1 Symmetrization: Proof of Lemma B4

These notes give an alternate proof of Lemma B4. We first repeat the preliminary material on positive dominance.

1.1 Positive Dominance

See [**S2**] and [**S3**] for more details about the material here. Let $G \in \boldsymbol{R}[x_1, ..., x_n]$ be a multivariable polynomial:

$$G = \sum_I c_I X^I, \qquad X^I = \prod_{i=1}^n x_i^{I_i}. \tag{1}$$

Given two multi-indices I and J, we write $I \preceq J$ if $I_i \leq J_i$ for all i. Define

$$G_J = \sum_{I \preceq J} c_I, \qquad G_\infty = \sum_I c_I. \tag{2}$$

We call G *weak positive dominant* (WPD) if $G_J \geq 0$ for all J and $G_\infty > 0$. We call G *positive dominant* if $G_J > 0$ for all J.

Lemma 1.1 (Weak Positive Dominance) *If G is weak positive dominant then $G > 0$ on $(0,1]^n$. If G is positive dominant then $G > 0$ on $[0,1]^n$.*

Proof: We prove the first statement. The second one has almost the same proof. Suppose $n = 1$. Let $P(x) = a_0 + a_1 x + ...$ Let $A_i = a_0 + ... + a_i$. The proof goes by induction on the degree of P. The case $\deg(P) = 0$ is obvious. Let $x \in (0,1]$. We have

$$P(x) = a_0 + a_1 x + x_2 x^2 + \cdots + a_n x^n \geq$$

$$x(A_1 + a_2 x + a_3 x^2 + \cdots a_n x^{n-1}) = xQ(x) > 0$$

Here $Q(x)$ is WPD and has degree $n - 1$.

Now we consider the general case. We write

$$P = f_0 + f_1 x_k + ... + f_m x_k^m, \qquad f_j \in \boldsymbol{R}[x_1, ..., x_{n-1}]. \tag{3}$$

Since P is WBP so are the functions $P_j = f_0 + ... + f_j$. By induction on the number of variables, $P_j > 0$ on $(0,1]^{n-1}$. But then, when we arbitrarily set the first $n-1$ variables to values in $(0,1)$, the resulting polynomial in x_n is WPD. By the $n=1$ case, this polynomial is positive for all $x_n \in (0,1]$. ♠

Polynomial Subdivision: Let $P \in \boldsymbol{R}[x_1,...,x_n]$ as above. For any x_j and $k \in \{0,1\}$ we define

$$S_{x_j,k}(P)(x_1,...,x_n) = P(x_1,...,x_{j-1},x_j^*,x_{j+1},...,x_n), \qquad x_j^* = \frac{k}{2} + \frac{x_j}{2}. \tag{4}$$

If $S_{x_j,k}(P) > 0$ on $(0,1]^n$ for $k=0,1$ then we also have $P>0$ on $(0,1]^n$.

Positive Numerator Selection: If $f = f_1/f_2$ is a bounded rational function on $[0,1]^n$, written in so that f_1, f_2 have no common factors, we always choose f_2 so that $f_2(1,...,1) > 0$. If we then show, one way or another, that $f_1 > 0$ on $(0,1]^n$ we can conclude that $f_2 > 0$ on $(0,1]^n$ as well. The point is that f_2 cannot change sign because then f blows up. But then we can conclude that $f > 0$ on $(0,1]^n$. We write $\text{num}_+(f) = f_1$.

1.2 Reduction to Two Halves

For ease of notation set $q_k = p_k''$. Let D be the set of configurations (q_0, q_1, q_3) such that

1. $512 q_{01} \in [416, 498]$
2. $512 q_{02} \in [0, 16]$.
3. $512 q_{12} \in [-465, -348]$.
4. $512 q_{32} \in [348, 465]$.
5. $q_{11} = q_{31} = 0$.

Lemma B4 does not involve the point p_2, so we ignore it. The subset $D \subset (\boldsymbol{R}^2)^3$ denotes the set of triples (q_0, q_1, q_3) which satisfy the conditions for inclusion in Υ''. This set is not meant to be confused with the set from the proof of Lemma B21, though it plays the same role in the proof here. We let $D_\pm \subset D$ denote those configurations with

$$\pm(q_{12} + q_{32}) \geq 0. \tag{5}$$

Obviously $D = D_+ \cup D_-$.

Lemma B4 is an immediate consequence of the following two results.

Lemma 1.2 (B41) *With respect to vertical symmetrization of configurations in D_+, the list $\{(0,1),(0,3)\}$ is good for all $s \geq 2$.*

Lemma 1.3 (B42) *With respect to vertical symmetrization of configurations in D_-, the list $\{(0,1),(0,3)\}$ is good for all $s \geq 13$.*

Lemma 1351 has a straightforward proof that is similar in spirit to the proof we gave for Lemma B21. Lemma 1352 is much trickier. As we remarked after stating Lemma B4, the result is false if we replace 13 by 9 in the statement. It took me quite some time to find a proof for Lemma B42.

