Extending the use of MaxSpec

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Spec and MaxSpec can be understood as functors from the category CRing to Top, and allows a geometric interpretation of rings. In particular, in classical algebraic geometery, it is used in order to study the geometry of zero sets of polynomials. Instead of polynomial rings, however, this note will explore the MaxSpec of rings that are naturally associated to topological spaces, manifolds and measure spaces.

1 MaxSpec and $C(X, \mathbb{R})$

We first recall the definition of MaxSpec and it's associated topology.

Definition 1.1. *Given a commutative ring R, $\operatorname{MaxSpec}(R) := \{m \subset R \mid m \text{ is a maximal ideal }\}$, with a topological structure called the Zariski topology. This topology is generated by sets of the form $V(I) := \{\mathfrak{m} \in \operatorname{MaxSpec}(R) \mid \mathfrak{m} \supset I\}$, where I is an ideal of R. Furthermore, there is a canonical basis of topology of $\operatorname{MaxSpec}(R)$, $\{X_f\}_{f \in R}$, where $X_f = \{\mathfrak{m} \in \operatorname{MaxSpec}(R) \mid \mathfrak{m} \not\ni f\}$.*

Definition 1.2. *Given a topological space X, $C(X, \mathbb{R}) := \{f : X \to \mathbb{R} \mid f \text{ is continuous }\}$, and has a natural ring structure induced by $\mathbb{R}$. Given $f, g \in C(X, \mathbb{R}), (f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ define a ring structure on $C(X, \mathbb{R})$.*

This note was inspired by the following (surprising) result in [1], and follows the proof sketch given in the problem statement.

Proposition 1.3. *Given a compact, Hausdorff space X, $\operatorname{MaxSpec}(C(X, \mathbb{R})) \cong X$*

Proof. Note that for $x \in X$, $\mathfrak{m}_x := \{f \in C(X, \mathbb{R}) \mid f(x) = 0\}$ is a maximal ideal. This follows from the fact that $\mathfrak{m}_x = \ker(\mathrm{eval}_x)$ where $\mathrm{eval}_x : C(X, \mathbb{R}) \to \mathbb{R}$ taking $f \mapsto f(x)$. We will show that in fact, $\mu : X \mapsto \operatorname{MaxSpec}(C(X, \mathbb{R}))$ taking $x \mapsto \mathfrak{m}_x$ is a homeomorphism.

We begin by proving surjectivity of μ. Let $\mathfrak{m} \in \operatorname{MaxSpec}(C(X, \mathbb{R}))$ and,

$$V(\mathfrak{m}) := \{x \in X \mid f(x) = 0,\ \forall f \in \mathfrak{m}\}.$$

Assume for a contradiction that $V(\mathfrak{m})$ is empty. Then, for any $x \in X$, there exists a function $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. By continuity of f_x, there is an open neighborhood $U_x \ni x$ on which f_x is nonvanishing and positive. If f_x is negative for some element in our neighborhood, then we may take $f_x^2 \in \mathfrak{m}$, which is guaranteed to be positive. Repeating this process for any element, we observe that $\bigcup_{x \in X} U_x = X$. By compactness of X there are a finite number of points $x_1, \ldots, x_n \in X$ such that $\bigcup_{i=1}^n U_{x_i} = X$. Therefore, we can define a function $f \in \mathfrak{m}$,

$$f := f_{x_1} + \cdots + f_{x_n}.$$

This is positive on X, since it is the sum of positive functions. However, if f vanishes nowhere, then $\frac{1}{f}$ is continuous on X, implying that f is a unit. This contradicts $f \in \mathfrak{m}$, so we conclude that $V(\mathfrak{m})$ is nonempty. If $x \in V(\mathfrak{m})$, then $\forall f \in \mathfrak{m},\ f(x) = 0$, so $\mathfrak{m} \subset \mathfrak{m}_x$. By maximality of $\mathfrak{m}_x$, $\mathfrak{m} = \mathfrak{m}_x$. We have shown that given an arbitrary maximal ideal $\mathfrak{m}$ of $C(X, \mathbb{R})$ there is an x such that $\mathfrak{m} = \mathfrak{m}_x$, and so we have shown surjectivity.

