GRAPHS, MATROIDS, AND MODULI SPACES

MELODY CHAN

Contents

Date: April 28, 2022.

These are incomplete notes for Math 2720T: Graphs, matroids, and moduli spaces, taught at Brown University, Spring 2022. The subsection headings give a rough idea of what we covered in the classroom, but some of them have remained unfilled in these typed notes. There are surely errors.

1. Engaged Pedagogy

Reference: *Teaching to Transgress: Education as the Practice of Freedom* by bell hooks.

> "The classroom remains the most radical space of possibility in the academy."

Question 1.1. Is it possible for a graduate topics course in graphs, matroids, and moduli spaces be about the practice of freedom, of liberation? Is transgression an appropriate word to describe graduate education in mathematics? Should I be teaching to transgress, should you be learning to transgress? And how, if so? What specific structures, what practices can this class build towards the end of liberation? What forms of liberation would an ideal graduate topics course bring you?

Question 1.2. bell hooks ties engaged pedagogy to self-actualization, bringing the whole self to the classroom. In what ways is it challenging to bring our whole selves into a mathematics classroom? In what ways can we rise to the challenge?

How does your thinking above relate to the *physical* classroom versus a virtual, Zoom classroom? We've spent a long time out of the physical classroom. What did we lose, and what specifically do we gain in convening in the physical classroom, as far as bringing our fully engaged, whole selves? (And what do we lose when we step away from Zoom and into the physical classroom?)

Part 1. Graphs

2. Basics. Spanning trees

Definition 2.1. A graph G is a pair of finite sets V and H, a fixed point free involution i on H, and a root map $r\colon H \to V$.

The **edge set** $E(G)$ is the set of pairs $\{h, i(h)\}$. Can relax i to be an arbitrary involution, to get a graph with legs $L(G)$. These form the objects of a groupoid $\mathcal{G}^\circ$ of graphs. There is a geometric realization functor $|\cdot|\colon \mathcal{G}^\circ \to \mathrm{Top}$.

Let $G = (V(G), H(G), i, r)$ be a graph. A subgraph is a choice of $V' \subseteq V$ and $H' \subseteq H$ such that $i(H') \subseteq H'$ and $r(H') \subseteq V'$. Informally: choose some of the vertices, and some edges between them. G is called *acyclic* if it contains no *cycle* as a subgraph, i.e., $H_1(G;\mathbb{Z}) = 0$. G is called a *tree* if it is connected and acyclic.

Say a subgraph H is spanning if $V(H) = V(G)$.

Proposition 2.2. Let G be a connected graph on n vertices. Let T be a spanning subgraph. The following are equivalent:

(1) T is minimal connected.
(2) T is maximal acyclic.
(3) T has $n-1$ edges and is connected.
(4) T has $n-1$ edges and is acyclic.

In these situations, say T is a **spanning tree.**

Exercise 2.3. Enumerate the spanning trees of K_4.

Proof. Orient the edges of G arbitrarily; now you have a CW-complex. Consider the cellular chain complex

$$\mathbb{Q}E \xrightarrow{\partial} \mathbb{Q}V.$$

Because G is connected, $\dim \mathrm{Im}(\partial) = n-1$; more generally

$$\dim \mathrm{Im}(\partial) = n - \dim H_0(G;\mathbb{Q}) = n - \#\{\text{connected components}\}.$$

Consider the set of vectors

$$S = \{\partial e : e \in E(T)\}.$$

By the same reasoning, a spanning subgraph T is

(1) connected iff S is a spanning set for $\mathrm{Im}(\partial)$,
(2) acyclic iff S is linearly independent.

The proposition then follows from linear algebra: these are four equivalent ways of saying that S is a basis for $\mathrm{Im}(\partial)$. □

Let $\mathcal{T}(G)$ denote the set of spanning trees of G. Evidently $\mathcal{T}$ is a functor $\mathcal{T}\colon \mathcal{G}^\circ \to \mathsf{Set}$.

Theorem 2.4. (Cayley's Theorem, actually due to Borchardt 1860)

$$|\mathcal{T}(K_n)| = n^{n-2}.$$

That is, the number of trees on vertex set $\{1, \ldots, n\}$ is n^{n-2}.

Remark 2.5. Possibly discuss two proofs: by Prüfer codes[1], and Joyal's proof of 1980 blogged about here.

Exploration 2.6. What is the S_n-representation on the n^{n-2} spanning trees of K_n?[2]

3. Parking functions

Definition 3.1. A parking function is a sequence $(b_1, \ldots, b_n)$ of nonnegative numbers such that if $c_1 \leq \cdots \leq c_n$ is the reordering, then $c_i < i$.

Give the interpretation of parking functions in terms of, well, parking. Cars are numbered $1, \ldots, n$, and parking spots are linearly ordered $0, \ldots, n-1$.

Example 3.2. The three parking functions for $n = 2$ are $01, 10$, and 00.

There is an amazingly slick proof that the number of parking functions is also $(n+1)^{n-1}$, due to Pollak 1974, see these slides.

There are known bijections between parking functions and spanning trees of K_{n+1}:

Exploration 3.3. Postnikov-Shapiro define G-parking functions for an arbitrary rooted graph (G, v) [PS04]. These are functions $b\colon V(G) \setminus \{v\} \to \mathbb{Z}_{\geq 0}$ such that for all $I \subseteq V(G) \setminus \{v\}$, there exists $v \in I$ such that $b(i) < \#$ edges at i leaving I. Verify that the case $G = K_{n+1}$ recovers parking functions of length n. Let $\mathrm{Park}(G)$ be the set of G-parking functions. Read [CP05] for a family of bijections between $\mathcal{T}(G)$ and $\mathrm{Park}(G)$, and see the references therein.

At this point we have three equinumerous sets: $\mathcal{T}(G)$, $S(G)$, and $\mathrm{Park}(G)$. Here's something worth exploring:

Is there a natural isomorphism between $\mathcal{T}$ and Park as functors

$$\mathsf{RootedGraphs}^{\circ} \to \mathsf{Set}?$$

If not (I believe not), add as little rigidifying data to get a natural isomorphism, e.g., try rooted ribbon graphs.

[1] See my notes for first-year undergraduates at here, §2.2, or [Sta99, §5.6]. Can Prüfer codes be related to any of the other bijections swirling around here? Note that they take as input a graph with total ordering on vertex set.

[2] This has surely been studied, but I asked a couple experts and don't know where. On the other hand, the S_n action on the $(n+1)^{n-1}$ *parking functions* of length n, denoted Park_n, is of central interest. See [Hai02], where Haiman proves that the space of *diagonal harmonics* DH_n, of dimension $(n+1)^{n-1}$, is $\mathrm{Park}_n \otimes \mathrm{sgn}_n$ as an S_n-representation and resolves a bunch of conjectures. The proof features the geometry of $\mathrm{Hilb}_n(\mathbb{C}^2)$.

4. The matrix tree theorem

Definition 4.1. Laplacian matrix of G. Let G be a connected graph on vertex set $\{1,\ldots,n\}$. Define a symmetric $n\times n$ matrix $\Delta = \Delta(G)$ by

$$\Delta_{i,j} = \begin{cases} -\#\{\text{edges } ij\} & \text{if } i \neq j \\ \#\{\text{nonloop edges at } i\} & \text{if } i = j. \end{cases}$$

Since row and column sums are all 0, Δ is not full rank. On the other hand, the *reduced* Laplacian $\Delta_i^{\mathrm{red}}(G) = \Delta^{\mathrm{red}}(G)$, obtained by deleting row i and column i for some i, is full rank. This follows immediately from an even more interesting statement:

Theorem 4.2. (Kirchhoff's Matrix Tree Theorem) For any[3] graph G,

$$\det \Delta^{\mathrm{red}}(G) = |\mathcal{T}(G)|.$$

Proof. Note $\partial\partial^t = \Delta$ and $\partial_i\partial_i^t = \Delta_i^{\mathrm{red}}$, where ∂_i is obtained from ∂ by deleting row i. Now use Cauchy-Binet. □

In the Matrix Tree theorem, we have an equality of numbers. One immediately wishes for a combinatorial proof. The right hand side is evidently the cardinality of a set. The left hand side is naturally the cardinality of a finite abelian group: the sandpile group of G, which we shall define.

Definition 4.3. Let G be a connected graph. Regard $\partial\colon \mathbb{Z}E \to \mathbb{Z}V, \Delta\colon \mathbb{Z}V \to \mathbb{Z}V$ as an integral map now.