1.3 Proof of Lemma B41

We adopt the convention that $[u,v]t = u(1-t) + vt$. We define map $\phi_\pm : [0,1]^4 \to (\boldsymbol{R}^2)^3$ as follows:

$$\phi(a,b,c,d) = (q_0(b,d), q_1(a,c), q_3(a,c)), \tag{6}$$

$$512 q_0(b,d) = ([416,498]b, 16d).$$

$$512 q_1(a,c) = (0, -[348,465]a \pm 59c).$$

$$512 q_3(a,c) = (0, +[348,465]a \pm 59c).$$

In these coordinates, the symmetrization operation is $(a,b,c,d) \to (a,b,0,0)$.

Lemma 1.4 (B411) $D_\pm \subset \phi_\pm([0,1]^4)$.

Proof: This is just like the proof of Lemma B2111. The only non-obvious point is why every pair (p_{12}, p_{32}) is reached by the map $\phi_\pm$. The essential point is that for configurations in $D_\pm$ we have $512|p_{12} + p_{32}| \leq 2 \times 59$. ♠

Following the same idea as in the proof of Lemma B21, we define

$$F_s = \Big(\|\Sigma^{-1}(q_0) - \Sigma^{-1}(q_1)\|^{-s} + \|\Sigma^{-1}(q_0) - \Sigma^{-1}(q_3)\|^{-s} \Big) \circ \phi_+(a,b,c,d) \tag{7}$$

Here Σ^{-1} is the inverse of stereographic projection. We want to show that $F_s(a,b,c,d) \geq F_s(a,b,0,0)$ for all $a,b,c,d \in [0,1]^4$ and all $s \geq 2$. By the Convexity Lemma, it suffices to prove this for $s = 2$. Define

$$\Phi(a,b,c,d) = \text{num}_+(F_2(a,b,c,d) - F_2(a,b,0,0)). \tag{8}$$

Let $\Phi|_{c=0}$ denote the polynomial we get by setting $c = 0$. We define other such symbols similarly. Let $\nabla\Phi$ denote the gradient of Φ.

Lemma 1.5 (B412) *$\Phi_{c=0}$ and $\Phi_{c=0}$ and $\nabla\Phi \cdot (0,0,1,1)$ are weak positive dominant and hence positive on $(0,1)^4$.*

Proof: The Mathematica file `LemmaB412.m` does this calculation. ♠

We conclude from Lemma B412 that Φ is positive on $(0,1)^4$. Hence $\Phi > 0$ on $(0,1)^4$. In this way, Lemma B41 is a consequence of Lemma B411 and B412.

1.4 Proof of Lemma B42

Just keep make the parallel nature of Lemmas B41 and B42 more clear we repeat Lemma B411.

Lemma 1.6 (B421) *$D_\pm \subset \phi_\pm([0,1]^4)$.*

Define

$$E_k = \|\Sigma^{-1}(q_k) - \Sigma^{-1}(q_0)\|^{-2}, \qquad k = 1,3. \tag{9}$$

Lemma B42 says that $E_1^t + E_3^t \geq 0$ on D_- as long as $t > 13/2$.

We could approach the proof of Lemma B42 just as we approached the proof of Lemma B41. The polynomial that appears in what would be the analogue of Lemma B412 has 102218 terms. I was able, finally, to show directly that this polynomial is positive on $(0,1]^4$ but the argument is rather complicated. I will sketch it at the end of this chapter. Here I will give a different approach. I found this alternate approach here after a lot of trial and error.

Lemma 1.7 (B422) *Suppose that x_1, y_1, x_2, y_2 are positive numbers with*

$$x_2 = y_2, \qquad 7x_1+8y_1 \geq 7x_2+8y_2, \qquad 3x_1^2+3y_1^2-4x_1y_1 \geq 3x_2^2+3y_2^2-4x_2y_2.$$

Then $x_1^t + y_1^t \geq x_2^t + y_2^t$ for all $t \geq 13/2$.

Lemma 1.8 (B423) *On D_-, the function $F = 7E_1 + 8E_3$ is not increased by vertical symmetrization.*

Proof: We define Φ in terms of F, exactly as in Equations 7 and 8, except that we use ϕ_- in place of ϕ_+. The calculation in the Mathematica file `LemmaB422.m` shows that the analogue Lemma B413 holds for this version of Φ. The rest of the proof is the same as for Lemma B413. ♠

Lemma 1.9 (B424) *On D_-, the function $F = 3E_1^2 + 3E_3^2 - 4E_1E_3$ is not increased by vertical symmetrization.*

Lemma B42 follows directly from Lemmas B2351, B422, B423, B424.

1.5 Proof of Lemma B421

By the Convexity Lemma, it suffices to prove this result for the exponent $t = 13/2$. All the functions involved are homogeneous, so without loss of generality it suffices to consider the case when $x_2 = y_2 = 1$. Let $\Lambda_1, \Lambda_2, \Lambda_3$ respectively denote the level sets

$$7x + 8y = 15, \qquad 3x^2 + 3y^2 - 4xy = 2, \qquad x^{13/2} + y^{13/2} = 2.$$

We show that if Λ_1, Λ_2 separate (x, y) from the origin then so does Λ_3.

