GRAPH COMPLEXES AND THE TOP-WEIGHT EULER CHARACTERISTIC OF $\mathcal{M}_{g,n}$

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Contents

These are notes for a lecture series at ATOM. The main reference for the material below is [CFGP19].

1. Graphs

Definition 1.1. A *graph* G is:

- finite sets $V(G)$ and $H(G)$, whose elements are called *vertices* and *half-edges*,
- a fixed-point free involution $s : H(G) \to H(G)$ (the "other half edge" map), and
- a map $r : H(G) \to V(G)$ ("root map").

So loops and multi-edges are allowed, in that they were never disallowed. Let $E(G) = H(G)/(x \sim s(x))$; its elements are *edges*.

An isomorphism $G \to G'$ is what it must be: a bijection $V(G) \to V(G')$ and $H(G) \to H(G')$ compatible with the involutions s, s' and the root maps r, r'.

Definition 1.2. An n-marking of G is a function $\{1, \dots, n\} \to V(G)$.

Definition 1.3. (G, m) is called *stable* if $|r^{-1}(v) \amalg m^{-1}(v)| \geq 3$ for all $v \in V(G)$, and *connected and genus* g if the geometric realization $|G|$ is connected and has Euler characteristic $1 - g$.

Remark 1.4. Argue that there are only finitely many isomorphism classes of stable, marked (G, m) of type (g, n).

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Exercise 1.5. Enumerate the isomorphism classes of stable, 2-marked graphs of genus 1.

How many, exactly? When $g = 0$, you can say a lot: the sequence for $n = 3, 4, 5, \ldots$ is

$1, 4, 26, 236, 2752, 39208, 660032, 12818912, 282137824, 6939897856, 188666182784, 5617349020544, \ldots$

(OEIS A000311. These are also counts of *phylogenetic trees* on $n-1$ taxa)

In complete generality, I don't know. For $n = 0$, at least, the sequence for $g = 2, 3, 4, 5, \ldots$ is

$$7, 42, 379, 4555, 69808, 1281678, 27297406, ??$$

(OEIS A174224, Maggiolo-Pagani, also see my thesis)

But a closely related enumerative problem–that of the "orbifold Euler characteristic" of Kontsevich's graph complex–has a beautiful closed formula. Moreover, this formula is closely related to the formula for the *top-weight Euler characteristic* of $\mathcal{M}_{g,n}$.

2. Kontsevich's graph complex

Fix $g, n \geq 0$ with $2g-2+n > 0$. Define the following rational chain complex $G^{(g,n)}$; it is an n-marked version of Kontsevich's graph complex $G^{(g)}$.[1] In degree p, $G_p^{(g,n)}$ is a $\mathbb{Q}$-vector space with generators

$$[G, m, \omega]$$

for (G, m) a connected, loopless[2] n-marked stable graph of genus g, and $\omega\colon E(G) \xrightarrow{\cong} \{0, \ldots, p\}$ a bijection. We impose relations

$$[G, m, \omega] = \operatorname{sgn}(\sigma)[G', m', \omega']$$

whenever there is an isomorphism $(G, m) \xrightarrow{f} (G', m')$ under which ω and ω' are related by permutation $\sigma \in S_{p+1}$.

The differential is a signed sum of edge contractions. Check $\partial^2 = 0$. This talk focuses on Euler characteristics, hence doesn't focus on the differential.

So $G^{(g,n)}$ is a finite chain complex. In fact:

Definition 2.1. Say a marked graph (G, m) is *alternating* if $\operatorname{Aut}(G, m) \to \operatorname{Sym} E(G)$ factors through the alternating subgroup $\operatorname{Alt} E(G)$.

Then $(G, m, w) = -(G, m, w) = 0$ for any non-alternating G. Hence $\dim G_p^{(g,n)} =$ number of alternating marked graphs of type (g, n) with $p+1$ edges, up to isomorphism.

Example 2.2. If G has parallel edges then G is not alternating. Hence if G is *covered by triangles* then $\partial[G, m, w] = 0$.