Let $\mathrm{Div}(G) = \mathbb{Z}V$, the **divisor group** on G.
Let $\mathrm{Div}^0(G) = \mathrm{Im}(\partial) \subset \mathbb{Z}V$, the divisors of degree 0.
Let $\mathrm{Prin}(G) = \mathrm{Im}(\Delta)$, the **principal divisors**.
The **sandpile group** $\mathrm{Pic}^0(G) = S(G)$ is the quotient

$$\mathrm{Pic}^0(G) = \frac{\mathrm{Div}^0(G)}{\mathrm{Prin(G)}}.$$

Said with less fanfare, $S(G)$ is the cokernel of $\Delta^{\mathrm{red}}\colon \mathrm{Div}^0(G) \to \mathrm{Div}^0(G)$.

The fact that $|S(G)| = \det \Delta^{\mathrm{red}}(G)$ must have some good conceptual proof, but the only one I can think of is via Smith normal forms.

Example 4.4.

(1) Compute $S(G)$ for the n-edge banana graph (2 vertices, n parallel edges).
(2) Compute $S(G)$ for the n-cycle C_n. Fun fact: $S(G) \cong S(G^*)$, and with an isomorphism that is canonical up to sign.

Exploration 4.5. Prove $\det \Delta^{\mathrm{red}}(K_n) = n^{n-2}$. In fact I assert that

$$S(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2},$$

[3] The interesting case is connected graphs. If G is not connected, both sides are 0.

noncanonically. This is a group-theoretic promotion of Cayley's theorem. This isomorphism cannot possibly be canonical with respect to $\mathrm{Aut}(K_n) = S_n$. Prove the existence of such an isomorphism, and quantify failure of canonicity.

Remark 4.6. Discuss torsor structures on spanning trees. To dig deeper, see [CCG15] especially the introduction; note this paper arose from a question of Jordan Ellenberg on MathOverflow [Ell]! Also [BW18].

5. Matrix Tree Theorem, weighted version

Lemma 5.1. (Weighted Cauchy-Binet theorem) Fix $d \le n$, let $A \in K^{d\times n}, B \in K^{n\times d}$. Let $D \in K[x_1,\dots,x_n]^{n\times n}$ be diagonal with $D_{i,i} = x_i$. Then

$$\det(ADB) = \sum_{S=\{i_1,\dots,i_d\}} \det(A_S)\det(B_S)\cdot x_S \in K[x_1,\dots,x_n]_d$$

where $x_S = x_{i_1}\cdots x_{i_d}$. In particular, $\det(ADB)$ is homogeneous of degree d and supported on squarefree monomials.

For later use, note the interpretation $A_S = A\cdot\pi_S$ where π_S denotes coordinate projection $K^n \to K^S$.

In particular, given graph G, let D_G be the diagonal $E(G)\times E(G)$ matrix with formal variables x_e on the diagonal.

Definition 5.2. Define the *edge-weighted Laplacian*

$$\Delta(\{x_e\}, G) = \Delta = \partial D \partial^T.$$

Explicitly,

$$\Delta(\{x_e\}, G)_{i,j} = \begin{cases} -\sum_{e=ij} x_e & \text{if } i \ne j \\ \sum_{e \text{ nonloop at } i} x_e & \text{if } i = j \end{cases}$$

Consider any reduction $\Delta^{\mathrm{red}}(x_e)$ obtained by deleting row and column corresponding to $e_0 \in E(G)$. Then a consequence of the weighted Cauchy-Binet theorem is

Corollary 5.3. (Weighted Matrix Tree Theorem)

$$\det \Delta^{\mathrm{red}}(\{x_e\}, G) = \sum_{T\in\mathcal{T}(G)} \prod_{e\in E(T)} x_e.$$

6. Dual Laplacians

We follow here the perspective of Bloch-Esnault-Kreimer [BEK06]; see also [Bro21]. Let G be a connected graph. Let $\mathcal{H}_G = \mathcal{H}\colon H_1(G;\mathbb{Z}) \hookrightarrow \mathbb{Z}E$ be the inclusion, regarded by abuse of notation also as a matrix with respect to any $\mathbb{Z}$-basis for H_1 and the basis E for $\mathbb{Z}E$.

Definition 6.1. The dual Laplacian of G is

$$\Lambda_G = \mathcal{H}_G^t D_G \mathcal{H}_G.$$

Definition 6.2. The *graph polynomial*, or *Symanzik polynomial*, of G is $\Psi_G = \det \Lambda_G \in K[x_e : e \in E(G)]$.

Note that although we chose arbitrarily a $\mathbb{Z}$-basis for $H_1(G;\mathbb{Z})$, Λ_G is independent of that choice, since another choice results in multiplication of $\det L_G$ by $(\pm 1)^2 = 1$.

Proposition 6.3. (Spanning tree formula for graph polynomial)

$$\Lambda_G = \sum_{T \in \mathcal{T}(G)} \prod_{e \notin E(T)} x_e.$$

This theorem is proved as a special case of the functional equation below, which I believe is due to [BEK06].

Definition 6.4. The graph hypersurface of G is the projective hypersurface

$$X_G = V(\Lambda_G) \subset \mathbb{P}^{|E(G)|-1}.$$

Let $\widehat{X_G} = V(\Lambda_G) \subseteq \mathbb{A}^{|E(G)|}$ be its affine cone, and let $Y_G = \mathbb{A}^{|E(G)|} \setminus V(\Lambda_G)$ be the complement in affine space.

7. A functional equation for polynomials of configurations

See [BEK06, §1]. Let K a field, E a finite set. KE shall denote the vector space with basis E. This chosen basis gives a choice of identification $KE \cong (KE)^\vee$.

Definition 7.1. A configuration is a linear subspace $i_V : V \hookrightarrow KE$.

Let $d = \dim V$, and let M_V be the matrix for i_V with respect to a chosen basis $v_1, \ldots, v_d$ for V. Define

$$\Psi_V(x_e) = \det(M_V^t D M_V),$$

where D is a diagonal matrix with $D_{i,i} = x_i$. Here we suppress the dependence on choice of basis for V, though note a different choice of basis just changes Ψ_V by a constant in $(K^*)^2$.

Now let W sit in the short exact sequence

$$0 \to V \xrightarrow{i_V} KE \to W \to 0,$$

with dual short exact sequence

$$0 \to W^\vee \xrightarrow{i_{W^\vee}} KE \to V^\vee \to 0,$$

where we identified $(KE)^\vee \cong KE$.

Let $s : W \to KE$ be any splitting of the first short exact sequence. Choose ordered bases for V and W, and an ordering of E furnishing an ordered basis of KE. The maps (i_V, s) produce an isomorphism

$$\det V \otimes \det W \xrightarrow{\cong} \det KE$$

which actually does not depend on choice of s. (Exercise). With respect to our choices of ordered bases of V, W, and KE, we get a 1×1 matrix $[c]$, for some $c \in K^*$, representing this isomorphism. Then, also with respect to our choices, the following equation holds:

Theorem 7.2.

$$\Psi_V(x_e) = c^2 \left(\prod_{e \in E} x_e \right) \Psi_{W^\vee}(x_e^{-1}).$$

Before proving Theorem 7.2, we use it to deduce Proposition 6.3.

Proof of Proposition 6.3.

$$0 \to H_1(G;\mathbb{Q}) \hookrightarrow \mathbb{Q}E \xrightarrow{\partial} \mathrm{Im}(\partial) \to 0$$

and its dual short exact sequence

$$0 \to \mathrm{Im}(\partial)^\vee \hookrightarrow \mathbb{Q}E \to H^1(G;\mathbb{Q}) \to 0.$$

We shall apply Theorem 7.2 to this setup to deduce the explicit formula for the graph polynomial. One term in Theorem 7.2 we may recognize as a reduced Laplacian. Indeed, pick any root vertex v, and write $V' = V \setminus \{v\}$. Then identify $\mathrm{Im}\,\partial \cong \mathbb{Q}V'$ for any root vertex v, by sending $v' - v \mapsto v'$. To be very formal, then we write $\partial_v \colon \mathbb{Q}E \twoheadrightarrow \mathbb{Q}V'$. Then

$$\Psi_{\mathrm{Im}\,\partial^\vee} = \det \partial_v^T D \partial = \det \Delta_v^{\mathrm{red}} = \sum_{T \in \mathcal{T}(G)} \prod_{e \in E(T)} x_e.$$

Next, note that any $\mathbb{Z}$-basis for $H_1(G;\mathbb{Z})$ together with any $\mathbb{Z}$-basis for $\mathrm{Im}\,\partial$, for example $\{v' - v : v' \in V'\}$, form a $\mathbb{Z}$-basis for $\mathbb{Z}E$. This says that, in our situation, the constant $c = \pm 1$ in the Theorem.