Now we show injectivity of μ. This requires Urysohn's Lemma, which states that for a compact, Hausdorff space X, and two closed disjoint subsets A, B there is a continuous function $f : X \to [0, 1]$ such that $f(a) = 0,\ \forall a \in A$ and $f(b) = 1,\ \forall b \in B$. Taking this lemma for granted (or checking it to be true by reading [3]), this gives us the injectivity condition, $x \neq y \implies \mathfrak{m}_x \neq \mathfrak{m}_y$. This is because $\{x\}, \{y\}$ are closed and disjoint when $x \neq y$, implying that there is a function $f \in C(X, \mathbb{R})$ such that $f(x) = 0$ and $f(y) = 1$. Therefore, $f \in \mathfrak{m}_x$, but $f \notin \mathfrak{m}_y$ as desired.

Now we show that this μ is a homeomorphism We have already discussed that X_f form a basis of topology for $\operatorname{MaxSpec}(C(X, \mathbb{R}))$, which can be written as,

$$X_f = \{\mathfrak{m} \in \operatorname{MaxSpec}(R) \mid \mathfrak{m} \not\ni f\} = \{\mathfrak{m}_x \in \operatorname{MaxSpec}(R) \mid \mathfrak{m}_x \not\ni f\}.$$

Consider the set $U_f = \{x \in X \mid f(x) \neq 0\}$. We claim that U_f forms a basis of topology. The zero function $0 : x \mapsto 0$ and the constant function $1 : x \mapsto 1$ imply that $X, \emptyset \in \{U_f \mid f \in C(X, \mathbb{R})\}$. All we need to show is that given U_f and U_g for every point $p \in U_f \cap U_g$ there is an U_h such that $p \in U_h \subset U_f \cap U_g$. But, $U_f \cap U_g$. However,

$$U_{fg} = \{x \in X \mid f(x)g(x) \neq 0\} = \{x \in X \mid f(x), g(x) \neq 0\} = U_f \cap U_g,$$

and so we conclude that U_f generate a basis of topology for X. Now, we show $\mu(U_f) = X_f$. Since $\mu(x) = \mathfrak{m}_x$

$$\mu(U_f) = \mu(\{x \in X \mid f(x) \neq 0\}) = \{\mathfrak{m}_x \mid \mathfrak{m}_x \not\ni f\} = X_f$$

We already know that μ is bijective, and any open set can be constructed from the basis of topology, so this proves μ is actually a homemorphism. $\square$

In hindsight, this proposition is not extremely surprising, as it is essentially attempting to extend the Weak Nullstellensatz from the ring of polynomials to the ring of continuous functions, but having to impose additional conditions. Recall that the Weak Nullstellensatz Theorem says that $\operatorname{MaxSpec}(\mathbb{C}[x_1, \ldots, x_n]) \cong \mathbb{C}^n$. In this case, $\mathbb{C}[x_1, \ldots, x_n]$ can be considered a subring of the much larger ring, $C(\mathbb{C}^n, \mathbb{C})$. Here, the parallels become a little clearer – we would like to claim that $\operatorname{MaxSpec}(C(\mathbb{C}^n, \mathbb{C}))$ is a recognizable space, but we are forced to consider $\operatorname{MaxSpec}(C(B_r(0)^n, \mathbb{R})) \cong B_r(0)^n$ instead. Here, $B_r(0) = \{z \in \mathbb{C} \mid |z| \leq r, r \in \mathbb{R}\}$.

Recall that we require that X be compact and Hausdorff, and the ring of functions evaluate to the real numbers in this proposition. However, it seems that these restrictions can be relaxed when considering compactifications.[1]

This proposition seems to imply that we can use tools from algebraic geometry study the zero locus of continuous functions on sufficiently nice topological spaces, such as compact manifolds. In fact, we can say more about the allowable functions on a manifolds, which we will explore in a later section.

Even though the proposition states that $\operatorname{MaxSpec}(C(X, \mathbb{R})) \cong X$, how strong is this equivalence? In order to rigorously pose and answer this question, we turn to category theory. We want to know if

[1]There are many different compactifications of a space and in fact, we can consider compactifications as functors. We can consider the category of these compactifications, with the morphisms being "inclusion." In other words, for two compactifications $\mathfrak{C}_0, \mathfrak{C}_1 : \text{HausTop} \to \text{HausCmptTop}$ there is an arrow between $\mathfrak{C}_0 \to \mathfrak{C}_1$ if $\mathfrak{C}_0(X) \subset \mathfrak{C}_1(X)$ for $X \in \text{HausTop}$. There is a final object in this category of compactifications, $\mathfrak{C}_{SC}$, called the Stone-Čech compactification. This is also called the maximal compactification. Using this, it seems as though we can extend the proposition as: Given a Hausdorff space X, $\operatorname{MaxSpec}(C(\mathfrak{C}_{CS}(X), \mathfrak{C}_{CS}(\mathbb{R})) \cong \mathfrak{C}_{CS}(X)$. I'd like to thank Riley Levy for introducing me to this maximal compactification.