Proposition 2.3. The top-weight Euler characteristic of $\mathcal{M}_{g,n}$ is the Euler characteristic of $G^{(g,n)}$:

$$\sum_p (-1)^p \dim G_p^{(g,n)},$$

[1] Be warned that there are several different flavors of graph complexes; see [Kon93]

[2] In fact it's OK to allow or disallow loops, either way.

or, better yet,

$$\sum_p (-1)^p [G_p^{(g,n)}]$$

as a *virtual* S_n*-representation*, meaning an element in the Grothendieck group of S_n-representations: a formal $\mathbb{Q}$-linear combination of S_n-representations.

Proof. Brief sketch of proof from [CGP]. □

3. Frobenius characteristics

Recall that the irreps of S_n are in bijection with partitions $\lambda \vdash n$; write V_λ for the Specht module corr to λ. Suppose you have a sequence $W_1, W_2, \dots$ where

$$W_n = \sum_{\lambda \vdash n} c_\lambda V_\lambda, \qquad c_\lambda \in \mathbb{Q}$$

is a virtual finite-dimensional S_n representation. You can encode all these simultaneously by an element of the (completed) ring of symmetric functions with $\mathbb{Q}$-coefficients, defined as follows:

Definition 3.1. Let

$$\Lambda = \lim_{\leftarrow} \mathbb{Z}[x_1, \dots, x_n]^{S_n}$$

denote the *ring of symmetric functions*, where the inverse limit above taken in the category of graded rings; it has elements like

$$x_1 x_2^2 + x_1^2 x_2 + x_1 x_3^2 + x_1^2 x_3 \cdots$$

Let $\widehat{\Lambda} = \lim_{\leftarrow} \mathbb{Z}[[x_1, \dots, x_n]]^{S_n}$ be its degree completion.

"Recall" that Schur functions s_λ provide a $\mathbb{Z}$-basis for this ring as a $\mathbb{Z}$-module. Define homogeneous and inhomogeneous power sum symmetric functions p_i and P_i, and define p_λ and P_λ. Remark that the p_λ are a $\mathbb{Q}$-basis for $\Lambda \otimes \mathbb{Q}$.

Definition 3.2. Let W be a finite-dimensional S_n-representation. Its **Frobenius characteristic**, denoted $\operatorname{ch} W$, is the degree n symmetric function defined in two equivalent ways:

(1)
$$\operatorname{ch} W = \sum c_\lambda s_\lambda,$$
for $W \cong \bigoplus c_\lambda V_\lambda$,

(2)
$$\operatorname{ch} W = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{Tr}_W(\sigma) \psi(\sigma),$$
where $\psi(\sigma) = p_{a_1} \cdots p_{a_\ell}$ for σ with cycles of length $a_1 \geq \cdots \geq a_\ell$.

Lemma 3.3. These are equal.

Proof. It's not implausible that something like this holds, since to know a representation is exactly to know its character. This lemma is equivalent to the combinatorial fact (omitted) that the transition matrix from the basis $\{s_\lambda : \lambda \vdash n\}$ to $\{p_\lambda : \lambda \vdash n\}$ of $\Lambda_n \otimes \mathbb{Q}$ is the character table of S_n. □

Definition 3.4. Similarly, if you have a sequence of virtual f.d. S_n-representations $W = \{W_n\}$ for $n = 0, 1, 2, \ldots$, define $\operatorname{ch} W$ by extending linearly; now it is an element of $\widehat{\Lambda}$. So W knows $\operatorname{ch} W$ and $\operatorname{ch} W$ knows W.

4. Main theorem

Let $g \geq 2$. Let z_g denote the Frobenius characteristic of $W = \{W_n\}$ with $W_n = \sum_p (-1)^p [G_p^{(g,n)}]$. So z_g knows, for fixed g and all n at once, all the Euler characteristics of the complex $G^{(g,n)}$, and it knows it as S_n-reps.