The formula for the graph polynomial follows. □

Proof of Theorem 7.2. It suffices to prove, for each $T \subseteq E$ of size $|E| - d = |E| - \dim V$, that

$$\mathrm{coeff}(\Psi_V, x_{E \setminus T}) = c^2 \cdot \mathrm{coeff}(\Psi_{W^\vee}, x_T).$$

Let $\alpha_{E\setminus T}$ and β_T be, respectively, the compositions

$$V \xrightarrow{i_v} KE \to K(E \setminus T) \qquad KT \to KE \twoheadrightarrow W,$$

regarding both as matrices with respect to the chosen bases of V, W, E. Then $\alpha_{E\setminus T}$ and β_T are each full rank iff the other is[4]. Moreover, if both are full rank, then $\pm c = \det \alpha_{E\setminus T} \det(\beta_T)^{-1}$. (The $\pm$ arises because it is not known whether listing all the elements of $E \setminus T$ and then listing the elements of T is the same, up to alternating permutation, as listing the elements of E in specified order.)

In any case, we are done once we note that the coefficient of $x_{E\setminus T}$ in Ψ_V is $(\det \alpha_{E\setminus T})^2$, and the coefficient of x_T in $\Psi_{W^\vee}$ is $(\det \beta_T^t)^2$.

□

[4]directly, or draw a diagram and use snake lemma

7.1. Basic properties of graph hypersurfaces.

Proposition 7.3. See, e.g., [Alu14, Proposition 4.4].

(1) Let G_1 and G_2 be connected graphs and let $G = G_1 \vee G_2$ be the 1-vertex join at any two vertices. Then
$$\Psi_G = \Psi_{G_1} \cdot \Psi_{G_2} \in \mathbb{Z}[x_e : e \in E(G_1) \amalg E(G_2)].$$
Therefore $Y_G = Y_{G_1} \times Y_{G_2}$. In particular, if G and G' are related by a sequence of 1-vertex cuts and joins, then $Y_G \cong Y_{G'}$.

(2) If G consists of a bridge joining graphs G_1 and G_2, then $Y_G \cong Y_{G_1} \times Y_{G_2} \times \mathbb{A}^1$.

(3) Let G be obtained from G' by a "Whitney flip." Then $Y_G \cong Y_{G'}$.

Proof. The second claim is a special case of the first. For the last one, convince yourself that there is a commuting diagram
$$\begin{array}{ccc} H_1(G;\mathbb{Z}) & \longrightarrow & \mathbb{Z}E(G) \\ \cong\downarrow & & \downarrow\cong \\ H_1(G';\mathbb{Z}) & \longrightarrow & \mathbb{Z}E(G') \end{array}$$
□

For purposes of blowing up to integrate forms on graph hypersurface complements, it is useful to know what coordinate linear subspaces are contained in graph hypersurfaces. For $S \subseteq E(G)$, let $L_S = V(x_e : e \in S)$. For which subsets S is $L_S \subseteq X_G$?

Proposition 7.4. $L_S \subseteq X_G$ iff S contains a cycle.

Proof. Suppose S contains a cycle. Then every spanning tree T misses some edge e_T of S, therefore $L_S \subseteq X_G$. Conversely, if S is acyclic, then S is contained in a spanning tree T. Then the point with $x_e = 0$ for $e \in E(T)$ and $x_e = 1$ otherwise is in L_S but not in X_G. (Note Ψ_G has all nonnegative coefficients!) □

Exercise 7.5. If G' is obtained from subdividing an edge of G, then how is $\widehat{X}_{G'}$ related to $\widehat{X}_G$? Dually, same question if G' is obtained from G by adding an edge parallel to a given edge.

8. Graph hypersurfaces and the Grothendieck ring

I found the survey by Aluffi [Alu14] to be very intuitive and helpful for my understanding, and much of this material is culled from there and other sources mentioned below.

Let k be a field. Define the Grothendieck ring of varieties $K_0(\mathrm{Var}_k))$, generated by quasiprojective varieties over k. Can actually make this definition definition over a base ring R, e.g., $R = \mathbb{Z}$, or indeed arbitrary base scheme S; a possible source is [CLNS18, §1.2]. Thus $K_0(\mathrm{Var}_{\mathbb{Z}})$ is, as an additive group, the free abelian group generated by $[X]$ for X a finite-type scheme over $\operatorname{Spec}\mathbb{Z}$, with relations
$$[X] = [Y] + [X \setminus Y]$$

for Y a closed subscheme of X. It becomes a ring via $[X] \cdot [Y] = [X \times_{\mathbb{Z}} Y]$.

Beyond the scope of this course is the still-partly-conjectural notion of a *motive* due to Grothendieck. Let k be any field.

Say a variety X is (naïve) mixed Tate if $[X]$ lies in the subring $\mathbb{Z}[\mathbb{L}]$, where $\mathbb{L} = [\mathbb{A}^1]$ is the Lefschetz (naïve) motive. Now if a variety X defined over $\operatorname{Spec} \mathbb{Z}$ is mixed Tate, then it must be *polynomially countable,* i.e., there is a polynomial $N_X(t) \in \mathbb{Z}[t]$ such that $\#X(\mathbb{F}_q) = N_X(q)$ for all finite fields $\mathbb{F}_q$.

For a while, physical evidence suggested that perhaps all graph hypersurfaces are mixed Tate. This would imply, as was conjectured informally by Kontsevich in a lecture in 1997 (see [Sta98], that all graph hypersurfaces were polynomially countable.

There was some early evidence by combinatorialists Stanley [Sta98] and Stembridge [Ste98], but a big disproof by Belkale-Brosnan, that uses matroids and appeals to Mnev universality [BB03]. Maybe we can return to this paper. There are some interesting questions in [Sta98]: one could ask if the hypersurface of a *regular matroid* is polynomially countable, but Stembridge shows that it is not for the matroid R_{10}. Stanley also asks whether it is at least true that graph hypersurfaces are quasipolynomial (determined by N possibly different polynomials, depending on the residue class of q mod N, for some N), but [BB03] disproves this.

Exploration 8.1. Later: classify those regular matroids that are polynomially countable, in view of Seymour's decomposition theorem for regular matroids.

An interesting more recent reference is [DSW21].

9. Kontsevich's graph complexes

Kontsevich [Kon93] defined six flavors of graph complexes

$$\text{commutative/associative/Lie} \times \text{even/odd},$$

the first having to do with vertex decorations, and the second having to do with a choice of a notion of orientation. In fact, the three choices can be replaced by any cyclic operad in a more general construction, as described in [CV03].

Why?[5] They arise, as Kontsevich sketches, from three infinite-dimensional Lie algebras $\ell_\infty, a_\infty, c_\infty$, sitting in a conjectural unifying framework of *noncommutative symplectic geometry*. Each of these contain a copy of the symplectic Lie algebra $\mathfrak{sp}(2n)$. The $\mathfrak{sp}(2n)$-invariants of the exterior algebra of the Lie algebra is related to the three respective flavors of graph complexes (with "even" orientation.) These graph complexes have precise relations with

- $H^*(\operatorname{Out}(F_n); \mathbb{Q})$, in the Lie case, via Culler-Vogtmann Outer Space;
- $H^*(\mathcal{M}_{g,m}/S_m; \mathbb{Q})$, in the associative case, via Penner's ribbon orbi-cellular decomposition of moduli spaces of marked graphs;
- Invariants of manifolds of odd dimension, in the commutative case, in a precise way that unfortunately I do not know.

[5]Helpfully sketched in [CV03].

The first mystery is that the commutative *odd* graph complex has recently been shown to be related to cohomology of moduli spaces of curves $\mathcal{M}_g$.

Definition 9.1. (Commutative, odd) graph complex $\mathcal{GC}$. This is a rational chain complex (for now). Generators are (G, ω) where G is a connected graph with minimum valence ≥ 3, and ω is an additive generator of $\wedge^{|E(G)|}\mathbb{Z}^{|E(G)|} \cong \mathbb{Z}$.

The degree of (G, ω) is $|E(G)| - 2|V(G)|$. Define the differential, of degree -1. Note that $\mathcal{GC}$ breaks up as

$$\mathcal{GC} = \bigoplus_{g \geq 2} \mathcal{GC}^{(g)}.$$

Theorem 9.2. [CGP21] There is a canonical surjection

$$H^{4g-6-k}(\mathcal{M}_g; \mathbb{Q}) \to H_k(\mathcal{GC}).$$

We can already compare with Looijenga's calculation of what is called in that paper the *Poincaré-Serre* polynomial of $\mathcal{M}_3$, which is $1 + t^2u^2 + t^6u^{12}$ [Loo93, 4.7]. This calculation encodes that the nonzero rational cohomology groups of $\mathcal{M}_3$ are

$$H^0(\mathcal{M}_3; \mathbb{Q}), H^2(\mathcal{M}_3; \mathbb{Q}), H^6(\mathcal{M}_3; \mathbb{Q}),$$

each of rank 1, pure of weights $0, 2$, and 12 respectively.