morphisms are also preserved, but this only makes sense in the context where $C_{\mathbb{R}} : \mathrm{Top} \to \mathrm{CRing}$ is a functor that takes $X \to C(X, \mathbb{R})$. We check that given a continuous function $f : X \to Y$, there is an induced map $f^* : C(Y, \mathbb{R}) \to C(X, \mathbb{R})$ by precomposing f. In other words, $f^*(g) = g \circ f$ for $g \in C(Y, \mathbb{R})$. We claim that this association of morphisms satisfies functoriality. For,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \ ,$$

given an element $h \in C(Z, \mathbb{R})$,

$$(g \circ f)^*(h) = h \circ g \circ f = f^*(h \circ g) = f^*(g^*(h)) = f^* \circ g^*(h).$$

Furthermore, $\mathrm{id}_X^*(f) = f \circ \mathrm{id}_X = f$ and so $\mathrm{id}_X^* = \mathrm{id}_{C(X,\mathbb{R})}$ as desired. We conclude that $C_{\mathbb{R}}$ is a contravariant functor. Now, we examine where morphisms $f : X \to Y$ of compact Hausdorff spaces are sent through the functors $C_{\mathbb{R}}$ and MaxSpec.

Proposition 1.4. *There is a natural transformation between*

$$\mathrm{MaxSpec} \circ C_{\mathbb{R}} : \mathrm{HausCmptTop} \to \mathrm{HausCmptTop}$$

and $1 : \mathrm{HausCmptTop} \to \mathrm{HausCmptTop}$ *the identity functor.*

Proof. We have already established that $C_{\mathbb{R}}$ sends $f \to f^*$ where $f^*(g) = g \circ f$. We have already studied how the functor MaxSpec acts on morphisms, $\mathrm{MaxSpec} : f^* \mapsto \bar{f}^*$ where $\bar{f}^* : \mathfrak{m}_x \mapsto f^{*-1}(\mathfrak{m}_x)$. Note that,

$$f^{*-1}(h) = \{g \mid g \circ f = h\}.$$

We conclude that

$$f^{*-1}(\mathfrak{m}_x) = f^{*-1}(\{h \mid h(x) = 0\}) = \{g \mid g \circ f = h, h(x) = 0\} = \{g \mid g \circ f(x) = 0\} = \mathfrak{m}_{f(x)}$$

Under the homeomorphisms μ_X, μ_Y, we conclude that $\mu_Y^{-1} \circ \bar{f}^* \circ \mu_X = f$. This implies that

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mu_X"'] & Y \arrow[d, "\mu_Y"] \\
\mathrm{MaxSpec}\, C(X, \mathbb{R}) \arrow[r, "\bar{f}^*"'] & \mathrm{MaxSpec}\, C(X, \mathbb{R})
\end{tikzcd}$$

which is the commutative diagram we desire for a natural transformation. □

A natural question to ask given this information is if there is a natural transformation between $C_{\mathbb{R}} \circ \mathrm{MaxSpec}$ and the identity functor on a subcategory of CRing. In other words, given a ring R with sufficiently nice properties, such as Noetherianity, does $C(\mathrm{MaxSpec}\,(R)\,, k) \cong R$ for k a field? This does not seem to hold. We check two examples.

Example 1.5. *Consider a local ring* $(R, \mathfrak{m})$. *By definition,* $\mathfrak{m}$ *is the only maximal ideal, and so* $\mathrm{MaxSpec}\,(R) = \{\mathfrak{m}\}$. $C(\{*\}) \cong \mathbb{R}$ *for a singleton set, but not all local rings are* $\mathbb{R}$. *We conclude that* $C(\mathrm{MaxSpec}\,(R)) \ncong R$ *for* R *an arbitrary local ring.*

Example 1.6. *By the Weak Nullstellensatz,* $\mathrm{MaxSpec}\,(\mathbb{C}\,[x_1, \ldots, x_n]) \cong \mathbb{C}^n$. *However,* $C(\mathbb{C}^n, \mathbb{R}) \ncong \mathbb{C}\,[x_1, \ldots, x_n]$, *since polynomials with complex coefficients generally do not evaulate to real values.*

2 Extending to Manifolds and Measure Spaces

Recalling the structure of the proof of propostion 1.3, we note that for surjectivity, we need compactness of X, continuity of f, and the fact that if $f \in \mathfrak{m}$, $f^2 \in \mathfrak{m}$ and f^2 is nonegative for any f. This last fact requires that the field we map to is $\mathbb{R}$. For surjectivity, we needed Urysohn's Lemma to show that there are functions that separate points, which itself required X to be Hausdorff and compact. In the last step, to show that μ was a homeomorphism, we used the bijectivity of μ and the fact that it sent one basis of topology to another. We claim that this argument extends to compact manifolds.