Theorem 4.1. [CFGP]

$$z_g = \sum_{k,m,r,s,a,d} \frac{(-1)^{k-r}(k-1)!B_r}{r!} m^{r-1} \cdot \prod_{p|(m,d_1,\ldots,d_s)} \left(1 - \frac{1}{p^r}\right) \frac{1}{P_m^k} \prod_{i=1}^{s} \frac{\mu(m/d_i)^{a_i} P_{d_i}^{a_i}}{a_i!}, \tag{1}$$

where the sum is over integers $k, m > 0$ and $r, s \geq 0$, and s-tuples of positive integers $a = (a_1, \ldots, a_s)$ and $d = (d_1, \ldots, d_s)$, such that

$$0 < d_1 < \cdots < d_s < m, \quad \text{and} \quad d_i \,|\, m; \tag{2}$$

$$a_1 + \cdots + a_s + r = k + 1; \tag{3}$$

$$a_1 d_1 + \cdots + a_s d_s + g - 1 = km, \tag{4}$$

and the product runs over primes p dividing m and all $d_1, \ldots, d_s$.

Other formula hold for $g = 0$, $g = 1$, now with the sum starting at $n = 3$ and $n = 1$ respectively.

Remark 4.2. By the comparison theorem to top-weight cohomology discussed in STAGOSAUR, this is also the Frobenius characteristic of the top-weight Euler characteristic of $\mathcal{M}_{g,n}$, the moduli of genus g, n-marked algebraic curves.

Remark 4.3. Contextual remarks. History of formula. Related results.

A faint whiff of the proof follows.

Proof. We have

$$z_g = \sum_{i,n \geq 0} \sum_{\sigma \in S_n} \frac{(-1)^i}{n!} \chi_{G_i^{(g,n)}}(\sigma) \psi(\sigma)$$

by definition. Here $\chi_{G_i^{(g,n)}}(\sigma)$ denotes the *character* of the S_n-rep $G_i^{(g,n)}$ at σ; recall the character is the trace of a matrix representing the action of σ.

By definition of trace, we have to:

1. Count marked graphs $[G, m]$ that are fixed by a given $\sigma \in S_n$. Each $[G, m]$ should be counted with coefficient $\psi(\sigma)$ times

$$\begin{cases} (-1)^{|E(G)|} & \text{if } G \text{ is alternating,} \\ 0 & \text{else.} \end{cases}$$

Cute observation: this is the same as counting with coefficients

$$\sum_{\tau \in \operatorname{Aut}(G,m)} (-1)^{|E(G)|} \cdot \frac{\operatorname{sgn} \tau_E}{|\operatorname{Aut}(G,m)|}.$$

(Explain why.) Moreover,

Theorem 4.4. [CGP19] The subcomplex of $G^{(g,n)}$ spanned by $[G, m, \omega]$ with m not injective is acyclic.

So we need only count $[G, m]$ with m injective. What do these look like? They look like n marked points sprinkled onto a genus g graph (as opposed to nontrivial trees of marked points sprouting off of a genus g graph). This is crucial to understanding $G^{(g,n)}$ for all n simultaneously: study unmarked graphs, then sprinkle marked points on. It allows us to reduce to:

2. Count pairs (G, τ) for G an *unmarked* genus g graph, and $\tau \in \operatorname{Aut}(G)$. Claim—this is not at all trivial, see [CFGP19, Proposition 3.2]—that such a pair (G, τ) should be counted with coefficient

$$(-1)^{|E(G)|} \cdot \operatorname{sgn} \tau_E \frac{P(\tau_V)P(\tau_E)}{P(\tau_H)},$$

where τ_V, τ_E, and τ_H denote the permutations on the sets $V(G)$, $E(G)$, $H(G)$ of G induced by τ, and $P(\rho) := P_{a_1} \cdots P_{a_s}$ for a permutation ρ with cycle type $a_1 \geq \cdots \geq a_s$.

(Do an example to try to get the idea across.)

3. Count stable orbigraphs (X, f), counted with coefficient ?, where ? is sketched below.

What's an orbigraph?

Definition 4.5. An *orbigraph* is a pair (X, f) where X is a graph and $f : V(X) \amalg E(X) \to \mathbb{Z}_{>0}$ is a function satisfying $f(r(x))|f([x])$ for all $x \in H(X)$.

Definition 4.6. An orbigraph (X, f) is *stable* if it satisfies

(i) $\operatorname{val}_X(v) > 0$ for all $v \in V(X)$,
(ii) if $\operatorname{val}_X(v) < 3$ then there exists an $h \in H(X)$ with $r(h) = v$ and $f([h]) > f(v)$.