Part 2. Moduli spaces of smooth and stable curves

10. The Deligne-Mumford-Knudsen compactification of $\mathcal{M}_{g,n}$

The material below comes directly from [Cha17, §2.1, 2.3] and the references therein. See [Cha21] for a more recent colloquium-style survey on algebraic and tropical moduli spaces.

10.1. **Stable curves.** Fix k an algebraically closed field. By a *curve* we shall mean a reduced, proper, connected scheme X of dimension 1 over k. The arithmetic genus of the curve is $h^1(X, \mathcal{O}_X)$. A *node* of X is a point $p \in X(k)$ with the property that $\widehat{\mathcal{O}}_{X,p} \cong k[\![x, y]\!]/(xy)$. A *nodal* curve is a curve whose only singularities, if any, are nodes.

Definition 10.1. (Stable n-pointed curves). A nodal, n-marked curve of genus g is $(X, p_1, \ldots, p_n)$, where $p_i \in X(k)$ are distinct nonsingular points of a genus g nodal curve X.

We say that a nodal, marked curve $(X, p_1, \ldots, p_n)$ is **stable** if $\mathrm{Aut}(X, p_1, \ldots, p_n)$ is finite, that is, there are only finitely many automorphisms of the curve X that fix each $p_1, \ldots, p_n$ pointwise. This is often equivalently stated as follows: $(X, p_1, \ldots, p_n)$ is stable if the restriction of $\omega_X(p_1 + \cdots + p_n)$ to every irreducible component of X is a line bundle of positive degree. Here ω_X denotes the dualizing sheaf of X.

Notice that all smooth curves of genus $g \geq 2$ already have only finitely many automorphisms. A smooth curve of genus 1 has finitely many automorphisms once one fixes one marked point; and a smooth curve of genus 0, also known as $\mathbb{P}^1$, has finitely many automorphisms once one fixes three marked points. So we could equally phrase the stability condition as follows:

Observation 10.2. For every irreducible component C of X, let $\phi\colon C^\nu \to C$ denote the normalization of C. An n-marked nodal curve $(X, p_1, \ldots, p_n)$ is stable if and only if

(1) for every component C of geometric genus 0,
$$|C \cap \{p_1, \ldots, p_n\}| + |\{q \in C^\nu : \phi(q) \in X^{\text{sing}}\}| \geq 3;$$
(2) for every component C of geometric genus 1,
$$|C \cap \{p_1, \ldots, p_n\}| + |\{q \in C^\nu : \phi(q) \in X^{\text{sing}}\}| \geq 1.$$

(The second condition sounds misleadingly general. You can trace through the definition yourself to see that it excludes only one additional case, the case that the whole of X is just a smooth curve of genus 1 with no marked points.)

Exercise 10.3. Let $g, n \geq 0$. Check that stable n-marked curves of genus g exist if and only if $2g - 2 + n > 0$.

10.2. **Dual graphs of stable curves.** We are working towards the goal of associating a graph, with some vertex decorations and some edge lengths, to a smooth curve X/K. The graph we are going to associate to X is the *dual graph* of the special fiber of a stable model for X. Basically, the dual graph of a stable curve Y is a combinatorial gadget that records:

- how many irreducible components Y has, and what their geometric genera are;
- how the irreducible components of Y intersect; and
- the way in which the n marked points are distributed on Y.

Now we will explain this completely, starting with the graph theory.

Conventions on graphs. All graphs will be finite and connected, with loops and parallel edges allowed. (Graph theorists would call such objects finite, connected *multigraphs*.) Remember that a graph G consists of a set of vertices $V(G)$ and a set of edges $E(G)$. Each edge is regarded as having two endpoints which are each identified with vertices of G, possibly the same.

Definition 10.4. (Vertex-weighted marked graphs). A *vertex-weighted, n-marked graph* is a triple (G, m, w) where:

- G is a graph;
- $w\colon V(G) \to \mathbb{Z}_{\geq 0}$ is any function, called a *weight function*, and
- $m\colon \{1, \ldots, n\} \to V(G)$ is any function, called an n-marking.[6]

[6]Another common setup for marking a tropical curve is to attach infinite rays to a graph, labeled $\{1, \ldots, n\}$. Our marking function m is obviously combinatorially equivalent.

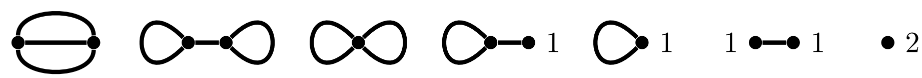


FIGURE 1. The seven genus 2 stable vertex-weighted graphs with no marked points. The vertices have weight zero unless otherwise indicated.

The *genus* of (G, m, w) is

$$g(G) + \sum_{v \in V(G)} w(v)$$

where

$$g(G) = |E| - |V| + 1$$

is the first Betti number of G, considered as a 1-dimensional CW complex, say.

Definition 10.5. (Stability for vertex-weighted marked graphs). With (G, m, w) as above, we'll say that (G, m, w) is *stable* if for every $v \in V(G)$,

$$2w(v) - 2 + \mathrm{val}(v) + |m^{-1}(v)| > 0.$$

Here $\mathrm{val}(v)$ denotes the graph-theoretic *valence* of the vertex v, which is defined as the number of half-edges incident to it.

Figure 1 shows the seven distinct stable vertex-weighted graphs of type $(g, n) = (2, 0)$.

Exercise 10.6. Find the five stable, 2-marked weighted graphs of genus 1.

Definition 10.7. (Dual graph of a stable curve). Let k be an algebraically closed field, and let $(Y, p_1, \ldots, p_n)$ be a stable, n-marked curve over k.

The *dual graph* of $(Y, p_1, \ldots, p_n)$ is the vertex-weighted, marked graph (G, m, w) obtained as follows.

- The vertices v_i of G are in correspondence with the irreducible components C_i of Y, with weights $w(v_i)$ recording the geometric genera of the components.
- For every node p of Y, say lying on components C_i and C_j, there is an edge e_p between v_i and v_j.
- The marking function $m\colon \{1, \ldots, n\} \to V(G)$ sends j to the vertex of G corresponding to the component of Y supporting p_j.

Note that by Observation 10.2, (G, m, w) is stable since (Y, p_i) is stable.

10.3. **Clutching and gluing maps.** The encyclopedic reference for moduli spaces of curves is [ACG11]. A useful reference for connections to tropical moduli spaces is [ACP15].

Proposition 10.8. We have morphisms

$$\kappa_{g_1,n_1,g_2,n_2} \colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

and

$$\gamma_{g,n} \colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$$

called *clutching* and *gluing* morphisms. Notice that the image of these morphisms is contained strictly in the boundary of the target compactified moduli space.

There are also forgetful maps $\pi_i \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ for $i = 1, \ldots, n+1$, that forget the marked point i *and stabilize.* Clutching, gluing, and forgetful morphisms taken together are called the tautological morphisms.

10.4. **The boundary strata $\mathcal{M}_{\mathbf{G}}$ of $\overline{\mathcal{M}}_{g,n}$.** The strata of the boundary $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ are naturally indexed by genus g, n-marked stable graphs $\mathbf{G} = (G, m, w)$, according to the dual graphs of the stable curves that they parametrize. I would now like to describe these strata, which we'll denote $Y_{\mathbf{G}}$. This description follows [ACP15, §3.4] and the correctness of this description is proved in [ACG11, §12.10].

Fix a combinatorial type $\mathbf{G} = (G, m, w)$. For each vertex v, let $n_v = \mathrm{val}(v) + |m^{-1}(v)|$ where $\mathrm{val}(v)$ is the valence of v. Let

$$\mathcal{M}_{\mathbf{G}} = \prod_{v \in V(G)} \mathcal{M}_{w(v),n_v}.$$

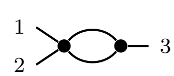
Using clutching and gluing maps, $\mathcal{M}_{\mathbf{G}}$ can be identified with the moduli space of n-marked genus g stable curves, *together with a chosen isomorphism of the dual graph with* $\mathbf{G}$. To get rid of that choice of isomorphism, we take the stack quotient $[\mathcal{M}_{\mathbf{G}} / \operatorname{Aut}(\mathbf{G})]$. The theorem is then that there is a canonical isomorphism

$$Y_{\mathbf{G}} \cong [\mathcal{M}_{\mathbf{G}} / \operatorname{Aut}(\mathbf{G})].$$

An explicit example is given in Example 10.9 below.

Example 10.9. Let's see everything at work in the following specific example of a stratum in $\overline{\mathcal{M}}_{1,3}$.

Let $\mathbf{G}$ be the combinatorial type below.



Consider the boundary stratum $Y_{\mathbf{G}}$ of $\overline{\mathcal{M}}_{1,3}$. Locally, it is a self-intersection of the boundary component whose dual graph is obtained from $\mathbf{G}$ by contracting either edge.

