An informal introduction to blow-ups

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Motivation.

A *variety over* $\mathbf{C}$, or a *complex algebraic variety*, means a locally ringed[1] topological space X, such that each point $P \in X$ has an open neighborhood U with $U \cong \operatorname{Spec} A$, where $A = \mathbf{C}[x_1, \dots, x_n]/I$ is a finitely generated $\mathbf{C}$-algebra which is also a domain, i.e. I is taken to be a prime ideal; we also require X to be Hausdorff when we view it with the analytic topology.[2] Good examples are *affine varieties* (i.e. $X = \operatorname{Spec} A$ with A as above), which we've dealt with extensively in 2520; the first non-affine examples are the complex projective space $\mathbf{P}^n_{\mathbf{C}}$ and its irreducible Zariski closed subsets, cut out by homogeneous prime ideals of $\mathbf{C}[x_0, \dots, x_n]$ provide a large class of examples, called *complex projective varieties*. For an introduction to projective varieties from the classical perspective of homogeneous coordinates, see [Rei90]. For scheme theory and the modern perspective on varieties, see [Har10].

It's natural to wonder how a variety X might fail to be a manifold, when we view X with the analytic (as opposed to Zariski) topology. The issue is that varieties may have *singularities*, which, in the analytic topology, are points that do not have neighborhoods homeomorphic to $\mathbf{C}^n$.

Definition 1. *Let X be a variety. Then $P \in X$ is* ***nonsingular point*** *if for any open affine neighborhood* $\operatorname{Spec} A$ *of P in X, we have that A_P is a regular local ring. Otherwise P is* ***singular****.*

For an example, consider the nodal cubic $C = \operatorname{Spec} \mathbf{C}[x,y]/(y^2 - x^3 - x^2)$. Then we've seen that the origin $(0,0)$, corresponding to $m = (x,y) \in C$, is a singular point of C, because m is not principally generated in the localization $(\mathbf{C}[x,y]/(y^2 - x^3 - x^2))_m$, while $\dim(\mathbf{C}[x,y]/(y^2 - x^3 - x^2))_m = 1$. Another example is given by the cone $\operatorname{Spec} \mathbf{C}[x,y,z]/(x^2 + y^2 - z^2) \subseteq \mathbf{A}^3$. This variety is singular at the origin $(0,0,0)$, since the maximal ideal $m = (x,y,z)$ cannot be generated by 2 elements in the localization $(\mathbf{C}[x,y,z]/(x^2+y^2-z^2))_m$, but $\dim(\mathbf{C}[x,y,z]/(x^2+y^2-z^2))_m = 2$. In general, the minimal number of generators of a maximal ideal m_P of a point $P \in X$ is equal to the dimension of the *Zariski cotangent space* m_P/m_P^2 by Nakayama's lemma, so A_P is regular if and only if $\dim_{\mathbf{C}} m_P/m_P^2 = \dim A_P = \dim X$, so geometrically, singular points are those where the tangent space has the wrong dimension.

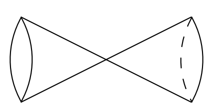


Figure 1: A conical singularity $\operatorname{Spec} \mathbf{C}[x,y,z]/(x^2 + y^2 - z^2)$

As it turns out, "most points" of a variety X will not be singular, in the sense that the set of singular points is Zariski closed:

[1] A ringed topological space is a space X together with a *sheaf of rings* $\mathcal{O}_X$: every open subset U gets a ring $\mathcal{O}_X(U)$, thought of as a ring of functions on U. There are some additional conditions that abstract our notions of how functions on a space should behave with respect to open covers. Saying *locally ringed* means that we are requiring that for any $P \in X$, the ring of germs of functions at P is a local ring. A good motivating example of a locally ringed space is that of a manifold together with its sheaf of differentiable functions: the stalk at a point P really is the ring of germs of functions at P; this is local with maximal ideal consisting of functions vanishing at P.

