory of FI.

Proposition. *The inclusion functor* $\mathrm{I} : \mathcal{F} \to \mathrm{FI}$ *is an equivalence of categories.*

Proposition. *The subcategory $\mathcal{F}$ of* FI *is a skeleton (i.e. an equivalent category where no two distinct objects are isomorphic) of* FI.

Fact. The morphisms in $\mathcal{F}$ can be generated by inclusions and endomorphisms of $[n]$ for all $n \in \mathbb{Z}_{\geq 0}$.

By studying the skeleton $\mathcal{F}$ of FI, we can gain structural information about FI.

Proposition. *The endomorphisms of* $[n]$ *in* FI *is symmetric group* S_n.

Proof. Since injection of a finite set to itself is surjective, all endomorphisms of $[n]$ are bijections from $[n]$ to itself. These are precisely permutations on n set elements and they form a group under composition. Hence $\mathrm{End}_{\mathrm{FI}}([n]) \cong S_n$. □

Definition. Let R be a commutative ring. An FI-module is a functor $V : \mathrm{FI} \to R\text{-mod}$.

Slogan: The image of an FI-module V is determined by the image of the skeleton $\mathcal{F}$ up to isomorphism. To describe a FI-module V, we only need to specify the image of $[n]$, inclusion maps and endomorphisms, namely S_n-actions, under V.

Denote $V([n])$ by V_n for all $n \in \mathbb{Z}_{\geq 0}$ and let $i_{n,m}$ be the inclusion maps $[n] \hookrightarrow [m]$ for $n \leq m$.

Example. Let $R = \mathbb{Z}$. The functor $V : \mathrm{FI} \to \mathrm{Ab}$ defined by $V_n = \mathbb{Z}$ and $V(f) = \mathrm{id}$ for all $f \in \mathrm{Mor}(\mathcal{F})$ is an FI-module.

Example. Let $R = \mathbb{Z}$. The functor $V : \text{FI} \to \text{Ab}$ defined by $V_n = \mathbb{Z}[x_1, \ldots, x_n]$ and $V(i_{n,m})$ being natural inclusions, is a FI-module. The symmetric group acts by permuting the variables.

Example. The functor $V : \text{FI} \to \text{Ab}$ defined by $V_n = \mathbb{Z}[S_n]$ and $V(i_{n,m})$ being natural inclusions, where S_n acts by conjugation, is an FI-module.

Non-example. The map $V : \text{FI} \to \text{Ab}$ defined by $V_n = \mathbb{Z}[S_n]$ and $V(i_{n,m})$ are natural inclusions, where S_n acts by left-multiplication, is not an FI-module.

References

[1] Jordan S. Ellenberg Thomas Church and Benson Farb. Fi-modules and stability for representations of symmetric groups. URL: `https://arxiv.org/pdf/1204.4533.pdf`.

[2] Jenny Wilson. An introduction to fi–modules and their generalizations. URL: `http://www.math.lsa.umich.edu/~jchw/FILectures.pdf`.