Let's describe $Y_{\mathbf{G}}$. I'll assume $\operatorname{char} k \neq 2$ in this example. According to the discussion above we have $\mathcal{M}_{\mathbf{G}} \cong \mathcal{M}_{0,4}$. Essentially, to give a stable curve C with dual graph $\mathbf{G}$ *along with a fixed identification of the two nodes of C with the two edges of* $\mathbf{G}$, we choose (up to projective equivalence) four distinct points p_1, p_2, q_1, q_2 on a $\mathbb{P}^1$, with the understanding that p_1 will be marked 1, p_2 marked 2, and q_1 and q_2 will be the two points of attachment of the other rational curve. Of course $\mathcal{M}_{0,4}$ is an honest variety: for example, fixing $p_1 = 0, p_2 = 1$, and $q_1 = \infty$ identifies $\mathcal{M}_{0,4}$ with $\mathbb{A}^1 - \{0, 1\}$.

Now $D_{\mathbf{G}}$ is then the stack quotient $[\mathcal{M}_{0,4}/(\mathbb{Z}/2\mathbb{Z})]$, where the action is the one that exchanges q_1 and q_2. You can work out that with the identification $\widetilde{\mathcal{M}_{\mathbf{G}}} = \mathbb{A}^1 - \{0, 1\}$ above, the action sends a to $1 - a$.[7]

Thus the quotient $\mathcal{M}_{\mathbf{G}}$ is a once-punctured plane with a $\mathbb{Z}/2\mathbb{Z}$-stacky point, corresponding to the fixed point $(0, 1, \infty, 1/2)$ of $\mathcal{M}_{0,4}$ under $\mathbb{Z}/2\mathbb{Z}$. It is the stacky point that produces monodromy: walking around it interchanges the analytic branches of the boundary divisor that meet along it.

10.5. **Beyond clutching and gluing.** We previously defined

$$\mathcal{M}_G := \prod_{v \in V(G)} \mathcal{M}_{w(v),L(v)},$$

and we may as well define

$$\overline{\mathcal{M}}_G := \prod_{v \in V(G)} \mathcal{M}_{w(v),L(v)}.$$

One might guess that $[\overline{\mathcal{M}}_G / \operatorname{Aut}(G)]$ is the closed stratum D_G, the moduli space of stable curves with dual graph admitting a sequence of edge contractions to G. This is close: in fact [ACG11, XII.10.11, p. 315] $[\overline{\mathcal{M}}_G / \operatorname{Aut}(G)]$ is canonically identified with the *normalization* of D_G.

By repeatedly clutching and gluing (and in any order), notice that for any stable graph $\mathbf{G}$ of type (g, n) we have a map

$$\overline{\mathcal{M}}_G \to \overline{\mathcal{M}}_{g,n}.$$

inducing maps on homology

$$H_*(\overline{\mathcal{M}}_G; \mathbb{Q}) \to H_*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}).$$

The data of

(1) a graded S_n-representation $H_*(\overline{\mathcal{M}}_{g,n})$ for each pair of nonnegative integers g, n satisfying $2g - 2 + n > 0$, and
(2) for each stable graph G of type (g, n), a morphism

$$H_*(\overline{\mathcal{M}}_G; \mathbb{Q}) \to H_*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$$

[7]We can use, e.g., the following convention for the *cross ratio*

$$\operatorname{cr}(a, b, c, d) = \frac{(a - d)(b - c)}{(a - b)(d - c)}$$

as a coordinate on $\mathcal{M}_{0,4}$. Then

$$\operatorname{cr}(0, 1, \infty, x) = x, \qquad \operatorname{cr}(0, 1, x, \infty) = 1 - x.$$

forms an example of a *modular operad.* But first, what is an operad?

Part 3. **Operads**

11. Operads

A great beginner's reference is [Val14], h/t Kaelin Cook-Powell.

Here is an analogy. Recall that a linear representation of a group G is a vector space V together with a homomorphism of groups $G \to \mathrm{GL}(V)$.

This definition allows one to isolate the notion of a group and study it in the abstract, separately from a specific action on some object in a category like a vector space. This is useful on a number of levels: a group G can be studied in the abstract. Then it can be related to a particular object in a category, like a vector space V, by specifying a particular group $\mathrm{GL}(V)$ and a homomorphism $G \to \mathrm{GL}(V)$.

Similarly, an operad isolates the notion of an operation on objects in a symmetric monoidal category. For example, how would you define an associative operation on V? Our goal is to define the notion of an *operad*, which isolates the very notion of "operation." Then, to describe an associative operation on V, we would do it as in this table of analogies:

group	operad
the group $\mathrm{GL}(V)$	the endomorphism operad $\mathrm{End}(V)$
a particular group G	a particular associative operad Ass
linear representation $G \to \mathrm{GL}(V)$	associative algebra $\mathrm{Ass} \to \mathrm{End}(V)$

We work over Vect_k for concreteness; note Vect_k can be replaced by any symmetric monoidal category.

Definition 11.1. An $\mathbb{S}$-module over Vect_k is a sequence $\{V_n\}_{n\geq 1}$ of right S_n-representations.

Definition 11.2. An *symmetric operad* or simply *operad* over Vect_k is

(1) an $\mathbb{S}$-module $\mathcal{P}$ over Vect_k,
(2) a *unit* $e\colon K \to \mathcal{P}(1)$ (regarded as picking out a unit $I = e(1) \in \mathcal{P}(1)$, and
(3) for all k and numbers $n_1, \dots, n_k \geq 1$, a map

$$\mu_{n_1,\dots,n_k,k}\colon \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \otimes \mathcal{P}(k) \to \mathcal{P}(n_1 + \cdots + n_k),$$

satisfying unital, associative, and equivariance axioms (omitted).

Remark 11.3. (An example of an equivariance axiom). Say f, g_1, and g_2 are functions $V \to V$ that are binary, binary, and ternary respectively. Then $\sigma = (12) \in S_2$ acts by

$$f^{(12)}(g_1(x_1,x_2,x_3), g_2(x_4,x_5)) = f(g_2(x_4,x_5), g_1(x_1,x_2,x_3)).$$

In other words, we have an equality of 5-ary functions

$$f^{(12)}(g_1(-,-,-), g_2(-,-)) = f(g_2(-,-), g_1(-,-,-))^{\tau}$$

where $\tau = \binom{1\ 2\ 3\ 4\ 5}{4\ 5\ 1\ 2\ 3} \in S_5$.

Exercise 11.4. Encode the example above as an equivariance axiom.

Example 11.5.

(1) The *commutative operad* Com has $\mathrm{Com}(n) = K$ as a trivial right S_n-module, for each n.
(2) The *associative operad* Ass has $\mathrm{Ass}(n) = K[S_n]$ as a right S_n-module, for each n.
(3) Let V be a vector space. The endomorphism operad $\mathrm{End}(V)$ is given by $\mathrm{End}(V)(n) = \mathrm{Hom}(V^{\otimes n}, V)$.

Exercise 11.6. Define a morphism of operads.

Definition 11.7. Let $\mathcal{P}$ be an operad. Then a *representation* of $\mathcal{P}$, also called a $\mathcal{P}$*-algebra* structure, is a morphism of operads $\mathcal{P} \to \mathrm{End}(V)$.

Example 11.8. A morphism of operads $\mathrm{Ass} \to \mathrm{End}(V)$ is exactly the data of an associative algebra structure on V, i.e., a bilinear operation $\times\colon V \times V \to V$ that is associative.

In one direction, a morphism $\mathrm{Ass} \to \mathrm{End}(V)$ restricts to $K[S_2] = \mathrm{Ass}(2) \to \mathrm{Hom}(V \otimes V, V)$ yielding a bilinear map f which is the image of $\mathrm{id} \in S_2$. It might seem like the morphism $\mathrm{Ass} \to \mathrm{End}(V)$ has many more layers of data, for example a ternary operation g which is the image of id in $K[S_3] = \mathrm{Ass}(3)$, a 4-ary operation, and so on. However, we have diagrams

$$\begin{array}{ccc} K[S_2] \otimes K[S_1] \otimes K[S_2] & \longrightarrow & K[S_3] \\ \downarrow & & \downarrow \\ \mathrm{End}(V)(2) \otimes \mathrm{End}(V)(1) \otimes \mathrm{End}(V)(2) & \longrightarrow & \mathrm{End}(V)(3) \end{array} \qquad \begin{array}{ccc} \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} & \longmapsto & \mathrm{id} \\ \downarrow & & \downarrow \\ f \otimes 1 \otimes f & \longmapsto & f(f(-,-),-) \end{array}$$

$$\begin{array}{ccc} K[S_1] \otimes K[S_2] \otimes K[S_2] & \longrightarrow & K[S_3] \\ \downarrow & & \downarrow \\ \mathrm{End}(V)(1) \otimes \mathrm{End}(V)(2) \otimes \mathrm{End}(V)(2) & \longrightarrow & \mathrm{End}(V)(3) \end{array} \qquad \begin{array}{ccc} \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} & \longmapsto & \mathrm{id} \\ \downarrow & & \downarrow \\ 1 \otimes f \otimes f & \longmapsto & f(-,f(-,-)) \end{array}$$

which show that $g = f(f(-,-),-) = f(-,f(-,-))$, and in particular that f is associative. And so on.