[2] This definition is meant to evoke that of complex manifolds, but in more precise language, we want X to be an integral scheme which is separated and locally of finite type over $\operatorname{Spec} \mathbf{C}$.

Proposition 1. *Let X be a variety. The set X^{sing} of singular points of X forms a Zariski closed subset of X.*

For complex varieties, Zariski closed subsets will have Lebesgue measure 0; this can be proven using e.g. Sard's theorem. Therefore, any complex variety is a measure 0 subset away from being a smooth complex manifold: just delete the singular locus X^{sing} and you are left with a manifold. In a sense, however, this is a "geometrically trivial" solution to the problem of replacing X by a manifold: for example, if X^{sing} is nonempty, then $X \setminus X^{\mathrm{sing}}$ will never be compact, nor will the map $X \setminus X^{\mathrm{sing}} \to X$ be a surjection. In algebraic geometry, we make the question of replacing X by a manifold $\tilde{X}$ more interesting by requiring that the map $\tilde{X} \to X$ be *proper*, which means it pulls back compact subsets to compact subsets.

Definition 2. *Let X be a variety. A* ***resolution of singularities*** *for X is a nonsingular variety $\tilde{X}$ together with a proper morphism $f : \tilde{X} \to X$ which is an isomorphism away from the singular locus: $f : \tilde{X} \setminus f^{-1}X^{\mathrm{sing}} \to X \setminus X^{\mathrm{sing}}$ is an isomorphism.*

Indeed, one can verify that properness, together with our assumption that X is irreducible, will imply that the map $\tilde{X} \to X$ is surjective. A major result of algebraic geometry is that resolutions of singularities exist over algebraically closed fields of characteristic 0:

Theorem 1 (Hironaka [Hir64])**.** *Let X be a variety over an algebraically closed field k of characteristic 0 (e.g. $k = \mathbf{C}$). Then there exists a resolution of singularities $f : \tilde{X} \to X$.*

The main operation used to resolve singularities is *blowing-up*, which is a particular geometric surgery operation one can do on varieties. In fact, Hironaka's theorem is proven by showing that $\tilde{X}$ is obtained from X via a finite sequence of blow-ups. It's worth mentioning that the above remains a major open problem when k is a field of characteristic p.

Blowing-up via universal property.

Blowing-up is a *universal* surgery operation on varieties. Before we define it via the universal property that it satisfies, we need to define a *hypersurface*. All results and definitions in this section are reformulations of those found in Chapter II of [Har10].

Definition 3. *An* ***irreducible divisor*** *D in a variety X is a codimension one subvariety which is locally cut out by one equation: for an affine open $\operatorname{Spec} A \subseteq X$, $D \cap \operatorname{Spec} A = \operatorname{Spec} A/(f)$ for some irreducible $f \in A$. A* ***hypersurface*** *H in X is a finite union of irreducible divisors, possibly with multiplicity, so $H \cap \operatorname{Spec} A = \operatorname{Spec} A/(f_1^{a_1} \cdots f_k^{a_k})$ where f_i are irreducible and $a_i \geq 1$.*

In the language of scheme theory, D as above is an *irreducible Cartier divisor* and H is an *effective Cartier divisor*. These definitions may seem a little opaque; it might seem more natural to just define a hypersurface to be a codimension one subvariety. Indeed, these notions coincide on a smooth variety.

Proposition 2. *Suppose X is a smooth variety. Then every irreducible codimension one subvariety is a hypersurface, i.e. locally cut out by a single equation.*

