Integral Appollonian Gaskets: This is a worksheet which will help you understand the integral appollonian Gaskets. There is a paper by Jeff Lagarias et. al. on integral appollonian circle packings which has all the material I am going to explain here.

A circle in $\boldsymbol{C} \cup \infty$ has 2 complementary regions, both disks. Assuming that the circle has radius r, the *curvature* of the bounded disk it bounds is $1/r$ and the curvature of the unbounded complementary disk is $-1/r$. The curvatures of the 2 "disks" bounding a straight line are both 0. Say that a *Descartes configuration* is a configuration of 4 mutually tangent disks which have disjoint interiors. Given such disks, with curvatures k_1, k_2, k_3, k_3, the famous Descartes formula says that

$$Q(r_1, r_2, r_3, r_4) = 2(r_1^2 + r_2^2 + r_3^2 + r_4^2) - (r_1 + r_2 + r_3 + r_4)^2 = 0. \quad (1)$$

1: Let C be a bounded disk with curvature k, and let r denote the distance from the center of C to the origin. Let $I(z) = \frac{1}{\bar{z}}$ be inversion. Prove that $I(C)$ has curvature $(k^2 r^2 - 1)/k$.

2: Let D_1 be the infinite disk lying to the left of the line $x = 1$. Let D_2 be the infinite disk lying to the right of the line $x = x_0$. Here $x_0 > 1$. Let D_3 and D_4 be two disks tangent to each other and mutually tangent to D_1 and D_2. (Picture a can of tennis balls with 2 tennis balls in it.) Using the formula in Equation 1, verify that Descartes' formula holds for the Descartes configuration $I(D_1), I(D_2), I(D_3), I(D_4)$.

3: Let $\mathcal{D}$ denote the space of all Descartes configurations. Let $\mathcal{U}$ denote the collection of Descartes configurations which arise in Problem 2, together with all the rotated/scaled images of these. Show that $\mathcal{U}$ contains an open subset of $\mathcal{D}$.

4: Let $f : \mathcal{D} \to \boldsymbol{R}$ denote the Descarte relation. We want to show that $f = 0$ on $\mathcal{D}$. We know already that $f = 0$ on $\mathcal{U}$. Observe also that $\mathcal{D}$ is connected because we allow disks bounded by straight lines. Argue that in suitable coordinates, f is an algebraic function. An algebraic function that vanishes on an open set must vanish everywhere. This proves the Descartes relation. (Hint: stereographic projection.)

5: Introduce the *Descartes form*

$$D(V, W) = \frac{Q(V + W) - Q(V) - Q(W)}{2}. \tag{2}$$

Note that $D(V, V) = Q(V)$. There is a matrix Δ associated to D, whose entries are $\Delta_{ij} = D(e_i, e_j)$ where the $(e)s$ are the standard basis vectors in $\boldsymbol{R}^4$. Prove the this matrix is

$$-\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}. \tag{3}$$

Hint: argue that the matrix is curculent, so you only have to compute one row.

6: Introduce the Hadamard Matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{4}$$

This is an amazing matrix: It is the matrix of an element of $SO(4)$ and $H = H^{-1}$. Prove that

$$H\Delta H^{-1} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \tag{5}$$

Thus $(\boldsymbol{R}^4, D)$ is just a rotated and scaled copy of $\boldsymbol{R}^{1,3}$.

7: Use Descartes' Formula to establish the *Soddy relation:* If D_1, D_2, D_3, D_4 and D_1, D_2, D_3, D_4' are two Descartes relations, then

$$k_4' = 2(k_1 + k_2 + k_3) - k_4. \tag{6}$$

8: Introduce the map $T_4 : \boldsymbol{R}^4 \to \boldsymbol{R}^4$ given by

$$T_4(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, 2(x_1 + x_2 + x_3) - x_4). \tag{7}$$

Use the previous exercise to show that T_4 is a D-preserving integral linear transformation such that T_4^2 is the identity. Prove that T_4 is reflection in the hyperplane which is D-perpendicular to $(0,0,0,1)$.

9: Let Γ denote the group generated by T_1, T_2, T_3, T_4, where T_k is defined just like T_4 except with the coords permuted. Interpret the previous exercise hyperbolically and argue that Γ is the group generated by reflections in a collection of 4 symmetrically placed totally geodesic planes in $\boldsymbol{H}^3$. These planes are part of a tetrahedron whose vertices are the permutations of $(1,1,1,-1)$. Draw how these planes intersect the ideal boundary of $\boldsymbol{H}^3$.

10: A *integral Descartes point* is a point of the form $(k_1, k_2, k_3, k_3) \in \boldsymbol{Z}^4$ consisting of the curvatures of disks in a Descartes configuration. In other words, the Descartes points are just the integer D-null vectors. Note that Γ preserves the integral Descartes points. This, the full orbit of an integral Descarte point consists entirely of other integral Descartes points. So, if you start with a Descartes configuration having D_1 as the outside of the unit disks and D_2, D_3, D_4 having integer curvatures, then you can fill in the interstitial disks to make the Appollonian packing so that all the disks have integral curvatures. Do this for the case when $(k_2, k_3, k_4) = (2,2,3)$. Use the Soddy relation to deduce the curvatures of the other disks going several layers in. Check your answers against pictures you can find on the internet.