Notes for Math 2720T: Graphs, matroids, and moduli spaces

1. ENGAGED PEDAGOGY

Reference: *Teaching to Transgress: Education as the Practice of Freedom* by bell hooks.

“The classroom remains the most radical space of possibility in the academy.”

**Question 1.1.** Is it possible for a graduate topics course in graphs, matroids, and moduli spaces be about the practice of freedom, of liberation? Is transgression an appropriate word to describe graduate education in mathematics? Should I be teaching to transgress, should you be learning to transgress? And how, if so? What specific structures, what practices can this class build towards the end of liberation? What forms of liberation would an ideal graduate topics course bring you?

**Question 1.2.** bell hooks ties engaged pedagogy to self-actualization, bringing the whole self to the classroom. In what ways is it challenging to bring our whole selves into a mathematics classroom? In what ways can we rise to the challenge?

How does your thinking above relate to the *physical* classroom versus a virtual, Zoom classroom? We’ve spent a long time out of the physical classroom. What did we lose, and what specifically do we gain in convening in the physical classroom, as far as bringing our fully engaged, whole selves? (And what do we lose when we step away from Zoom and into the physical classroom?)

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Part 1. Graphs

2. Basics

Definition 2.1. A graph $G$ is a pair of finite sets $V$ and $H$, a fixed point free involution $i$ on $H$, and a root map $r : H \to V$.

The edge set $E(G)$ is the set of pairs $\{ h, i(h) \}$. Can relax $i$ to be an arbitrary involution, to get a graph with legs $L(G)$. These form the objects of a groupoid $\mathcal{G}^\circ$ of graphs. There is a geometric realization functor $| \cdot | : \mathcal{G}^\circ \to \text{Top}$.

3. Spanning trees

Let $G = (V(G), H(G), i, r)$ be a graph. A subgraph is a choice of $V' \subseteq V$ and $H' \subseteq H$ such that $i(H') \subseteq H'$ and $r(H') \subseteq V'$. Informally: choose some of the vertices, and some edges between them. $G$ is called acyclic if it contains no cycle as a subgraph, i.e., $H_1(G; \mathbb{Z}) = 0$. $G$ is called a tree if it is connected and acyclic.

Say a subgraph $H$ is spanning if $V(H) = V(G)$.

Proposition 3.1. Let $G$ be a connected graph on $n$ vertices. Let $T$ be a spanning subgraph. The following are equivalent:

1. $T$ is minimal connected.
2. $T$ is maximal acyclic.
3. $T$ has $n-1$ edges and is connected.
4. $T$ has $n-1$ edges and is acyclic.

In these situations, say $T$ is a spanning tree.

Exercise 3.2.Enumerate the spanning trees of $K_4$.

Proof. Orient the edges of $G$ arbitrarily; now you have a CW-complex. Consider the cellular chain complex

$$\mathbb{Q}E \xrightarrow{\partial} \mathbb{Q}V.$$  

Because $G$ is connected, $\dim \text{Im}(\partial) = n - 1$; more generally,

$$\dim \text{Im}(\partial) = n - \dim H_0(G; \mathbb{Q}) = n - \# \{ \text{connected components} \}.$$  

Consider the set of vectors

$$S = \{ \partial e : e \in E(T) \}.$$  

By the same reasoning, a spanning subgraph $T$ is

1. connected iff $S$ is a spanning set for $\text{Im}(\partial)$,
2. acyclic iff $S$ is linearly independent.

The proposition then follows from linear algebra: these are four equivalent ways of saying that $S$ is a basis for $\text{Im}(\partial)$.

Let $\mathcal{T}(G)$ denote the set of spanning trees of $G$. Evidently $\mathcal{T}$ is a functor $\mathcal{T} : \mathcal{G}^\circ \to \text{Set}$. 

□
Theorem 3.3. (Cayley’s Theorem, actually due to Borchardt 1860)

\[ |\mathcal{T}(K_n)| = n^{n-2}. \]

That is, the number of trees on vertex set \{1, \ldots, n\} is \(n^{n-2}\).

Remark 3.4. Possibly discuss two proofs: by Prüfer code\(^1\) and Joyal’s proof of 1980 blogged about [here].

Exploration 3.5. What is the \(S_n\)-representation on the \(n^{n-2}\) spanning trees of \(K_n\)?