This proposition shows that the existence of codimension one subvarieties which are not *locally principal* (cut out by one equation) is an indication that our variety has singularities. The converse is not true, however; it may well be the case that a variety has all of its codimension one subvarieties as hypersurfaces, but still has singularities. But,in both of our singular examples above, we are able to find codimension one subvarieties which are not locally principal. On the nodal cubic $C = \operatorname{Spec} R$ where $R = \mathbf{C}[x,y]/(y^2 - x^3 - x^2)$, the origin O is a codimension one subvariety. It is given by the intersection of C with $(x = 0)$ and $(y = 0)$, i.e. $O = \operatorname{Spec} R/(x,y)$. But the maximal ideal $m = (x,y)$ of the local ring $(\mathbf{C}[x,y]/(y^2 - x^3 - x^2))_m$ is not principally generated, so there is no neighborhood $\operatorname{Spec} R_f$ of O where it is cut out by only one equation. However, the subvariety $2O = \operatorname{Spec} R/(x^2, y)$ is a hypersurface, because the ideal (x^2, y) of R can be principally generated by x^2. This is the statement that any polynomial in R vanishing only on O must vanish with multiplicity. Similarly, one can show that there are lines in the cone $\operatorname{Spec} \mathbf{C}[x,y,z]/(x^2 + y^2 - z^2)$ which cannot be cut out by one equation.

Given input data (X, Z) where X is a variety and Z is a subvariety, the blowing-up $\operatorname{Bl}_Z X$ should be thought of as the *canonical* way of replacing Z with a hypersurface, in the sense that it satisfies an appropriate universal property.

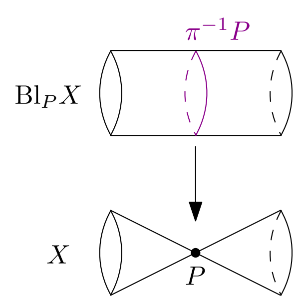
Definition 4. *Let X be a variety and Z a subvariety. The* ***blowing-up of** X **along** Z*, if it exists, is a variety $\operatorname{Bl}_Z X$ together with a morphism $\pi : \operatorname{Bl}_Z X \to X$, such that* (a) *$\pi^{-1}Z$ is a hypersurface and* (b) *$\operatorname{Bl}_Z X \to X$ is final with respect to this property. The subvariety Z is called the* ***center*** *of the blow-up.*

Since we defined it via a universal property, the blow-up, if it exists, is unique up to unique isomorphism. While we won't prove it, the blow-up in fact exists for any pair of input data (X, Z):

Proposition 3. *Let X be a variety and Z a subvariety. Then the blow-up $\operatorname{Bl}_Z X \to X$ exists.*

In fact, blow-ups apply in much greater generality; one can blow-up any subscheme of a Noetherian scheme. We will say something about the construction of the blow-up in the next section, but for now we collect some nice properties that it satisfies. We can see from the universal property that if we take Z to be a hypersurface, then $\operatorname{Bl}_Z X = X$, since X together with the identity map $X \to X$ clearly satisfies the universal property. So for example, without getting our hands dirty with any constructions, we know a lot of trivial blow-ups. Strangely, we know that the blow-up of the nodal cubic C at the origin O will be nontrivial, while the blow-up at $2O$ will be trivial. In fact, a dense affine chart of the blow-up $\operatorname{Bl}_O C$ is isomorphic to the normalization $\operatorname{Spec} \mathbf{C}[x,y,z]/(y^2 - x^3 - x^2, zx - y)$.

In our other running example, if one puts $P = (0,0,0)$ in affine 3-space, and takes $X = \operatorname{Spec} \mathbf{C}[x,y,z]/(x^2 + y^2 - z^2)$ to be the cone, one sees that a dense affine chart of $\operatorname{Bl}_P(X)$ is isomorphic to $\operatorname{Spec} \mathbf{C}[X,Y,Z]/(X^2 + Y^2 - 1)$, which is the cylinder, and that the inverse image of P in this chart is the circle $\operatorname{Spec} \mathbf{C}[X,Y,Z]/(X^2 + Y^2 - 1, Z)$. See [Hau] for the details behind this particular example and [EH09] for many more examples of blow-ups.