11.1. **Operads as monoids in $\mathbb{S}$-modules.** It's worth mentioning a more algebraic definition of an operad, as a monoid in the category of $\mathbb{S}$-modules. Let k be a field of characteristic 0.

Definition 11.9. An $\mathbb{S}$-module is a functor $\mathbb{B} \to \mathsf{Vect}_k$ where $\mathbb{B}$ is the groupoid of finite sets and isomorphisms.

(Alternatively, take $\mathbb{B}$ to be the skeletal category consisting of one object $\{1, \ldots, n\}$ for every integer $n = 0, 1, \ldots$ and all isomorphisms. However, *not* passing to the skeleton has some advantages.) It is also interesting to replace Vect_k with GrVect_k, as is done in [Get95] or [GK98]; or Set; in the latter case we have exactly a *combinatorial species*.

Definition 11.10. Define a monoid structure on $\mathbb{S}$-modules which we shall call plethysm: given $\mathbb{S}$-modules $\mathcal{V}, \mathcal{W}$, let $\mathcal{V} \circ \mathcal{W}$ be the $\mathbb{S}$-module given by

$$(\mathcal{V} \circ \mathcal{W})(n) := \bigoplus_{d \geq 0} \left(\mathcal{V}(d) \otimes \bigoplus_{f\colon [n] \to [d]} \bigotimes_{i=1}^{d} \mathcal{W}(f^{-1}(i)) \right)_{S_d} .$$

and define the unit $\mathbb{S}$-module $\mathbf{1}$ to be the trivial S_1-representation in arity 1 and 0 in other arities.

Then an operad is a monoid in the monoidal category of $\mathbb{S}$-modules. In other words, an operad is an $\mathbb{S}$-module $\mathcal{P}$, and morphisms of $\mathbb{S}$-modules

$$\rho\colon \mathcal{P} \circ \mathcal{P} \to \mathcal{P}, \qquad u\colon \mathbf{1} \to \mathcal{A}$$

satisfying associativity and unit axioms.

Exercise 11.11. Spell out the equivalence of the two definitions.

Example 11.12. Operad of little discs D^k. This is a topological operad, with $D^k(n)$ being the topological space parametrizing arrangements of n labelled nonoverlapping k-discs inside a unit k-disc. What is the operadic structure?

See the survey [Val14, Theorem 10] and the references to [Boardman-Vogt] and [Ma] therein for the following:

Exercise 11.13. Any k-fold loop space $\Omega^k(Y)$ is an algebra over D^k.

Recall the loop space $\Omega(Y, y_0)$, the space of based maps $\mathrm{Map}(S^1, Y)$, topologized by the compact-open topology. It is itself a based space, based at the constant loop. Iterated loop spaces $\Omega^k(Y) = \Omega(\Omega(\cdots(Y))$, and more specifically iterated *deloopings* play a central role in the construction of groups $K_i(R)$ associated to a ring R in algebraic K-theory.

Notice that (dropping basepoints from the notation whenever convenient)

$$\mathrm{Map}(S^1, \mathrm{Map}(S^1, Y)) \cong \mathrm{Map}(S^1 \wedge S^1, Y),$$

where $S^1 \wedge S^1 = (S^1 \times S^1)/(S^1 \bigvee S^1) \cong S^2$ is the smash product on pointed topological spaces. In other words, there is a smash/map adjunction, in an appropriately restricted category of *weakly Hausdorff, compactly generated* (WHCG) spaces.

Theorem 11.14. Any connected topological space X that is an algebra over D^k is homotopy equivalent to some k-fold loop space $\Omega^k(Y)$.

12. Cyclic operads

Possible reference: [Get95, §2.2, p. 6], [GK95]. A cyclic $\mathbb{S}$-module is an $\mathbb{S}$-module $\mathcal{P}$ with an action of S_{n+1} on each $\mathcal{P}(n)$ extending the action of S_n. Write $v^* = (01 \cdots n) \cdot v$. Write $\mathcal{P}((n)) = \mathcal{P}(n-1)$, so $\mathcal{P}((n))$ has a right S_n action.

Definition 12.1. A cyclic operad is an operad $\mathcal{P}$ which is simultaenously an cyclic $\mathbb{S}$-module, such that $1^* = 1$ and

$$a(1, \ldots, 1, b)^* = b^*(a^*, 1, \ldots, 1). \tag{1}$$

(Draw a picture.) Side remark: if we replace Vect_k by GrVect_k then the equality above is replaced by

$$a(1,\dots,1,b)^* = b^*(a^*,1,\dots,1)\cdot(-1)^{|a||b|}.$$

Example 12.2. Suppose V has nondegenerate symmetric bilinear pairing $\langle -,-\rangle$. Then $\mathrm{End}(V)$ is a cyclic operad, given by the right S_{n+1} action on

$$\mathrm{Hom}(V^{\otimes n+1},k) \cong \mathrm{Hom}(V^{\otimes n},\mathrm{Hom}(V,k)) = \mathrm{Hom}(V^{\otimes n},V) = \mathrm{End}(V)(n),$$

where the pairing $\langle -,-\rangle$ provides the identification between V and $\mathrm{Hom}(V,k)$.

For example, if $T \in \mathrm{Hom}(V,V)$ then what is $T^* \in \mathrm{Hom}(V,V)$? We have

$$\begin{gathered}\mathrm{Hom}(V,V) \xrightarrow{\cong} \mathrm{Hom}(V,\mathrm{Hom}(V,k)) \xrightarrow{\cong} \mathrm{Hom}(V\otimes V,k)\\ T \longmapsto (v \mapsto \langle -,Tv\rangle) \longmapsto (v\otimes w \mapsto \langle w,Tv\rangle).\end{gathered}$$

And

$$\begin{gathered}\mathrm{Hom}(V,V) \leftarrow \mathrm{Hom}(V,\mathrm{Hom}(V,k)) \leftarrow \mathrm{Hom}(V\otimes V,k),\\ T^* \longleftarrow (v \mapsto (w \mapsto \langle v,Tw\rangle)) \longleftarrow (v\otimes w \mapsto \langle v,Tw\rangle)\end{gathered}$$

where T^* is given by

$$\langle T^*v,w\rangle = \langle v,Tw\rangle$$

for all v,w. Thus T^* is exactly the *adjoint* of T, in the familiar case that V is an inner product space, and condition (1) amounts to the fact that

$$(TS)^* = S^*T^*.$$

Exercise 12.3. More generally, if $T \in \mathrm{Hom}(V^{\otimes n},V)$, then $T^* \in \mathrm{Hom}(V^{\otimes n})$ is the unique linear map satisfying

$$\langle v_n, T^*(v_0\otimes\cdots\otimes v_{n-1})\rangle = \langle(v_0, T(v_1\otimes\cdots\otimes v_n)\rangle.$$

Definition 12.4. An symmetric pairing $\langle -,-\rangle$ on an associative k-algebra V is *invariant* if for all $v_i \in V$,

$$\langle v_0, v_1v_2\rangle = \langle v_0v_1,v_2\rangle, \quad \text{equivalently} \quad \langle v_0,v_1v_2\rangle = \langle v_2,v_0v_1\rangle,$$

by symmetry of the pairing.

By iterating, the condition in Definition 12.4 is equivalent to

$$\langle v_0, v_1\cdots v_n\rangle = \langle v_n, v_0\cdots v_{n-1}\rangle \tag{2}$$

for all $v_i \in V$.

Now turn the question on its head: suppose we have given to us a vector space V and a nondegenerate symmetric pairing $\langle -,-\rangle$. What would it mean to put an associative structure on V with respect to which the pairing $\langle -,-\rangle$ is invariant?

It would be exactly the specification of an associative product

$$f\colon V\otimes V \to V$$

such that the n-fold product map $f_n\colon V^{\otimes n}\to V$ satisfies (2), in other words

$$f_n^* = f_n.$$

In other words, we expect the cyclic operad Ass to be given in arity n by

$$\mathrm{Ass}(n) = \begin{cases} \mathrm{Ind}_{C_{n+1}}^{S_{n+1}} \mathbf{1} & \text{if } n \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(Recall that a transitive right (say) action of G on X is isomorphic to the action of G on the right cosets of $\mathrm{Stab}(x_0)$, for some $x_0 \in X$.)