4. Parking functions

Definition 4.1. A parking function is a sequence \((b_1, \ldots, b_n)\) of nonnegative numbers such that if \(c_1 \leq \cdots \leq c_n\) is the reordering, then \(c_i < i\).

Give the interpretation of parking functions in terms of, well, parking. Cars are numbered \(1, \ldots, n\), and parking spots are linearly ordered \(0, \ldots, n-1\).

Example 4.2. The three parking functions for \(n = 2\) are 01, 10, and 00.

There is an amazingly slick proof that the number of parking functions is also \((n+1)^{n-1}\), due to Pollak 1974, see [these slides].

There are known bijections between parking functions and spanning trees of \(K_n+1\):

Exploration 4.3. Postnikov-Shapiro define \(G\)-parking functions for an arbitrary rooted graph \((G,v)\) \([PS04]\). These are functions \(b: V(G) \setminus \{v\} \to \mathbb{Z}_{\geq 0}\) such that for all \(I \subseteq V(G) \setminus \{v\}\), there exists \(v \in I\) such that \(b(i) < \# \text{ edges at } i \text{ leaving } I\). Verify that the case \(G = K_{n+1}\) recovers parking functions of length \(n\). Let \(\text{Park}(G)\) be the set of \(G\)-parking functions. Read [CP05] for a family of bijections between \(\mathcal{T}(G)\) and \(\text{Park}(G)\), and see the references therein.

At this point we have three equinumerous sets: \(\mathcal{T}(G), S(G),\) and \(\text{Park}(G)\). Here’s something worth exploring:

Is there a natural isomorphism between \(\mathcal{T}\) and \(\text{Park}\) as functors \(\text{RootedGraphs} \to \text{Set}\)?

If not (I believe not), add as little rigidifying data to get a natural isomorphism, e.g., try rooted ribbon graphs.

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\(^1\)See my notes for first-year undergraduates at [here], §2.2, or [Sta99] §5.6. Can Prüfer codes be related to any of the other bijections swirling around here? Note that they take as input a graph with total ordering on vertex set.

\(^2\)This has surely been studied, but I asked a couple experts and don’t know where. On the other hand, the \(S_n\) action on the \((n+1)^{n-1}\) parking functions of length \(n\), denoted \(\text{Park}_n\), is of central interest. See [Hai02], where Haiman proves that the space of diagonal harmonics \(\text{DH}_n\), of dimension \((n+1)^{n-1}\), is \(\text{Park}_n \otimes \text{sgn}_n\) as an \(S_n\)-representation and resolves a bunch of conjectures. The proof features the geometry of \(\text{Hilb}_n(\mathbb{C}^2)\).
5. THE MATRIX TREE THEOREM

Definition 5.1. Laplacian matrix of \( G \). Let \( G \) be a connected graph on vertex set \( \{1,\ldots,n\} \). Define a symmetric \( n \times n \) matrix \( \Delta = \Delta(G) \) by

\[
\Delta_{i,j} = \begin{cases} 
-\# \{ \text{edges } ij \} & \text{if } i \neq j \\
\# \{ \text{nonloop edges at } i \} & \text{if } i = j.
\end{cases}
\]

Since row and column sums are all 0, \( \Delta \) is not full rank. On the other hand, the \textit{reduced} Laplacian \( \Delta_{\text{red}}(G) = \Delta_{\text{red}}(G) \), obtained by deleting row \( i \) and column \( i \) for some \( i \), is full rank. This follows immediately from an even more interesting statement:

Theorem 5.2. (Kirchhoff’s Matrix Tree Theorem) For any graph \( G \),

\[
\det \Delta_{\text{red}}(G) = |T(G)|.
\]

Proof. Note \( \partial \partial^t = \Delta \) and \( \partial_i \partial_i^t = \Delta_i^{\text{red}} \), where \( \partial_i \) is obtained from \( \partial \) by deleting row \( i \). Now use Cauchy-Binet. \( \square \)

In the Matrix Tree theorem, we have an equality of numbers. One immediately wishes for a combinatorial proof. The right hand side is evidently the cardinality of a set. The left hand side is naturally the cardinality of a finite abelian group: the sandpile group of \( G \), which we shall define.

Definition 5.3. Let \( G \) be a connected graph. Regard \( \partial : \mathbb{Z}E \to \mathbb{Z}V, \Delta : \mathbb{Z}V \to \mathbb{Z}V \) as an integral map now.

