MATH 2520 ALGEBRA SPRING 2020
ALL PROBLEM SETS

Problem Set 1. Due Monday January 27

Please submit **this problem set only** to me by email. Starting with Problem Set 2, please submit hard copies, LaTeX strongly preferred.

1. Please tell me the following about yourself:
   (a) Your name
   (b) Your pronouns. For example, mine are she/her/hers.
   (c) A hobby or activity you enjoy
   (d) Anything else you'd like to tell me or ask me.
2. Optional, to inform my planning: what undergraduate or graduate classes have you taken in algebra, geometry, or topology?
3. Read the following articles.
   (a) this article by Federico Ardila: `https://bit.ly/35HihI4`[1]
   (b) this speech by Francis Su: `https://bit.ly/2jCqtSQ`

   I am curious to know: **What does "doing mathematics ethically" mean to you?** In fact, what does it mean to you in your life and your decisions as a student right now?

   This question is an invitation to recognize the power you carry as a mathematician, and the privilege and responsibility that comes with it. When you enter a scientific career, you do not leave yourself at the door. You can choose how to use that power. My hope is that you will always continue to think about this in your work.[2]

Problem Set 2. Due Monday February 3, hard copy, in class

1. For each of the following categories, determine the initial object, if one exists. Please explain.
   (a) Fields of characteristic p for a fixed prime p, with field homomorphisms.
   (b) Fields of any characteristic, with field homomorphisms.

[1] If you are interested in more resources, please check out

`http://math.sfsu.edu/federico/ethicsinmath.html`

[2] Source: F. Ardila-Mantila, CAT(0) *geometry, robots, and society*, preprint, 2019.

(c) For a fixed topological space X, the category $\mathrm{Open}(X)^{\mathrm{op}}$. Here $\mathrm{Open}(X)$ has objects all open subsets of X and a single morphism from U to V if $U \subseteq V$, and the superscript $^{\mathrm{op}}$ denotes the opposite category.

(d) The coslice category $x \downarrow \mathcal{C}$, for any category $\mathcal{C}$ and object x of $\mathcal{C}$.

(e) For R a commutative ring and $x \in R$, the category whose objects are homomorphisms $R \xrightarrow{f} S$ where S is a commutative ring and $f(x) = 0$, with a morphism from $R \xrightarrow{f} S$ to $R \xrightarrow{f'} S'$ for ring homomorphisms $S \xrightarrow{\varphi} S'$ making the triangle below commute.

$$\begin{array}{ccc} & R & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f'} \\ S & \xrightarrow{\varphi} & S' \end{array}$$

(f) Let $\mathrm{FdVect}_{\mathrm{k}}$ denote the groupoid of finite dimensional k-vector spaces, with *invertible* linear maps.

Recall that we may regard a set C as a category which we shall denote $\underline{C}$, with objects elements of C and with no nonidentity morphisms.

Now consider the category whose objects are functors $\mathrm{FdVect}_{\mathrm{k}} \xrightarrow{F} \underline{C}$ for C a set, and whose morphisms are commuting triangles

$$\begin{array}{ccc} & \mathrm{FdVect}_{\mathrm{k}} & \\ {\scriptstyle F}\swarrow & & \searrow{\scriptstyle F'} \\ \underline{C} & \xrightarrow{\varphi} & \underline{C'} \end{array}$$

2. The following categories $\mathcal{C}$ have natural forgetful functors $\mathcal{C} \to \mathrm{Set}$ sending an object to its underlying set. Identify left adjoints to these forgetful functors. Brief explanations are fine.

(a) abelian groups

(b) posets. Objects are posets $(P, \leq_P)$, and morphisms from $(P, \leq_P)$ to $(Q, \leq_Q)$ are functors from P to Q regarded as categories. That is, a morphism is a map $f\colon P \to Q$ such that $x \leq_P x'$ implies $f(x) \leq_Q f(x')$.

