Here are some hyperbolic geometry problems:

1: Write down the conditions on the numbers a, b, c, d such that the linear fractional transformation

$$T(z) = \frac{az + b}{cz + d}$$

preserves the unit disk.

2: Suppose that Q is a quadratic form in $\boldsymbol{R}^n$ and $T : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a linear transformation such that $Q(T(V), T(V)) = Q(V, V)$ for all nonzero vectors V. Prove that T preserves the quadratic form Q. Use this property to flesh out the argument in class that there are linear transformations of $\boldsymbol{R}^3$ which preserve the standard Lorentz form and do not fix the point $(0, 0, 1)$.

3.: Let $A(r)$ denote the area of a hyperbolic disk of radius r. Prove that $A(r) \geq A \exp(Br)$ for some positive constants A and B. In other words, the area grows exponentially fast in the hyperbolic plane. Hint: don't try for a formula; just get an estimate.

4: Prove that there is some constant C such that the area of any geodesic triangle in the hyperbolic plane is at most C. (In fact, you can take $C = \pi$.)

5: Prove that all the metric disks in the Klein model are ellipses. Which ellipses arise this way?

6: Any open convex domain Ω in the plane has a canonical metric on it called the *Hilbert metric*. The distance between points $b, c \in \Omega$ is defined as

$$\log \frac{\|a - c\| \|b - d\|}{\|a - b\| \|c - d\|}.$$

Here a, d lie in the boundary of Ω and a, b, c, d appear in order on the same line segment. Prove that this really does define a metric on Ω. The main step is the triangle inequality, but with some effort you can reduce to the case of hexagons.

7: Prove that when you put the Hilbert metric on the open unit disk, the result is isometric to the Klein model of the hyperbolic plane.

8: Two metric spaces X and Y are *Lipschitz equivalent* if there is a bijective map $f : X \to Y$ and a constant $K \geq 1$ such that

$$d(f(x_1), f(x_2)) \in [1/K, K]d(x_1, x_2)$$

for all $x_1, x_2 \in X$. Suppose you equip a convex polygon with its Hilbert metric. Is the resuting space bi-lipschitz equivalent to the hyperbolic plane?

9: Consider the metric space X you get by gluing together equilateral triangles 7 around per vertex and then putting the intrinsic metric on this space. That is, the distance between pairs of points is the length of the shortest path in the space joining them. Prove that the resulting space is lipschitz equivalent to the hyperbolic plane. Hint: consider a corresponding tiling of the hyperbolic plane by geodesic equilateral triangles.

10: (Famous unsolved problem) Is it possible to build X, the space from Problem 9, in 3-dimensional space while keeping the triangles rigid and not allowing any self-intersections?