For $j = 1, 2, 3$ let $f_j(m)$ be the value where the line $y = mx$ intersects Λ_j in the positive quadrant. Here we take $m > 0$. There is a unique such point because, by homogeneity, each of our functions is increasing on the line $y = mx$ as we move away from the origin. Letting $m = y/x$, we see that Λ_j separates (x, y) from the origin if and only if $f_j(m) \leq x$. By hypothesis, we have $f_j(m) \leq x$ for $m = 1, 2$.

Lemma 1.10 (B4211) *$f_3(m) < f_2(m)$ when $m \in (1, 9/8]$.*

Proof: Both $f_2(m)$ and $f_3(m)$ are positive for m positive. We compute

$$\beta(t) = f_2^{26}(t^2) - f_3^{26}(t^2) = \left(\frac{2}{3 - 4t^2 + 3t^4}\right)^{13} - \left(\frac{2}{1 + t^{13}}\right)^4.$$

We want to see that this expression is positive on $(1, 9/8]$. This is the same as showing that $\gamma(u) = \beta(1 + u/8)$ is positive on $(0, 1]$. We compute that $\text{num}_+(\gamma)$ is a weak positive dominant polynomial of degree 50. ♠

Lemma 1.11 (B4212) *$f_3(m) < f_1(m)$ when $m \in [9/8, \infty)$.*

Proof: Both $f_1(m)$ and $f_3(m)$ are positive for m positive. We compute

$$\beta(t) = f_1^{13}(t^2) - f_3^{13}(t^2) = \left(\frac{15}{7 + 8t^2}\right)^{13} - \left(\frac{2}{1 + t^{13}}\right)^2.$$

We want to see that this expression is positive on $[9/8, \infty)$. We check that $\gamma(u) = \beta(9/8 + u)$ is a degree 26 polynomial all coefficients positive and a positive constant term. ♠

Our two lemmas combine to prove our lemma in case $m \geq 1$. Suppose that $m < 1$. Then $y < x$. But then $8x + 7y > 7x + 8y \geq 15$. Since $7x + 8y \geq 15$ and $8x + 7y \geq 15$ we see that $x + y \geq 2$. But $x^t + y^t \geq 2$ for all $t \geq 1$ by the Convexity Lemma. This completes the proof.

1.6 Proof of Lemma B423

Let F be as in Lemma B423. We define Φ in terms of F just as in Equations 7 and 8, except that we use ϕ_- in place of ϕ_+. Let $\Psi = \Phi|_{a=0}$. Lemma B425 follows from these three facts.

1. The function $\partial\Phi/\partial a$ is positive on $(0, 1]^4$.
2. The subdivision $S_{d,1}(\Psi)$ is positive on $[0, 1]^3$.
3. The subdivision $S_{d,0}(\Psi)$ is positive on $[0, 1]^3$.

The Mathematica file `LemmaB4231and2.m` calculates that the first of these functions is weak positive dominant and the second is positive dominant. This takes care of the first two items. Now we deal with the third.

Given any polynomial Q, we define $[Q]$ to be the sum of the absolute values of the negative coefficients. If $c^s d^t$ divides Q then $Q + [Q]c^s d^t \geq 0$ on $[0, 3]$ because all the partial sums in Equation 2 are non-negative and then we get the claimed result from the same argument as in the proof of the Weak Positive Dominance lemma.

Let $P = S_{d,0}(\Psi)$. Our polynomial P has no monomials of the form $c^s d^t$ with $s + t < 2$. Given a monomial μ we define $\langle P, \mu\rangle$ to be the sum of all the terms of P which are divisible by μ. We now define

$$P_1 = P - Q - [Q]c^2, \qquad Q = \langle P_0, c^3\rangle. \tag{10}$$

$$P_2 = P_1 - Q - [Q]d^2, \qquad Q = \langle P_1, d^3\rangle. \tag{11}$$

$$P_3 = P_2 - Q - [Q]cd, \qquad Q = \langle P_2, c^2 d\rangle. \tag{12}$$

$$P_4 = P_3 - Q - [Q]cd, \qquad Q = \langle P_2, cd^2\rangle. \tag{13}$$

By construction $P_4 \leq P$ on $[0,1]^3$ and moreover

$$P_4(b,c,d) = p_A(b)c^2 + p_B(b)cd + p_C(d)c^2. \tag{14}$$

here P_A, P_B, P_C are polynomials in b alone. Define the discriminant

$$\Delta = 4p_A p_C - p_B^2. \tag{15}$$

This is again a polynomial in b alone. The Mathematica file `Lemma B4233.m` computes that $S_{b,0}(\Delta)$ and $S_{b,1}(\Delta)$ are weak positive dominant. Hence $\Delta > 0$ on $(0,1)$. Hence P_4 cannot vanish on $(0,1)^3$. Evaluating at a single point, we see that in fact $P_4 \geq 0$ on $[0,1]^3$. But then $P \geq 0$ on $[0,1]^3$ as well.