(c) topological spaces

(d) commutative rings (we always require that our rings have 1 and that morphisms send 1 to 1)

(e) rings (still with 1 as above)

Problem Set 3. Due Monday February 10, hard copy, in class

1. Explain the following constructions as functors from one category to another, and identify them as left adjoints in an adjoint pair. Brief explanations are fine.

   (a) The *abelianization* $G/[G,G]$ of a group G. Here $[G,G] = \langle ghg^{-1}h^{-1} : g,h \in G\rangle$ denotes the commutator subgroup of G.

   (b) The torsion-free quotient A/T of an abelian group A. Here T denotes the elements of A of finite order.

   (c) The group ring $\mathbb{Z}[G]$ of a group G

2. Regarding a group G as a groupoid with one object, a functor $F\colon G \to \mathrm{Set}$ is more commonly known as an action of G on a set. What are the limit and colimit of F, in terms of this action?

3. Let $\mathcal{C}$ be a small category. Prove that $\mathcal{C}$ is equivalent to a poset iff for all objects x, y in $\mathcal{C}$, $|\mathrm{Mor}_{\mathcal{C}}(x,y)| = 0$ or 1.

4. Define a functor $\mathcal{Z}\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ sending a topological space X to the set of closed subsets of X. (What does $\mathcal{Z}$ do on morphisms?) Is $\mathcal{Z}$ representable?

5. Define a functor $F\colon \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ sending a set S to

   $$\{(A,B,C) \in \mathcal{P}(S) \times \mathcal{P}(S) \times \mathcal{P}(S) : A \subseteq C, B \subseteq C\}.$$

   (What does F do on morphisms?) Is F representable?

6. Do Vakil Exercise 1.6.E for covariant functors. In other words: let $F\colon \mathcal{A} \to \mathcal{B}$ be a (covariant) additive functor of abelian categories. Verify that if F is exact (meaning both left and right exact), then $FA' \to FA \to FA''$ is exact in $\mathcal{B}$ whenever $A' \to A \to A''$ is exact in $\mathcal{A}$.

Problem Set 4. Due Wednesday February 19, hard copy, in class

1. (Adjointness of extension and restriction) Let $f\colon A \to B$ be a ring homomorphism, M an A-module, N a B-module. Exhibit a bijection

   $$\mathrm{Mor}_{B\text{-mod}}(B \otimes_A M, N) \cong \mathrm{Mor}_{A\text{-mod}}(M, N).$$

   You should also convince yourself that this is a *natural* bijection in both M and N, but you don't have to prove naturality.

2. Let M be an A-module and $I \subset A$ an ideal. Prove[3] that

   $$M \otimes_A A/I \cong M/IM$$

   as A/I-modules. Here $IM = \langle r \cdot m\colon r \in I, m \in M\rangle \subseteq M$.

[3] Use universal properties, or consider the exact sequence $I \to A \to A/I \to 0$, as suggested in Vakil 9.2.E.

3. Let $f\colon R \to S$ be a ring homomorphism. Prove, without using the construction of tensor products of algebras but only using universal properties, that $S \otimes_R R[t] \cong S[t]$ as R-algebras.

4. Let M be a finitely generated R-module, and $f\colon M \to M$ an R-linear map. Regard M as an $R[t]$-module by setting $t \cdot m = f(m)$.

   (a) Prove that $(t)M = M$ iff f is an isomorphism.

   (b) Does (a) still hold without the hypothesis that M is finitely generated?

5. Consider the ring $R = k[x^2, y^2] \subset k[x, y]$, and let $f = x^3+y^3-1$. Find $n > 0$ and $r_1, \ldots, r_n \in R$ such that
$$f^n + r_1 f^{n-1} + \cdots + r_n = 0 \in k[x, y].$$

6. For R a ring and $a, b \in R$, prove $R_{ab} \cong (R_a)_{b/1}$.

Problem Set 5. Due Monday February 24, hard copy, in class (2+2+1+3 points)

1. (Vakil Exercise 9.2.F) Suppose $\phi\colon B \to A$ is a ring morphism, and $S \subseteq B$ is a multiplicative subset of B. Prove that there is an isomorphism of rings
$$\phi(S)^{-1}A \cong A \otimes_B (S^{-1}B).$$

2. (Chinese Remainder Theorem) Let R be a ring, and let $Q_1, \ldots, Q_n$ be ideals of R such that $Q_i + Q_j = R$ for all $i \neq j$. Show[4] that $R/(\cap_i Q_i) \cong \prod_i R/Q_i$ as follows:

   (a) Consider the map of rings $\phi\colon R \to \prod_i R/Q_i$ obtained from the n projection maps $R \to R/Q_i$. Show that $\ker \phi = \cap_i Q_i$.