The above figure is somewhat impressionistic (for example, it's a two dimensional slice of something that has four dimensions), but indicates that the blow-up is "the same" as $X \setminus P$ outside of $\pi^{-1}P$; indeed if one subtracts the circle from the cylinder, one obtains a variety isomorphic to $X \setminus P$. This is not an accident, as the first part of the following proposition implies.

Proposition 4 (Cool properties of the blow-up)**.** *Let (X, Z) be a pair consisting of a variety and a subvariety. Then the blow-up $\pi : \operatorname{Bl}_Z X \to X$ satisfies the following:*

(i) π restricts to an isomorphism $\operatorname{Bl}_Z X \setminus \pi^{-1}Z \cong X \setminus Z$;

(ii) π is a proper morphism;

(iii) if $Y \subseteq X$ is another subvariety, then $\operatorname{Bl}_{Y\cap Z} Y = \overline{\pi^{-1}(Y \setminus Z)} \subseteq \operatorname{Bl}_Z(X)$; in particular $\operatorname{Bl}_{Y\cap Z} Y$ is a closed subvariety of $\operatorname{Bl}_Z(X)$;

(iv) if X and Z are both nonsingular, then $\pi^{-1}Z \cong \mathbf{P}(\mathcal{N}^\vee_{Z\subset X})$, the projectivization of the conormal bundle of Z in X.

As part (i) indicates, the blow-up leaves X unchanged outside Z, so we are justified in calling it a surgery operation; what we get at the end is "mostly the same as X," or the same outside some measure zero subset in the complex case, as remarked earlier. Part (ii) implies the following: if we blow-up a singular X and get nonsingular X', then X' counts as a resolution of singularities in the sense of Definition 2. So both the nodal cubic and the conical singularity above were resolved by one blow-up.

Part (iii) gives us a way of understanding some blow-ups in terms of others. For example, if we can compute the blow-up $\operatorname{Bl}_P \mathbf{A}^n$ of $\mathbf{A}^n$ at a point P, then we can compute the blow-up of any affine variety X containing P by taking the closure $\overline{\pi^{-1}(X \setminus P)}$ in $\operatorname{Bl}_P \mathbf{A}^n$. Part (iv) is supposed to be an indication that the *geometry of the blow-up is well understood*, at least when we are blowing up a smooth variety along a smooth center. In particular, if X is nonsingular of dimension n and Z is a nonsingular codimension ℓ subvariety, then the vector space fibers of the conormal bundle have dimension ℓ, so part (iv) implies that not only do we know that $\pi : \operatorname{Bl}_Z X \to X$ is an isomorphism outside of Z, but $\pi^{-1}Z$ is a fiber bundle over Z with fiber $\mathbf{P}^{\ell-1}$. One can use information about this conormal bundle to relate the cohomology ring of $\operatorname{Bl}_Z X$ to that of X. Since blow-ups of nonsingular subvarieties of nonsingular varieties are best understood, it's fruitful to study blow-ups of a singular variety Y by first finding an embedding $Y \subseteq X$ where X is nonsingular and then exploiting (iii).

Now that we have some rough idea of the geometry of blow-ups, and we know that Hironaka has proven that any variety X is brought to a nonsingular variety $\tilde{X}$ by a finite sequence of blow-ups, we have an answer to our question as to how far a complex variety is from a smooth complex manifold: it is a finite sequence of contractions of hypersurfaces away.

A little bit about the construction.

Classically, affine varieties were just viewed as irreducible zero sets of polynomials in affine space, and the blow-up of $Z \subseteq \mathbf{A}^n$ with Z given by m equations was constructed by writing down suitable algebraic equations for a subvariety $\operatorname{Bl}_Z(\mathbf{A}^n) \subseteq \mathbf{A}^n \times \mathbf{P}^{m-1}$, i.e. a system of polynomial equations in $n + m$ variables, homogeneous in the last m variables. In the modern setting, we do essentially the same thing via the Proj construction, which is a systematic (but not functorial) way of turning certain graded k-algebras into subschemes of $\mathbf{P}^n_k$. Blowing-up a subvariety Z of a general variety X reduces to blowing-up $Z \cap \operatorname{Spec} A$ for each affine open $\operatorname{Spec} A \subseteq X$ and then gluing, so here we'll focus on the affine case. Chapter 5 of [Eis08] is the main source for this section, with the results on the Proj construction coming from [Har10]. If R is a Noetherian ring and $J \subseteq R$ is any ideal, defining a closed subscheme $Z = V(J)$ of $X = \operatorname{Spec} R$, the construction of $\operatorname{Bl}_Z X$ uses the **blow-up algebra**