Let \( \text{Div}(G) = \mathbb{Z}V \), the \textbf{divisor group} on \( G \).

Let \( \text{Div}^0(G) = \text{Im}(\partial) \subset \mathbb{Z}V \), the divisors of degree 0.

Let \( \text{Prin}(G) = \text{Im}(\Delta) \), the \textbf{principal divisors}.

The \textbf{sandpile group} \( \text{Pic}^0(G) = S(G) \) is the quotient

\[
\text{Pic}^0(G) = \frac{\text{Div}^0(G)}{\text{Prin}(G)}.
\]

Said with less fanfare, \( S(G) \) is the cokernel of \( \Delta^{\text{red}} : \text{Div}^0(G) \to \text{Div}^0(G) \).

The fact that \( |S(G)| = \det \Delta_{\text{red}}(G) \) must have some good conceptual proof, but the only one I can think of is via Smith normal forms.

Example 5.4.

(1) Compute \( S(G) \) for the \( n \)-edge banana graph (2 vertices, \( n \) parallel edges).

(2) Compute \( S(G) \) for the \( n \)-cycle \( C_n \). Fun fact: \( S(G) \cong S(G^*) \), and with an isomorphism that is canonical up to sign.

Exploration 5.5. Prove \( \det \Delta_{\text{red}}(K_n) = n^{n-2} \). In fact I assert that

\[
S(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2},
\]

3The interesting case is connected graphs. If \( G \) is not connected, both sides are 0.
noncanonically. This is a group-theoretic promotion of Cayley’s theorem. This isomorphism cannot possibly be canonical with respect to Aut(K_n) = S_n. Prove the existence of such an isomorphism, and quantify failure of canonicity.

**Remark 5.6.** Discuss torsor structures on spanning trees.

6. **Matrix Tree Theorem, weighted version**

**Lemma 6.1.** (Weighted Cauchy-Binet theorem) Fix d ≤ n, let A ∈ K^{d×n}, B ∈ K^{n×d}. Let D ∈ K[x_1, ..., x_n]^{n×n} be diagonal with D_{i,i} = x_i. Then

\[ \det(ADB) = \sum_{S=\{i_1, ..., i_d\}} \det(A_S) \det(B_S) \cdot x_S \in K[x_1, ..., x_n]_d \]

where x_S = x_{i_1} \cdots x_{i_d}. In particular, \( \det(ADB) \) is homogeneous of degree d and supported on squarefree monomials.

For later use, note the interpretation \( A_S = A \cdot \pi_S \) where \( \pi_S \) denotes coordinate projection \( K^n \to K^S \).

In particular, given graph G, let \( D_G \) be the diagonal \( E(G) \times E(G) \) matrix with formal variables \( x_e \) on the diagonal.

**Definition 6.2.** Define the edge-weighted Laplacian

\[ \Delta(\{x_e\}, G) = \Delta = \partial D \partial^T. \]

Explicitly,

\[ \Delta(\{x_e\}, G)_{i,j} = \begin{cases} -\sum_{e=ij} x_e & \text{if } i \neq j \\ \sum_{e \text{ nonloop at } i} x_e & \text{if } i = j \end{cases} \]

Consider any reduction \( \Delta^{\text{red}}(x_e) \) obtained by deleting row and column corresponding to \( e_0 \in E(G) \). Then a consequence of the weighted Cauchy-Binet theorem is

**Corollary 6.3.** (Weighted Matrix Tree Theorem)

\[ \det \Delta^{\text{red}}(\{x_e\}, G) = \sum_{T \in \mathcal{T}(G)} \prod_{e \in E(T)} x_e. \]

7. **Dual Laplacians**

Sources: [Bro21], and especially Bloch-Esnault-Kreimer [BEK06]. Let G be a connected graph. Let \( \mathcal{H}_G = \mathcal{H} : H_1(G; \mathbb{Z}) \hookrightarrow \mathbb{Z}E \) be the inclusion, regarded by abuse of notation also as a matrix with respect to any \( \mathbb{Z} \)-basis for \( H_1 \) and the basis \( E \) for \( \mathbb{Z}E \).