   (b) Let $\mathfrak{m}$ be a maximal ideal of R. Show that the hypothesis that $Q_i + Q_j = R$ for all $i \neq j$ means that at most one of the Q_i is contained in $\mathfrak{m}$. Now use [A&M Proposition 3.9] to show that ϕ is surjective.

Let A be an abelian group. An A-graded ring is a ring R with a direct sum decomposition
$$R = \bigoplus_{a \in A} R_a$$
of abelian groups, such that $R_a R_b \subseteq R_{a+b}$. The elements of R_a are called *homogeneous* of degree a, and an ideal $I \subseteq R$ is called *homogeneous* if it is generated by homogeneous elements.[5]

3. Prove that I is homogeneous iff for every element $f \in I$, the homogeneous components of f are in I.

4. Suppose A admits a total order $\leq$ such that $a \leq a'$ implies $a + b \leq a' + b$ for all $b \in A$.

[4] This exercise appears in Eisenbud's *Commutative Algebra*.

[5] Adapted from Eisenbud Exercise 2.14, 2.15. Two good examples are (1) the $\mathbb{Z}$-grading on $S[x_1, \ldots, x_n]$ by total degree, and (2) the $\mathbb{Z}^n$-grading on $S[x_1, \ldots, x_n]$ given by $\deg x_1^{a_1} \cdots x_n^{a_n} = (a_1, \ldots, a_n)$.

(a) Prove that the radical of a homogeneous ideal is homogeneous.

(b) Let $I \neq (1)$ be a homogeneous ideal. Suppose that for any homogeneous elements $f, g \in R$, $fg \in I$ implies $f \in I$ or $g \in I$. Prove that I is prime.

Problem Set 6. Due Monday March 2, hard copy, in class

1. A *monomial* is a polynomial of the form $x_1^{a_1} \cdots x_n^{a_n}$. A *monomial* ideal in $k[x_1, \ldots, x_n]$ is one generated by monomials. In other words, an ideal is *monomial* if it is homogeneous with respect to the $\mathbb{Z}^n$-grading $\deg x_1^{a_1} \cdots x_n^{a_n} = (a_1, \ldots, a_n)$.

   (a) Which monomial ideals are prime?

   (b) Which monomial ideals are radical?

2. Let $S \subset A$ be a multiplicative subset of a ring A, and let $f\colon A \to S^{-1}A$ denote the localization map. Is it ever the case that the ideals of $S^{-1}A$ are in bijection with the ideals of A, via contraction—other than the case that f is an isomorphism $A \to A = S^{-1}A$?

3. Let $f\colon A \to B$ be a ring homomorphism. Suppose $x, y \in B$ satisfy monic polynomials over A of degrees m and n respectively. Does it follow that xy satisfies a monic polynomial over A of degree mn?

4. A&M Exercise 5 on p. 44

5. A&M Exercise 2 on p. 67.

Problem Set 7. Due Monday March 9, hard copy, in class (2+3+1.5+1.5+2 points)

1. (Noether normalization) Exhibit, with proof, a finite injective map

$$k[t_1, \ldots, t_d] \to k[x, y, z]/(xy, xz, yz)$$

   for some $d \geq 0$. Draw a picture.

2. For any ideal I of a ring R and $f \in R$, define the *saturation* to be the ideal

$$(I : f^\infty) = \{g \in R : gf^N \in I \text{ for some } N\}.$$

   The following problem shows that "saturation erases zeroes of f." Let k be an algebraically closed field. Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal and $f \in k[x_1, \ldots, x_n]$. Show that

$$Z((I : f^\infty)) = \overline{Z(I) \setminus Z(f)}.$$

   Here the overline denotes closure in the Zariski topology. Also, make up your own nontrivial example, and draw a picture.