Give example of an orbigraph associated to (G, τ).

Anyways, here's the idea: replace the count of (G, τ) by counting orbigraphs $(X, f) = G/\tau$. Each (X, f) needs to be weighed with the number of ways that (X, f) arises as G/τ. This idea of passing to the quotient is inspired by Gorsky's proof [Gor14]. (Note, however, orbigraphs are a bit more complicated than orbicurves in the sense that in a graph quotient, nontrivial stabilizers can appear on codimension 0 subgraphs.)

4. Show that the total contributions of lots of stable orbigraphs (X, f) are zero. How we do that is omitted here. The ones that are left look quite a lot like marked graphs, with varying genera and numbers of marked points. Then (fast forwarding much of the proof) the final step is the remarkable fact (Kontsevich, Penner, see [Ger04]) that for each (g, n) with $2g - 2 + n > 0$,

$$\sum_G \frac{(-1)^{|E(G)|}}{|\mathrm{Aut}\, G|} = (-1)^{n+1} \frac{(g+n-2)!}{g!} \cdot B_g. \tag{5}$$

where the sum is over connected, stable genus g, n-marked graphs. □

5. Counting with groupoids

In steps 1, 2, 3 above, *things were being counted with automorphisms.* It is quite helpful to conceptualize each counting problem in terms of "orbi-summation" over groupoids, and each reduction from one counting problem to the next as a *pushforward along morphisms of groupoids.* Without making any claim of originality for this circle of ideas, we isolate this combinatorial technique below.

Definition 5.1. A groupoid $\mathcal{G}$ is a category in which all morphisms are isomorphisms. Say it's a finite groupoid if it is equivalent to a category with finitely many (objects and) morphisms. For such a $\mathcal{G}$, write $\pi_0(\mathcal{G})$ for the set of isomorphism classes.

Example 5.2. Fix n. There's a finite groupoid $\mathcal{D}_n$ whose objects are all regular n-gons of side length 1, and whose morphisms are all isometries. It has one isomorphism class, and for any objects P, Q there are $2n$ morphisms from P to Q.

Let $\mathcal{G}$ be a finite groupoid, let V be a rational vector space (you are welcome to take $V = \mathbb{Q}$ throughout), and let $f : \pi_0(\mathcal{G}) \to V$ be a function.

Definition 5.3. We define the orbisum of f by

$$\int_{\mathcal{G}} f = \sum_{[x] \in \pi_0(\mathcal{G})} \frac{f(x)}{|\,\mathrm{Aut}(x)|} \in V.$$

In particular, the rational number $\int_{\mathcal{G}} 1$ is the *groupoid cardinality* of $\mathcal{G}$.

Example 5.4. $\mathcal{D}_n$ has groupoid cardinality $1/2n$. To be cheeky, we say "there are exactly $1/2n$ regular n-gons of side length 1."

Definition 5.5. If $F : \mathcal{G} \to \mathcal{H}$ is a functor between finite groupoids and $f : \pi_0(\mathcal{G}) \to V$ is a function, we define the push-forward $(F_* f) : \pi_0(\mathcal{H}) \to V$ by the formula

$$(F_* f)([h]) = \int_{(F \downarrow h)} f, \tag{6}$$

where the subscript denotes the "comma category" $(F \downarrow h)$.

Here, regard f as a function on $\pi_0(F \downarrow h)$ by composing with the natural map $\pi_0(F \downarrow h) \to \pi_0(\mathcal{G})$. "Recall" the comma category $(F \downarrow h)$: objects are pairs (g, ϕ) with g an object of $\mathcal{G}$ and $\phi : F(g) \to h$ a morphism in $\mathcal{H}$, and morphisms $(g, \phi) \to (g', \phi')$ are morphisms $j \colon g \to g'$ in $\mathcal{G}$ such that $\phi' \circ F(j) = \phi$. Then

$$\boxed{\textstyle\int_{\mathcal{G}} f = \int_{\mathcal{H}} F_* f.}$$

Remark 5.6. Reframe steps 1, 2, 3 as pushforwards of orbisummations along appropriate morphisms from

(1) The groupoid of (stable, connected) genus g, n-marked graphs (G, m);
(2) The groupoid of genus g, unmarked graphs G, together with $\tau \in \mathrm{Aut}(G)$;
(3) The groupoid of orbigraphs.