$$\operatorname{Bl}_J(R) = R \oplus J \oplus J^2 \oplus \cdots$$

of R along J; in fact, this algebra "contains the data of the blow-up" $\operatorname{Bl}_Z X$, in the following sense.

Proposition 5. *Let $X = \operatorname{Spec} R$ and $Z = V(J)$ a closed subscheme. Then the blow-up of X along Z is given by*

$$\operatorname{Bl}_Z(X) = \operatorname{Proj} \operatorname{Bl}_J(R),$$

together with the morphism

$$\operatorname{Bl}_Z(X) \to X$$

induced by the inclusion $R \to \operatorname{Bl}_J(R)$.

The Proj construction makes projective schemes out of graded rings which are generated in degree one; we saw in 2520 that the blow-up algebra has this property. Since R is Noetherian, $J = (f_1, \ldots, f_m)$ is a finitely generated ideal, and therefore $\operatorname{Bl}_J(R)$ is a finitely generated R-algebra; indeed there is a surjective graded morphism $R[T_1, \ldots, T_m] \to \operatorname{Bl}_J R$ taking T_i to f_i, which in turn gives a closed immersion $\operatorname{Proj} \operatorname{Bl}_J R \to \operatorname{Proj} R[T_1, \ldots, T_m] = \mathbf{P}_R^{m-1}$. Thus we see that the modern construction realizes the blow-up as a closed subset of $\mathbf{P}_R^{m-1} = \mathbf{P}_{\mathbf{Z}}^{m-1} \times_{\mathbf{Z}} \operatorname{Spec} R$ (if R is a finitely generated k-algebra, this is the same as $\mathbf{P}_k^n \times_k \operatorname{Spec} R$), which is a hint that this construction can be reconciled with classical one. Upon taking a quotient of the blow-up algebra by the ideal $J \operatorname{Bl}_J(R)$, we get the algebra

$$\operatorname{Bl}_J(R)/J \operatorname{Bl}_J(R) = \operatorname{gr}_J R = R/J \oplus J/J^2 \oplus J^2/J^3 \oplus \cdots$$

called the **associated graded ring of R with respect to J** (we considered this in 2520 in the special case where R is local and J is its maximal ideal). Therefore there is a surjective, graded morphism of graded rings $\operatorname{Bl}_J(R) \to \operatorname{Bl}_J(R)/J \operatorname{Bl}_J(R)$. We thus get a closed immersion

$$\operatorname{Proj} \operatorname{gr}_J(R) \hookrightarrow \operatorname{Proj} \operatorname{Bl}_J(R).$$

Using standard properties of Proj, one can show that

$$\begin{tikzcd} \operatorname{Proj} \operatorname{gr}_J(R) \arrow[r] \arrow[d] & \operatorname{Proj} \operatorname{Bl}_J(R) \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

is a fibered diagram, so that the fiber over Z in the blow-up is in fact $\operatorname{Proj} \operatorname{gr}_J(R)$. If Z is nonsingular, this is exactly $\mathbf{P}(\mathcal{N}^\vee_{Z \subset X})$, the projectivization of the conormal bundle. In fact, whenever we have an algebraic vector bundle E on X, its set of global sections M forms an R-module, and the projectivization $\mathbf{P}(E)$ is given by $\operatorname{Proj}(\operatorname{Sym}(M))$, where $\operatorname{Sym}(M)$ is the *symmetric algebra* of M, defined in a 2520 homework assignment. So, since we know that the fiber over Z in the blow-up is $\operatorname{Proj} \operatorname{gr}_J(R)$, we can see that this is the projectivization of the conormal bundle by first realizing that the global sections of the conormal bundle are given by J/J^2, and then realizing (via a detour on *regular sequences*) that since Z is nonsingular, $\operatorname{Sym}^d(J/J^2) = J^d/J^{d+1}$, so in fact $\operatorname{gr}_J(R) = \operatorname{Sym}(J/J^2)$.