3. Classify the points of the following spaces, and describe the closure of each point in the Zariski topology. Draw pictures as well as you can. Just the answers are fine; no proofs needed.

   (a) $\operatorname{Spec} \mathbb{C}[x, y]/(xy)$
   (b) $\operatorname{Spec} \mathbb{C}[x, y]/(x^2 y)$
   (c) $\operatorname{Spec} (\mathbb{C}[x, y]/(xy))_{(x,y)}$
   (d) $\operatorname{Spec} \mathbb{C}[\![x]\!]$

4. For the following ring maps $\phi \colon R \to S$, determine the corresponding maps $f \colon \operatorname{Spec} S \to \operatorname{Spec} R$ are, e.g., by describing what f does on each point of $\operatorname{Spec} S$. Draw pictures as well as you can. Just the answers are fine; no proofs needed.

   (a) The $\mathbb{C}$-algebra map $\mathbb{C}[t] \to \mathbb{C}[x, y]/(xy)$ sending t to x.
   (b) The $\mathbb{C}$-algebra map $\mathbb{C}[t] \to \mathbb{C}[x, y]/(xy)$ sending t to $x + y$.
   (c) The natural map $\mathbb{R}[x] \to \mathbb{C}[x]$.

5. A&M Exercise 17 on p. 12. We should have already proved that X_f are a basis of open sets, so you don't need to do that.

Problem Set 8. Due Monday March 30 by 2pm sharp, by email to

`melody_chan+2520@brown.edu`

(1+3+2+2+2 points)

1. A&M Exercises 19 and 20 on p. 13.

2. For each of the following rings R, describe the distinguished opens in $\operatorname{Spec} R$, and the value of the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ on the distinguished opens. Do it as simply and explicitly as possible. No proofs are needed.

   (a) $\mathbb{C}[x]/(x^3 - x^2)$
   (b) $\mathbb{C}[\![x]\!]$
   (c) $(\mathbb{C}[x, y]/(xy))_{(x,y)}$

3. Give an example, with proof, of an ideal having exactly three associated primes p_0, p_1, and p_2 with $p_0 \subsetneq p_1 \subsetneq p_2$. Draw a picture as well as you can.

4. Let Δ be a *simplicial complex* on $\{1, \ldots, n\}$. That is, Δ is a set of subsets of $\{1, \ldots, n\}$, closed under taking subsets. A *facet* is a maximal element of Δ.

   Let $A = k[x_1, \ldots, x_n]$, and let

$$I_\Delta = \langle \prod_{i \in \sigma} x_i : \sigma \subseteq [n], \sigma \notin \Delta \rangle.$$

Prove that

$$I_\Delta = \bigcap_S \langle x_i : i \in [n] \backslash S \rangle ,$$

as S runs over all facets of Δ, is a primary (in fact prime) decomposition of I_Δ. Example: if Δ has facets $\{1\}, \{2\}, \{3\}$ then we obtain $(xy, yz, xz) = (x, y) \cap (y, z) \cap (x, z)$.

Problem Set 9. Due Monday April 6 by 2pm sharp, by email to

`melody_chan+2520@brown.edu`

(2+0+3+2+2 points)

1. (Normalization of the Whitney umbrella) Compute, with proof, the normalization of the domain

$$\mathbb{C}[x, y, z]/(zy^2 - x^2).$$

Draw a picture of the normalization map as well as you can; try just drawing the map on the real points.

2. (Ungraded, don't hand in) Make sure you are happy with the statement that a module is projective iff it's a direct summand of a free module.

3. Exercise 4.11 on p. 136 from Eisenbud's *Commutative Algebra with a View Toward Algebraic Geometry*.[6]
   - For part (a), you may prove the first sentence only.
   - For part (b), you may use Proposition 2.10 without proof.