Exercise 12.5. Turn the expectation above into a definition of the cyclic operad Ass by defining operadic composition.

Remark 12.6. To be fair, the above doesn't quite fit our definition of a cyclic operad because it has no unit. It is, rather, an example of a non-unital operad.

Definition 12.7. A *non-unital operad* is an $\mathbb{S}$-module together with operadic compositions

$$\circ_i \colon \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1)$$

, satisfying associativity and equivariance. This shall be useful

Example 12.8. The commutative cyclic operad is

$$\mathrm{Com}(n) = \begin{cases} \mathbf{1} & \text{if } n \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

13. Graph complexes from cyclic operads

There are now functors (even,odd) from cyclic operads to chain complexes. The details are left to [CV03], which describe the even case, which was the case that [Kon93] focuses on.

The chain complex $\mathcal{G}^{\mathcal{P}}$ associated to an operad $\mathcal{P}$ in rational vector spaces is the direct sum, over (even resp. odd) oriented graphs G (connected, minimum vertex valence ≥ 3), of the vector spaces

$$\otimes_{v \in V(G)} \mathcal{P}(\mathrm{Leg}(v)),$$

subject to appropriate relations for isomorphisms $(G, \mathrm{or}) \to (G', \mathrm{or}')$. There are two orientation conventions:

- even, in which an orientation of a graph is defined to be an orientation of the vector space $H_1(G; \mathbb{R}) \oplus \mathbb{R}^{E(G)}$,
- odd, in which an orientation of a graph is defined to be an orientation of $\mathbb{R}^{E(G)}$.

The differential is a signed sum of non-loop contractions *in which the operad $\mathcal{P}$ governs the "merging" of vertices along a contracted edge.*

Because edge-contraction preserves first Betti number (genus), the complex $\mathcal{G}^{\mathcal{P}}$ splits as a direct sum

$$\mathcal{G}^{\mathcal{P}} = \bigoplus_g \mathcal{G}^{\mathcal{P},g}.$$

One wishes for the Euler characteristics of the complexes $\mathcal{G}^{\mathcal{P},g}$. That is hard. What is achievable is to compute the orbifold Euler characteristics of these complexes. Kontsevich asserts that it is reasonable to expect that this is asymptotically the same—most graphs, one expects, have no automorphisms. However, recently, Borinsky-Vogtmann give a refined perspective on the asymptotics.

Definition 13.1. Define the orbifold Euler characteristic: I take this to mean the sum

$$\sum_{[G]}(-1)^{|V(G)|}\left(\frac{\dim\left(\bigotimes_{v\in V(G)}\mathcal{P}(v)\right)_{\mathrm{Aut}(G)}}{|\operatorname{Aut} G|}\right)$$

over isomorphism classes of connected graphs of first Betti number g. (If $\mathcal{P}$ vanishes in arity ≤ 1 then only graphs with 3-valent vertices can contribute to the sum, and the sum can thus be shown to be finite.

Exercise 13.2. Compute the orbifold Euler characteristic of the Commutative and Associative graph complexes in genus 2.

14. Interlude: Fulton-MacPherson compactifications

Part 4. Mixed Hodge structures and the weight spectral sequence

Definition 14.1. Let R be a subring of $\mathbb{R}$; think of $R = \mathbb{Z}$, or $R = \mathbb{Q}$ or $\mathbb{R}$. An *R-Hodge structure*, pure of weight n, is a finitely generated R-module H and a decomposition

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=n} H^{p,q}$$

such that $\overline{H^{p,q}} = H^{q,p}$.

A morphism $(H, H^{p,q}) \to (H', H'^{p,q})$ between Hodge structures, both pure of weight n, is a map $f\colon H \to H'$ such that the complexification $f_{\mathbb{C}}\colon H_{\mathbb{C}} \to H'_{\mathbb{C}}$ sends $H^{p,q}$ to $(H')^{p,q}$.

Exercise 14.2. Given $u \in \mathbb{C}\setminus\mathbb{R}$, define a Hodge structure of weight 1 and dimension 2 temporarily denoted H_u:

$$H_u = \mathbb{Z}^2 \qquad (H_u)_{\mathbb{C}} = H^{1,0} \oplus H^{0,1} = \langle(1,u)\rangle \oplus \langle(1,\overline{u})\rangle$$

Classify all weight 1, dimension 2 Hodge structures supported in bidegrees $(1,0)$ and $(0,1)$.

Discussion of Hodge theorem.

15. Mixed Hodge structures

Definition 15.1. An R-mixed Hodge structure is

(1) a finitely generated R-module H,
(2) an ascending *weight* filtration on H

$$\cdots \subset W_j \subset W_{j+1} \subset \cdots = H$$

(3) a decending *Hodge* filtration on $H_{\mathbb{C}}$

$$H_{\mathbb{C}} \cdots F^p \supset F^{p+1} \supset \cdots$$

such that for each k,

$$\mathrm{Gr}_k^W H := W_k/W_{k-1}$$

with filtration on $(\mathrm{Gr}_k^W H)_{\mathbb{C}}$ induced by $F^\bullet$ is a pure Hodge structure of weight k.

A morphism of these is a map $H \to H'$ respecting the filtrations. These form an abelian category, but *not* semisimple, as explored in the exercise below.

Exercise 15.2. (Good first exercise adapted from S. Howe's helpful talk notes on mixed Hodge structures). Compute the abelian group

$$\mathrm{Ext}^1_{\mathbb{Q}\text{-MHS}}(\mathbb{Q}(-1), \mathbb{Q}).$$

Here $\mathbb{Q}(-n)$ is the 1-dimensional pure rational Hodge structure of weight $2n$, so $H^{n,n} \cong \mathbb{Q}$. (Actually by convention/for reasons that haven't yet been explained, one takes $H^{n,n} = (2\pi i)^n\mathbb{Q}$.)

Brief background: for H a MHS, the functors $\mathrm{Ext}^i(H, -)\colon \mathrm{MHS} \to \mathrm{AbGp}$ are the right derived functors of $\mathrm{Hom}(H, -)$, and the elements of $\mathrm{Ext}^1(H, H')$ are in bijection with isomorphism classes of extensions

$$0 \to H' \to E \to H \to 0.$$

Let $E = (H, W_\bullet, F^\bullet)$ be a rational MHS with

$$0 \to \mathbb{Q} \to E \to \mathbb{Q}(-1) \to 0.$$

(1) Argue that up to isomorphism, we may take

$$W_{-1} = 0, W_0 = W_1 = \mathbb{Q} \times 0 = \mathbb{Q}, W_2 = \mathbb{Q}^2.$$

(2) Write $\mathbb{C} = \mathbb{C} \times 0 \subset \mathbb{C}^2$. Argue that

$$F^0 \cap \mathbb{C} = \mathbb{C},\ F^1 \cap \mathbb{C} = 0,$$
$$\dim F^1 = 1, \quad F^2 = 0.$$

(3) What are all possibilities for F^1?
(4) Conclude the problem.

16. Spectral sequences

Let's work over $\mathbb{Q}$-vector spaces for a warm fuzzy feeling, but may freely replace this category with R-modules or any abelian category.

A (cohomological) *spectral sequence* $E_\bullet^{p,q}$ is:

- a vector space $E_r^{p,q}$ for each $r \in \mathbb{Z}_{\geq 0}$ and each $p, q \in \mathbb{Z}$;
- For each r, a differential
$$\partial_r \colon E_r^{p,q} \to E_r^{p-r,q+r-1}$$
with $\partial_r^2 = 0$,
- For each r, p, q, an isomorphism
$$E_{r+1}^{p,q} \cong \ker(\partial_r \colon E_r^{p,q} \to E_r^{p-r,q+r-1}) / \operatorname{im}(\partial_r \colon E_r^{p+r,q-r+1} \to E_r^{p,q}).$$

The last bullet point is often suppressed.

That is *all*. The remarkable part is how ubiquitous spectral sequences are in geometry/topology:

Proposition 16.1. Suppose $X_0 \subset X_1 \subset \cdots \subset X_n = X$ is a filtration of a space X by closed subspaces.

Then there is a spectral sequence in compactly supported cohomology

$$E_1^{p,q} = H_c^{p+q}(X_p; \mathbb{Q}) \Rightarrow E_\infty^{p,q} = H_c^{p+q}(X).$$

Proposition 16.2. More generally, suppose X is a stratified space $X = \bigcup_{\alpha \in P} S_\alpha$, where $(P, \leq)$ is the poset of strata (bigger strata are *lower* in P. Let $\sigma \colon P \to \mathbb{Z}$ be any strictly increasing function on P. For example, σ could measure codimension of strata, with appropriate constant shift.