Proposition 2.1. *For a compact, C^r–manifold X, define $C^r(X, \mathbb{R})$ to be functions from $X \to \mathbb{R}$ that are continuously differentiable r times.*

$$\operatorname{MaxSpec}\left(C^0(X, \mathbb{R})\right) \cong \cdots \cong \operatorname{MaxSpec}\left(C^r(X, \mathbb{R})\right) \cong X$$

This proposition essentially states that MaxSpec does not detect smoothness of functions.

Proof. $\mathfrak{m}_x^{(r)} := \{f \in C^r(X, \mathbb{R}) \mid f(x) = 0\}$ still is a maximal ideal since it is the kernel of the surjective ring homomorphism eval_x. We will prove that the map $\mu_r : x \mapsto \mathfrak{m}_x^{(r)}$ is still a homeomorphism by mostly appealing to the proof of Proposition 1.3. Since C^r functions are also continuous, the C^r functions map into $\mathbb{R}$, and X is compact, we conclude that μ_r is surjective. For injectivity of μ_r, we appeal to the Smooth Urysohn's Lemma, proven in [2]. To prove that μ_r and it's inverse is continuous, we appeal again to the proof of Proposition 1.3, as it will be identical. We conclude the proposition is true. $\square$

The most surprising fact about this proposition are the homeomorphisms,

$$\operatorname{MaxSpec}\left(C^0(X, \mathbb{R})\right) \cong \cdots \cong \operatorname{MaxSpec}\left(C^r(X, \mathbb{R})\right).$$

Yet, there is a proposition that says that the space of smooth (or differentiable) functions is dense in the space of continuous functions on a compact space. This might be enough evidence to believe this proposition.

Looking at differentiablilty, it might also be worth looking at the extention of the proposition to include integrability, but the argument does not seem to hold. We investigate this in detail. Given a compact measure space $(X, \mathscr{A}, m)$, the natural ring associated to this measure space is $\mathcal{L}^\infty(X)$.[2] Recall that this is defined to be,

$$\mathcal{L}^\infty(X) := \{f \mid ||f||_\infty < \infty\}, \text{ where } ||f||_\infty = \inf\{C \geq 0 \mid |f(x)| \leq C \text{ for almost every } x \in X\}.$$

When we try to apply the proof method in Proposition 1.3, injectivity certainly holds, since we can separate points with indicator functions. For $x \neq y$, there are two disjoint open sets U_x, U_y that separate these points. These sets are measurable if m is a Borel measure, or even better, a Radon measure. We can then take the indicator function on U_x to get our desired separating function. However, when trying to show injectivity, there is some trouble. A $\mathcal{L}^\infty$ function is not guaranteed to be continuous, and furthermore, if f is not continuous, we are not guaranteed that the maximal ideal that f is contained in has continuous functions that sufficiently approximate f.

[2]Unfortunately, $\mathcal{L}^p(X)$ is not a ring in general. Consider the following function, on $X = [0, 1]$ with the Lebesgue measure.

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

For this function, $f \in \mathcal{L}^1(X)$, but $f^2 \notin \mathcal{L}^1(X)$. Taking roots of this function, we can construct the same type of counterexample for each $\mathcal{L}^p(X), p < \infty$.

One possible fix is to change the ring entirely by switching the multiplication on the ring from value-wise multiplication to convolution. The downside to this is that the convolution generally does not have a unit, making the ring (or "rng" if you would like) harder to work with.

References

[1] Atiyah and Macdonald. *Introduction to Commutative Algebra.* Addison-Wesley, Reading, Massachusetts, 1969.

[2] Peter Petersen. *Manifold Theory.* http://www.math.ucla.edu/~petersen/manifolds.pdf

[3] Jonathan Simon. *Urysohn's Lemma.* http://homepage.divms.uiowa.edu/~jsimon/COURSES/M132Fall07/UrysohnLemma_v5.pdf