4. Study on your own Artinian rings, A&M pages 89-90. Now do Problems 2 and 3 on page 92.[7]

Problem Set 10 on the next page.

[6] Available from `https://search.library.brown.edu/catalog/b8048798` on the library website. The solution is outlined in a hint on p. 729; feel free to study the hint, but make sure to fill in all the details on your own. In the hint, P denotes the unique maximal ideal of R.

[7] For problem 3, I think the hint "use the fact that A is of finite length" can be rephrased as "use (8.6)."

Problem Set 10. Due Monday April 13 by 1pm sharp, by email to

`melody_chan+2520@brown.edu`

(2+5+2+2 points)

1. (Tor localizes) Let B be a flat A-algebra; for example, $B = S^{-1}A$ could be a localization of A. Prove
$$B \otimes_A \operatorname{Tor}_i^A(M, N) \cong \operatorname{Tor}_i^B(B \otimes_A M, B \otimes_A N)$$
for any A-modules M and N.

2. Eisenbud Exercise 6.2 on p. 172. Some comments:
   - For step (a), see the hint at the back of the book.
   - Step (b) is already done: we proved that flatness is a local property in A&M Proposition 3.10.
   - For step (c), I think "i implies ii" should say "i implies iii."

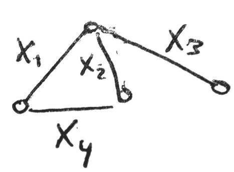
3. Let R be a principal ideal domain and $f\colon R \longrightarrow S = R[x_1, \ldots, x_n]/I$ a finite type R-algebra. Prove that S is flat over R iff $f^{-1}(p) = 0$ for every associated prime p of I.

   Which of the three ring maps in Problem Set 7, Problem 4 are flat? (Make sure your answers agree with the pictures you drew there.)

4. Let Δ be a simplicial complex. Recall we can regard simplicial complexes as topological spaces (via geometric realization), as explained in Wednesday's class presentation.

   Let $X_1, \ldots, X_t$ be subcomplexes of Δ with $\bigcup_{i=1}^t X_i = \Delta$ and $\bigcap_{i=1}^t X_i = \emptyset$. Suppose that for all nonempty $I \subseteq \{1, \ldots, t\}$, $\bigcap_{i \in I} X_i$ is either empty or contractible.

   Let P be the poset whose elements are the subspaces of Δ the form $\left\{\bigcap_{i \in I} X_i \mid I \subseteq \{1, \ldots, t\}\right\}$, ordered by reverse inclusion. So P has unique minimal and maximal elements Δ and $\emptyset$, respectively. Prove[8]
$$\chi(\Delta) - 1 = \mu_P(\Delta, \emptyset).$$



(An example to try.)

[8]This is a form of the "Nerve lemma," or computing Euler characteristics by a beefed up inclusion-exclusion.

Problem Set 11. Due Monday April 20 by 1pm sharp Eastern Time, by email to

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(5+4+2 points)

1. Let $I = (f_1, \dots, f_m)$ be an ideal of $A = k[x_1, \dots, x_n]$, where k is a field. Let $T = k[\epsilon]/(\epsilon^2)$. Given $g_1, \dots, g_m \in A$, prove that

$$T \longrightarrow R = T[x_1, \dots, x_n]/(f_1 + \epsilon g_1, \dots, f_m + \epsilon g_m)$$

makes R flat over T iff there exists an A-module map $I \to A/I$ taking f_i to g_i for each i.

Either do your own thing, or follow the hints below.

(a) Prove that for an arbitrary T-module M,

$$(\epsilon) \otimes_T M \cong M/\epsilon M.$$

Also, prove that a T-module M is flat iff $(\epsilon) \otimes_T M \to M$ is injective.

(b) Using (a), prove that for any $f \in A$, we have $f \in I$ iff $\epsilon \otimes f = 0$ in $(\epsilon) \otimes_T R$.

(c) Suppose $T \to R$ is flat, and define $\phi \colon I \to A/I$ by $\phi(\sum a_i f_i) = \sum a_i g_i + I$. Using (b), prove ϕ is well-defined.