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Appendix A. Exercises

Exercises: graphs and graph complexes.

(1) For $g \geq 3$, the *wheel* W_g is the graph obtained from a g-cycle C_g by adding a new vertex connected to the g vertices of C_g. Show W_g, with no marking function, is alternating iff g is odd. Show that $\partial[W_g, \omega] = 0$ for arbitrary ω.

(2) Write down the chain complex $G^{(1,3)}$.

(3) Write down the chain complex $G^{(1,4)}$.

Your last two results should agree with the following theorem: for $n \geq 3$, the chain complex $G^{(1,n)}$ has homology concentrated in degree $n-1$, of rank $(n-1)!/2$ [CGP19]. Check this.

Exercises: Frobenius characteristics. This exercise is related, in spirit, to the proof of the formula for z_1 found in [CFGP19, Proposition 1.5].

Let $\mathrm{Conf}_n(X)$ denote the configuration space of n ordered, distinct marked points on a topological space X. Let $W_n = H^0(\mathrm{Conf}_n(S^1); \mathbb{Q})$, and let $W = \{W_n\}_{n \geq 1}$. Prove that

$$\mathrm{ch}\, W = -\sum_{k \geq 1} \frac{\phi(k)}{k} \log(1 - p_k).$$

Either do your own thing,[3] or try the steps below.

(4) What is the homotopy type of $\mathrm{Conf}_n(S^1)$? What is $\dim H^0(\mathrm{Conf}_n(S^1); \mathbb{Q})$? How does S_n act on $H^0(\mathrm{Conf}_n(S^1); \mathbb{Q})$?

(5) Fix n. For which permutations $\sigma \in S_n$ is the character of this action at σ nonzero?

(6) For $d|n$, count the number of permutations in S_n that are a product of d disjoint n/d-cycles.

(7) Fix σ a product of d disjoint n/d-cycles, compute the character of this action at σ. You might use the Euler ϕ function.

(8) Compute the Frobenius characteristic of W_n as

$$\mathrm{ch}\, W_n = \frac{1}{n} \sum_{d|n} \phi(n/d) p_{n/d}^d.$$

(9) By summing over all n and over d dividing n, and setting $k = n/d$, conclude the formula for $\mathrm{ch}\, W$ above.

Exercises: Groupoid cardinality.

(10) What is the groupoid cardinality of the groupoid of all groups of order 4?

(11) A finite group G acting on a finite set X yields a groupoid with object set X and a morphism $x \to gx$ for every (g, x). Show that the cardinality of this groupoid is $|X|/|G|$.

[3] E.g., using [CFGP19, Lemma 9.1].

(12) This is an exercise on the number of degree d covering spaces of a fixed genus g graph. Fix R_g a graph with 1 vertex and g loops.[4]

Consider the groupoid $\mathcal{C}_{d,g}$ of "degree d covers of R_g." Precisely,

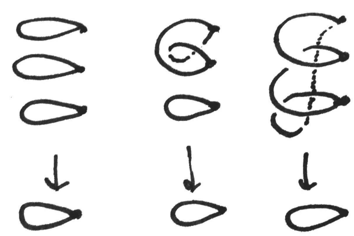
- objects are morphisms[5] $G \to R_g$, where G is a graph and $G \to R_g$ is a degree d covering space when regarded as a map of CW complexes.
- for $f\colon G \to R_g$ and $f'\colon G' \to R_g$, a morphism $f \to f'$ is a commuting diagram

$$\begin{array}{ccc} G & \xrightarrow{\cong} & G' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ R_g & = & R_g \end{array}$$

Show that

$$\int_{\mathcal{C}_{d,g}} 1 = (d!)^{g-1}.$$

Example: the three isomorphism classes of objects in $\mathcal{C}_{3,1}$ are drawn below, with 6, 2, and 3 automorphisms respectively. Thus $\int_{\mathcal{C}_{d,g}} 1 = \frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1$.



