Notes on Pythagorean Triples, by Rich Schwartz:

Primitive Pythagorean Triples: A *primitive Pythagorean triple* (or *PPT* for short) is a triple of integers (a, b, c) such that $a^2+b^2 = c^2$, and the greatest common divisor of (a, b, c) is 1. We alllow the signs to be both positive and negative. We do not consider $(0, 0, 0)$ a PPT. Thus, $c \not0$ for any PPT.

Map to the Circle: We identify a PPT (a, b, c) with the point $(a/c, b/c)$ on the unit circle. The map $(a, b, c) \to (a/c, b/c)$ is 2-to-1 because $(-a, -b, -c)$ gets mapped to the same point. We declare two such PPTs *equivalent* if they are equal or negatives of each other. This the map to the unit circles is injective on equivalence classes. The 8 PPTs

$$\pm(1, 0, 1), \qquad \pm(-1, 0, 1), \qquad \pm(0, 1, 1), \qquad \pm(0, -1, 1).$$

lie in 4 equivalence classes and give rise to 4 points on the unit circle. These points are the vertices of an ideal square, when the unit circle is considered the ideal boundary of the Klein model of the hyperbolic plane. Call these the *basic* PPTs. We call two basic PPTs *adjacent* if they are consecutive on the ideal square. This $(1, 0, 1)$ and $(0, 1, 1)$ are adjacent but $(1, 0, 1)$ and $(-1, 0, 1)$ are not.

Lorentzian Point of View: There is another point of view. We let L denote the Lorentz form:

$$L((x_1, y_1, z_1),(x_2, y_2, z_3)) = x_1x_2 + y_1y_2 - z_1z_2.$$

A vector V is called *null* if $L(V, V) = 0$. The PPTs are precisely the primitive integer null vectors.

Hyperbolic Square Tiling: Consider the two consecutive PPTs $(1, 0, 1)$ and $(0, 1, 1)$. The vector $W = (1, 1, 1)$ has the property that $L(X, W) = 0$ where X is either of the PPTs just listed. Note also that $L(W, W) = 1$. The map

$$I(V) = V - 2L(V, W)W \tag{1}$$

is an L-preserving linear transformation which fixes $(1, 0, 1)$ and $(0, 1, 1)$. This map is just like a Euclidean reflection, except that it is defined relative to L rather than the dot product. The equation is

$$L(x, y, z) = (-x - 2y + 2z, -2x - y + 2z, -2x - 2y + 3z).$$

For instance $L(-1,0,1) = (3,4,5)$. Because L is given by an integer matrix and $L = L^{-1}$, the map L acts on the set of PPTs.

The projective action of L on the Klein model is a hyperbolic isometry which fixes the geodesic connecting $(1,0,1)$ to $(0,1,1)$. What we just did for one of the sides of our ideal square, we can do for the other sides as well. Call the 4 isometries I_1, I_2, I_3, I_4. Let Γ denote the group generated by these isometries. The Γ-orbit of the initial ideal square gives rise to a tiling of the hyperbolic plane by ideal squares. By construction, the vertices of this tiling are all the unit-circle images of PPTs.

Exhaustion: The above construction realizes the PPTs as vertices of a tiling of the hyperbolic plane by ideal squares. Now we prove that every PPT is such a vertex.

Lemma 0.1 *Every PPT is a vertex of the tiling.*

Proof: This kind of argument is known as the method of descent in number theory. Just looking at the third coordinate of the map I_j, we have the formula

$$I(x,y,z) = (\cdot, \cdot, 3z \pm 2x \pm 2y).$$

Suppose we have some PPT which does not project to a vertex of the tiling. By symmetry, we can assume that all the signs are all positive. Let η be the PPT with minimal third coordinate which has this property. Letting $\eta = (x,y,z)$, we have

$$z < x + y < \sqrt{2}z.$$

But then $|3z - 2x - 2y| < z$. Let I be the isometry whose third coordinate is $3z - 2x - 2y$. From our bound above, we see that the third coordinate of $I(\eta)$ is smaller in absolute value than that of η. Moreover, $I(\eta)$ is not a vertex of our tiling because I permutes the vertices of the tiling. The PPT we get by making all the signs of $I(\eta)$ positive violates our choice of a minimal counter example. ♠