(d) Conversely, suppose given an A-module map $\phi \colon I \to A/I$ with $\phi(f_i) = g_i$. Suppose $\epsilon f = 0 \in R$ for $f \in A$. Write

$$\epsilon f = \sum (a_i + \epsilon b_i)(f_i + \epsilon g_i),$$

equate constant and ϵ terms, and use $\phi(f_i) = g_i$ to deduce $f \in I$.

2. (First order deformations.) A flat map $T \to R$ as above is called a *first-order deformation* of A/I; you showed that $\mathrm{Hom}_A(I, A/I)$ is the space of first-order deformations.

   Let $A = k[x_1, \dots, x_n]$ and let $I = (x_1, \dots, x_n)^2$. Then the space of first-order deformations of A/I is a k-vector space. Compute, with proof, the dimension of this vector space in terms of n.[9]

3. Let $S = k[x_1, \dots, x_n]$. Write down a free resolution of the graded S-module $k = S/(x_1, \dots, x_n)$. Compute the Betti numbers $\beta_{i,j}$ of k, and compute $\dim_k \mathrm{Tor}_i^S(k, k)$. Feel free to do this just for $n = 4$ if you prefer. No proofs are needed; just the answers are fine.

[9] This vector space is exactly the tangent space at I to the Hilbert scheme of $d = n + 1$ points in $\mathbb{A}^n$. The computation for $n > 2$ furnishes examples of singular points of such a Hilbert scheme. Ask your friendly neighborhood algebraic geometer for more details.

Problem Set 12. Due Monday April 27 by 1pm sharp Eastern Time, by email to

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(3+3+2+2+2 points)

1. Let $S = k[x_1, \dots, x_n]$, and let I and J be homogeneous ideals of S. Prove, or disprove,

$$d(S/(I \cap J)) = \max d(S/I), d(S/J).$$

2. Let Δ be a *simplicial complex* on $\{1, \dots, n\}$. An element $F \in \Delta$ is called a *face*, and its *dimension* is $\dim F = |F| - 1$. The dimension of Δ is the maximum dimension of a face in Δ. Let f_d denote the number of faces in Δ of dimension d.

   Let $S = k[x_1, \dots, x_n]$, and recall the definition of I_Δ from Problem Set 8.

   (a) Prove[10] that the Poincaré series of S/I_Δ is

   $$F_{S/I_\Delta}(t) = \frac{1}{(1-t)^n} \sum_{i \geq 0} f_{i-1} t^i (1-t)^{n-i}.$$

   (b) Using (a), compute $d(S/I_\Delta)$ in terms of Δ. Verify agreement with Problem 4 on Problem set 8.

3. Let $B = k[x_1, \dots, x_d]/I$ and $m = (x_1, \dots, x_d)$, and let $A = B_m$. Prove that $G_m(A)$ is isomorphic as a graded ring to

$$k[x_1, \dots, x_d]/\langle \mathrm{in}(f) \colon f \in I \rangle.$$

   Here, $\mathrm{in}(f)$ denotes the sum of all terms of f of lowest degree.

4. Let $S = k[x_1, \dots, x_n]/(f)$. Prove[11] that

$$\Omega_{S/k} \cong (\textstyle\bigoplus_{i=1}^n S dx_i)/(df)$$

   where $df = \sum \frac{\partial f}{\partial x_i} dx_i$.

5. A&M Exercise 1 on p. 125.

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[10] Possible hint: prove the following lemma. For $\sigma \subseteq \{1, \dots, n\}$, say a monomial in the x_i is *supported* on σ if the variables appearing are exactly $\{x_i \colon i \in \sigma\}$. Then

$$\sum_{i \geq 0} t^i \cdot \{\#\text{degree } i \text{ monomials supported on } \sigma\} = \frac{t^{|\sigma|}}{(1-t)^{|\sigma|}}.$$

For more, see Stanley's *Combinatorics and commutative algebra*.

[11] A proof sketch was given in the class presentation on April 17. For a reference, see Eisenbud p 386.