(13) Consider now the groupoid $\widetilde{C}_{d,g}$ of degree d covers of *all* roses with g petals. Precisely,

- Objects are morphisms $G \to R$, where G is a graph, R is a graph that is isomorphic to R_g, and $G \to R$ is a degree d covering space when regarded as a map of CW complexes.
- for $f\colon G \to R$ and $f'\colon G' \to R'$, a morphism $f \to f'$ is a commuting diagram

$$\begin{array}{ccc} G & \xrightarrow{\cong} & G' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ R & \xrightarrow{\cong} & R' \end{array}$$

Compute the groupoid cardinality of this groupoid.

[4] R_g is sometimes called a "rose with g petals."

[5] A *morphism* of graphs $G = (V, H, i, r) \to G' = (V', H', i', r')$ is a pair of maps $V \to V'$ and $H \to H'$ compatible with i, i' and r, r'. That is, it sends vertices to vertices and edges to edges, preserving vertex-edge incidence.

Appendix B. Exercise hints and solutions

Hints, solutions: graphs and graph complexes.

(1) If G has parallel edges then $[G, m, w] = 0$ in $G^{(g,n)}$. Hence, for any G that is *covered by triangles*—meaning every edge is in a triangle—$\partial[G, m, w] = 0$.[6] [7]

(2) The chain complex $G^{(1,3)}$ should be rank 1, concentrated in degree 2.

(3) The chain complex $G^{(1,4)}$ should be rank 9 in degree 3 and rank 6 in degree 2.

Hints, solutions: Frobenius characteristics.

(4) **Solution:** Disjoint union of $(n-1)!$ circles; S_n acting on left cosets of C_n.

(5) **Solution:** Exactly those that are a product of n/d disjoint d-cycles, for some $d|n$.

(6) **Solution:** $n!/(d!(n/d)^d)$.

(7) **Solution:** $\phi(n/d)\cdot(d-1)!(n/d)^{d-1}$.

Hints, solutions: Groupoid cardinality.

(10) **Solution:** $2/3$.

(11) This is the orbit-stabilizer lemma.

(12) Identify $\mathcal{C}_{d,g}$ as having the same groupoid cardinality as[8] the groupoid whose objects are

$$\{(\sigma_1, \ldots, \sigma_g) : \sigma_i \in S_d\}$$

with a morphism $(\sigma_1, \ldots, \sigma_g) \to (\tau\sigma_1\tau^{-1}, \ldots, \tau\sigma_g\tau^{-1})$ for each $\tau \in S_d$.

(13) Define a functor $F\colon \widetilde{C}_{d,g} \to \mathcal{R}_g$, where $\mathcal{R}_g$ is the groupoid of graphs isomorphic to R_g. Identify $(F \downarrow R_g)$ with a familiar groupoid.
Solution: $(F \downarrow R_g)$ is equivalent to $\mathcal{C}_{d,g}$. The answer is $\frac{(d!)^{g-1}}{2^g \cdot g!}$.

References

[CFGP19] Melody Chan, Carel Faber, Søren Galatius, and Sam Payne, *The S_n-equivariant top weight Euler characteristic of $\mathcal{M}_{g,n}$*, arXiv:1904.06367, 2019.

[CGP19] Melody Chan, Søren Galatius, and Sam Payne, *Topology of moduli spaces of tropical curves with marked points*, arXiv:1903.07187, 2019.

[Ger04] Ferenc Gerlits, *The Euler characteristic of graph complexes via Feynman diagrams*, arXiv:0412094v2, 2004.

[6] In this situation $[G, m, w]$ represents a homology class in the graph complex. But detecting whether or not this homology class is nonzero is much harder! The proof that W_g is nonzero in homology of the graph complex, for g odd, is covered in STAGOSAUR.

[7] It would be great if there were lots of (G, m) that were alternating, covered by triangles, and of maximal homological degree in $G^{(g,n)}$; then you'd get a nonzero homology class in $G^{(g,n)}$ (and hence a class in top-weight cohomology of $\mathcal{M}_{g,n}$.) Actually, there are only two. One of them is W_3; find the other!

[8] Indeed, equivalent as a category to

[Gor14] Eugene Gorsky, *The equivariant Euler characteristic of moduli spaces of curves*, Adv. Math. **250** (2014), 588–595. MR 3122177

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