Finally, we can see that $\mathrm{Bl}_Z X \to X$ is an isomorphism outside of $\pi^{-1}Z$ as follows. Let $f \in J$, that is f vanishes on Z, so that the distinguished affine open $\operatorname{Spec} R_f \subseteq \operatorname{Spec} R$ is disjoint from Z. Set $S = \{1, f, f^2, \ldots\}$

$$S^{-1}\mathrm{Bl}_J(R) = S^{-1}R \oplus S^{-1}J \oplus S^{-1}J^2 \oplus \cdots = S^{-1}R \oplus S^{-1}R \oplus \cdots \cong S^{-1}R[t],$$

so by standard properties of Proj,

$$\pi^{-1}\operatorname{Spec} R_f = \operatorname{Proj} S^{-1}\mathrm{Bl}_J R = \operatorname{Proj} S^{-1}R[t] \cong \operatorname{Spec} S^{-1}R = \operatorname{Spec} R_f.$$

Since $X \setminus Z$ can be covered by distinguished open affines of the form $\operatorname{Spec} R_f$, this proves that π restricts to an isomorphism $\mathrm{Bl}_Z X \setminus \pi^{-1}Z \to X \setminus Z$.

Worked example: blow-up of $\mathbf{A}^n$ at the origin.

Set $R = k[x_1, \ldots, x_n]$ and take $X = \mathbf{A}^n = \operatorname{Spec} R$ and $I = (x_1, \ldots, x_n)$. Then the blow-up algebra of R along I is given by

$$S = R \oplus I \oplus I^2 \oplus \cdots,$$

with a surjective map of graded rings $R[T_1, \ldots, T_n] \to S$ by $T_i \mapsto x_i$ **in degree** 1. What is the kernel of this map? We note that elements of the form $x_jT_i - x_iT_j$ are mapped to 0, and indeed this is everything, since $x_1, \ldots, x_n$ are algebraically independent. Therefore $\mathrm{Bl}_P \mathbf{A}^n = \operatorname{Proj} R[T_1, \ldots, T_n]/(x_jT_i - x_iT_j) \subseteq \mathbf{P}^n_R = \mathbf{P}^n_k \times_k \operatorname{Spec} R$. The blow-up is covered by the affine charts $D_+(T_s) = \operatorname{Spec} R[T_1/T_s, \ldots, T_n/T_s]/(x_jT_i/T_s - x_iT_j/T_s)$. But since $x_j/x_i = T_j/T_i$ we have that $\mathrm{Bl}_P \mathbf{A}^n$ is covered by n affine charts

$$D_+(T_s) = \operatorname{Spec} k[x_1, \ldots, x_n, x_1/x_s, \ldots, x_n/x_s].$$

Note that each of these charts $D_+(T_s)$ has a morphism to $\mathbf{A}^n$ induced by the inclusion of coordinate rings

$$k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n, x_1/x_s, \ldots, x_n/x_s].$$

In each case, the extension of the ideal $(x_1, \ldots, x_n)$ is principal. This is a manifestation of the universal property of blowing-up: we replaced the origin $(0, \ldots, 0)$ corresponding to the ideal $(x_1, \ldots, x_n)$ with a hypersurface, i.e. a subvariety locally cut out by a principal ideal.

Singularities in the news media.

See `https://www.nytimes.com/2019/02/14/science/math-algorithm-valentine.html`

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