Then there is a spectral sequence

$$E_1^{p,q} = \bigoplus_{\sigma(\beta)=-p} H_c^{p+q}(S_\beta); \mathbb{Z}) \Rightarrow H_c^{p+q}(X; \mathbb{Z}).$$

16.1. The Philip Hall Theorem and Petersen's spectral sequence.

Part 5. Matroids

Source: course notes from a graduate course on matroids by P. Seymour.

Definition 16.3. Definition of a matroid on ground set S by way of independence. Base, circuit, rank.

Proposition 16.4. Submodularity of rank.

Definition 16.5. Let $A, B \subseteq S$. Then A *spans* B if $r(A \cup B) = r(A)$.

Proposition 16.6. *A spans B if and only if for all $x \in B \setminus A$, there is a circuit $C_x \subseteq A \cup \{x\}$ containing x.*

Proof. Suppose $r(A \cup B) = r(A)$. Then for all $x \in B \setminus A, I \cup \{x\}$ is dependent, so contains a circut C_x that necessarily contains x since it can't lie entirely in I.

Conversely, suppose $r(A \cup B) > r(A)$. Let I be maximal independent in A. Then the hypothesis implies that there is some $x \in B \setminus A$ such that $I \cup \{x\}$ is independent. Now suppose for a contradiction there were some circuit $C_x = J \cup \{x\}$ for some independent $J \subseteq A$. Extend J to an independent set J' of size $|I| + 1$ inside $A \cup \{x\}$; this is possible since $I \cup \{x\}$ is one such set. Now J' can't contain x since J' can't contain C_x; therefore $J' \subseteq A$. But then $|J'| > |I|$, contradicting that I was maximal independent in A. □

Proposition 16.7. The two base exchange properties.

Proposition 16.8. Let $\mathcal{B} \subseteq \mathcal{P}(S)$. Then $\mathcal{B}$ is the set of bases of a matroid if and only if

(1) $\mathcal{B} \neq \emptyset$,
(2) all elements of $\mathcal{B}$ have the same size,
(3) Given $B_1, B_2 \in \mathcal{B}$ and given $x_1 \in B_1 \setminus B_2$, there exists $x_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x_1\}) \cup \{x_2\} \in \mathcal{B}$.

Proof. The forward direction was already proved. For the backwards direction, let $\mathcal{I}$ be the set of subsets of elements of $\mathcal{B}$. Let $X, Y \in \mathcal{I}$, with $|X| < |Y|$. The goal is to show that there is some $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$.

First, choose $B_2 \supseteq Y$ a base, and choose $B_1 \supseteq X$ another base subject to the property that $B_1 \cap B_2$ is maximal.

Suppose for a moment that there exists $y \in B_1 \cap Y$ with $y \notin X$. Then $X \cup \{y\} \in \mathcal{I}$, since $X \cup \{y\}$ is a subset of $\mathcal{B}$, and we win. So we may assume instead that $B_1 \cap Y \subseteq X$.

Now suppose for a moment that there is some $x_1 \in B_1 \setminus B_2$ with $x_1 \notin X$. Then there exists some $x_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x_1\}) \cup \{x_2\} \in \mathcal{B}$. This new base still contains X by design, and has larger intersection with B_2 than B_1 did, contradicting the choice of B_1. Therefore we may assume instead that $(B_1 \setminus B_2) \subseteq X$.

Now draw a Venn diagram with B_1 and B_2, and use it to compare the sizes of X and Y. The point is that Y has at most as many elements as X *within* $B_1 \cap B_2$, because $Y \cap B_1 \subseteq X$. And Y has at most as many elements as X *within* $B_1 \Delta B_2$, because in fact every single element of $B_1 \setminus B_2$ is in X. Conclude that $|Y| \leq |X|$, contradiction.

□

References

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths, *Geometry of algebraic curves. Volume II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 268, Springer, Heidelberg, 2011, With a contribution by Joseph Daniel Harris. MR 2807457

[ACP15] Dan Abramovich, Lucia Caporaso, and Sam Payne, *The tropicalization of the moduli space of curves*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 4, 765–809. MR 3377065

[Alu14] Paolo Aluffi, *Generalized Euler characteristics, graph hypersurfaces, and Feynman periods*, Geometric, algebraic and topological methods for quantum field theory, World Sci. Publ., Hackensack, NJ, 2014, pp. 95–136. MR 3204960

[BB03] Prakash Belkale and Patrick Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, Duke Math. J. **116** (2003), no. 1, 147–188. MR 1950482

[BEK06] Spencer Bloch, Hélène Esnault, and Dirk Kreimer, *On motives associated to graph polynomials*, Comm. Math. Phys. **267** (2006), no. 1, 181–225. MR 2238909

[Bro21] Francis Brown, *Invariant differential forms on complexes of graphs and Feynman integrals*, 2021.

[BW18] Matthew Baker and Yao Wang, *The Bernardi process and torsor structures on spanning trees*, Int. Math. Res. Not. IMRN (2018), no. 16, 5120–5147. MR 3848228

[CCG15] Melody Chan, Thomas Church, and Joshua A. Grochow, *Rotor-routing and spanning trees on planar graphs*, Int. Math. Res. Not. IMRN (2015), no. 11, 3225–3244. MR 3373049

[CGP21] Melody Chan, Søren Galatius, and Sam Payne, *Tropical curves, graph complexes, and top weight cohomology of $\mathcal{M}_g$*, J. Amer. Math. Soc. **34** (2021), 565–594.

[Cha17] Melody Chan, *Lectures on tropical curves and their moduli spaces*, Moduli of curves, Lect. Notes Unione Mat. Ital., vol. 21, Springer, Cham, 2017, pp. 1–26. MR 3729073

[Cha21] ———, *Moduli spaces of curves: classical and tropical*, Notices Amer. Math. Soc. **68** (2021), no. 10, 1700–1713. MR 4323846

[CLNS18] Antoine Chambert-Loir, Johannes Nicaise, and Julien Sebag, *Motivic integration*, Progress in Mathematics, vol. 325, Birkhäuser/Springer, New York, 2018. MR 3838446

[CP05] Denis Chebikin and Pavlo Pylyavskyy, *A family of bijections between G-parking functions and spanning trees*, J. Combin. Theory Ser. A **110** (2005), no. 1, 31–41. MR 2128964

[CV03] James Conant and Karen Vogtmann, *On a theorem of Kontsevich*, Algebr. Geom. Topol. **3** (2003), 1167–1224. MR 2026331

[DSW21] Graham Denham, Mathias Schulze, and Uli Walther, *Matroid connectivity and singularities of configuration hypersurfaces*, Lett. Math. Phys. **111** (2021), no. 1, Paper No. 11, 67. MR 4205801

[Ell] Jordan Ellenberg, *"What is the sandpile torsor?"*, http://mathoverflow.net/questions/83552.

[Get95] E. Getzler, *Operads and moduli spaces of genus* 0 *Riemann surfaces*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 199–230. MR 1363058

[GK95] E. Getzler and M. M. Kapranov, *Cyclic operads and cyclic homology*, Geometry, topology, & physics, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995, pp. 167–201. MR 1358617

[GK98] ———, *Modular operads*, Compositio Math. **110** (1998), no. 1, 65–126. MR 1601666

[Hai02] Mark Haiman, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149** (2002), no. 2, 371–407. MR 1918676

[Kon93] Maxim Kontsevich, *Formal (non)commutative symplectic geometry*, The Gel′fand Mathematical Seminars, 1990–1992, Birkhäuser Boston, Boston, MA, 1993, pp. 173–187. MR 1247289

[Loo93] E. Looijenga, *Cohomology of $\mathcal{M}_3$ and $\mathcal{M}_3^1$*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., vol. 150, Amer. Math. Soc., Providence, RI, 1993, pp. 205–228.

[PS04] Alexander Postnikov and Boris Shapiro, *Trees, parking functions, syzygies, and deformations of monomial ideals*, Trans. Amer. Math. Soc. **356** (2004), no. 8, 3109–3142. MR 2052943

[Sta98] Richard P. Stanley, *Spanning trees and a conjecture of Kontsevich*, Ann. Comb. **2** (1998), no. 4, 351–363. MR 1774974

[Sta99] ———, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR 1676282

[Ste98] John R. Stembridge, *Counting points on varieties over finite fields related to a conjecture of Kontsevich*, Ann. Comb. **2** (1998), no. 4, 365–385. MR 1774975

[Val14] Bruno Vallette, *Algebra + homotopy = operad*, Symplectic, Poisson, and noncommutative geometry, Math. Sci. Res. Inst. Publ., vol. 62, Cambridge Univ. Press, New York, 2014, pp. 229–290. MR 3380678

Department of Mathematics, Brown University, Box 1917, Providence, RI 02912
E-mail address: melody_chan@brown.edu