**Definition 7.1.** The dual Laplacian of G is

\[ \Lambda_G = \mathcal{H}_G^T D_G \mathcal{H}_G. \]

**Definition 7.2.** The graph polynomial of G is \( \Psi_G = \det \Lambda_G \in K[x_e: e \in E(G)]. \)

Note that although we chose arbitrarily a \( \mathbb{Z} \)-basis for \( H_1(G; \mathbb{Z}) \), \( \Lambda_G \) is independent of that choice, since another choice results in multiplication of \( \det L_G \) by \((\pm1)^2 = 1\).
Proposition 7.3. (Spanning tree formula for graph polynomial)

\[ \Lambda_G = \sum_{T \in \mathcal{T}(G)} \prod_{e \notin E(T)} x_e. \]

This theorem is proved as a special case of the functional equation below, which I believe is due to [BEK06].

Definition 7.4. The graph hypersurface of \( G \) is \( V(\Lambda_G) \subset \mathbb{A}^{|E(G)|} \).

8. A functional equation for polynomials of configurations

See [BEK06, §1]. Let \( K \) a field, \( E \) a finite set. \( KE \) shall denote the vector space with basis \( E \). This chosen basis gives a choice of identification \( KE \cong (KE)^\vee \).

Definition 8.1. A configuration is a linear subspace \( i_V : V \hookrightarrow KE \).

Let \( d = \dim V \), and let \( M_V \) be the matrix for \( i_V \) with respect to a chosen basis \( v_1, \ldots, v_d \) for \( V \). Define

\[ \Psi_V(x_e) = \det(M_V^t D M_V), \]

where \( D \) is a diagonal matrix with \( D_{i,i} = x_i \). Here we suppress the dependence on choice of basis for \( V \), though note a different choice of basis just changes \( \Psi_V \) by a constant in \( (K^*)^2 \).

Now let \( W \) sit in the short exact sequence

\[ 0 \to V \xrightarrow{i_V} KE \to W \to 0, \]

with dual short exact sequence

\[ 0 \to W^\vee \xrightarrow{i_W^\vee} KE \to V^\vee \to 0, \]

where we identified \( (KE)^\vee \cong KE \).

Let \( s : W \to KE \) be any splitting of the first short exact sequence. Choose ordered bases for \( V \) and \( W \), and an ordering of \( E \) furnishing an ordered basis of \( KE \). The maps \( (i_V, s) \) produce an isomorphism

\[ \det V \otimes \det W \cong \det KE \]

which actually does not depend on choice of \( s \). (Exercise). With respect to our choices of ordered bases of \( V, W \), and \( KE \), we get a \( 1 \times 1 \) matrix \( [c] \), for some \( c \in K^* \), representing this isomorphism. Then, also with respect to our choices, the following equation holds:

Theorem 8.2.

\[ \Psi_V(x_e) = c^2 \left( \prod_{e \in E} x_e \right) \Psi_{W^\vee}(x_e^{-1}). \]

Before proving Theorem 8.2, we use it to deduce Proposition 7.3.
Proof of Proposition 7.3

\[ 0 \to H_1(G; \mathbb{Q}) \to \mathbb{Q}E \xrightarrow{\partial} \text{Im}(\partial) \to 0 \]

and its dual short exact sequence

\[ 0 \to \text{Im}(\partial)^\vee \to \mathbb{Q}E \to H^1(G; \mathbb{Q}) \to 0. \]

We shall apply Theorem 8.2 to this setup to deduce the explicit formula for the graph polynomial. One term in Theorem 8.2 we may recognize as a reduced Laplacian. Indeed, pick any root vertex \( v \), and write \( V' = V \setminus \{v\} \). Then identify \( \text{Im} \partial \cong \mathbb{Q}V' \) for any root vertex \( v \), by sending \( v' \mapsto v' \). To be very formal, then we write \( \partial_v : \mathbb{Q}E \to \mathbb{Q}V' \). Then

\[ \Psi_{\text{Im} \partial} = \det \partial_v D \partial = \det \Delta_v^{\text{red}} = \sum_{T \in \mathcal{T}(G)} \prod_{e \in E(T)} x_e. \]

Next, note that any \( \mathbb{Z} \)-basis for \( H_1(G; \mathbb{Z}) \) together with any \( \mathbb{Z} \)-basis for \( \text{Im} \partial \), for example \( \{v' - v : v' \in V'\} \), form a \( \mathbb{Z} \)-basis for \( \mathbb{Z}E \). This says that, in our situation, the constant \( c = \pm 1 \) in the Theorem.

The formula for the graph polynomial follows. \( \square \)

Proof of Theorem 8.2

References


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