Euler's Formula:

The purpose of these notes is to explain Euler's famous formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{1}$$

1 Powers of e: First Pass

Euler's equation is complicated because it involves raising a number to an imaginary power. Let's build up to this slowly.

Integer Powers: It's pretty clear that $e^2 = e \times e$ and $e^3 = e \times e \times e$, and so on. For any positive integer p, we have $e^p = e \times ... \times e$, a total of p times. For negative integers, the definition is also pretty clear. For instance $e^{-2} = 1/(e \times e)$ and $e^{-3} = 1/(e \times e \times e)$. And so on.

Fractional Powers: How would you define $e^{2/3}$. This really means the cube root of $e \times e$. So $e^{2/3}$ is the number y such that $y \times y \times y = e \times e$. Using this system, it is pretty clear that you can make good sense of $e^{p/q}$ where p/q is any positive fraction. You can make sense of negative fractions using the formula $e^{-p/q} = 1/e^{p/q}$.

Real Powers: If a is a positive real number, then you could define e^a to be the limit of expressions of the form e^{p_n/q_n}, where p_n/q_n is a sequence of rational numbers converging to a. To give an example of what I'm talking about, let $a = \sqrt{2}$.

- $17/12$ is close to $\sqrt{2}$ and $e^{17/12} = 4.1233529...$
- $41/29$ is closer to $\sqrt{2}$ and $e^{41/29} = 4.111521...$
- $577/408$ is closer to $\sqrt{2}$ and $e^{577/408} = 4.113259$
- $1393/985$ is closer to $\sqrt{2}$ and $e^{1393/985} = 4.1132488$.

And so on. Taking better and better rational number approximations to e, you add more and more digits of accuracy. Once you know what e^a is for any positive a, you could define $e^{-a} = 1/e^a$.

2 Powers of e: Second Pass

Problems: This approach above has three problems. First, what is e in the first place? Second, for irrational numbers, why do the rational approximations really converge to something? Third, even if the whole approach works perfectly, it doesn't get us any closer to understanding what it means to raise e to an imaginary power. Now we're going to take an approach based on differential equations.

A Special Function: Below we're going to construct a (continuous and differentiable) function f such that

- $f(0) = 1$.
- $f'(x) = f(x)$ for all x.
- f is always positive.

Once you know such a function exists, you define

$$e^a = f(a).$$

This gives a clean definition of e^a for all values of a.

Properties: The main problem in this definition is: How to we know that the operation $a \to e^a$ is anything at all like exponentiation? I mean, we could have made up any crazy function f and then made the above definition. We need to justify the choice of f.

Well, we know that

$$e^0 = f(0) = 1.$$

That's a good start, because it matches what we know about exponentiation. But, this is a pretty weak property.

A more serious thing we can show is that $f(x)$ is the inverse to $\ln(x)$.

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)} = \frac{f(x)}{f(x)} = 1.$$

This calculation just uses properties of f and the chain rule. Integrating, we get $\ln(f(x)) = x + C$. You can find C by plugging in $x = 0$. This gives $C = \ln(f(0)) = \ln(1) = 0$. So, $\ln(f(x)) = x$, as advertised.

If you know all about the natural log function, the last calculation will completely satisfy you. But you might want an explanation *from scratch*, one that doesn't just explain one function in terms of another equally mysterious one. What would really clinch things is if we could prove, just from properties of f, that

$$e^{a+b} = e^a e^b.$$

This is the same as showing that $f(a+b) = f(a)f(b)$. Once you have the exponent rule, you can recover all the good things from the first pass above, but avoid the problems: e is defined as $f(1)$. Then, for instance,

$$e^3 = f(3) = f(1+1+1) = f(1)f(1)f(1) = e \times e \times e.$$

At the same time, the hard cases, like when a is irrational, are handled automatically: e^a is just $f(a)$.

Proof of Exponent Rule: We want to prove that $f(a+b) = f(a)f(b)$ just using the three properties of f. Consider the new function $g(x) = f(a+x)$. Note that $g'(x) = f'(a+x) = f(a+x) = g(x)$. Consider the derivative of g/f:

$$\frac{d}{dx}\frac{g(x)}{f(x)} = \frac{g'(x)f(x) - f'(x)g(x)}{f^2(x)} = \frac{g(x)f(x) - f(x)g(x)}{f^2(x)} = 0.$$

So, the function $g(x)/f(x)$ is constant. That means that

$$\frac{f(a+b)}{f(b)} = \frac{g(b)}{f(b)} = \frac{g(0)}{f(0)} = \frac{f(a)}{f(0)} = f(a).$$

Rearranging this gives the exponent law.

More Problems: We have a great definition of e^a for all real numbers a, but we haven't shown that the function f exists. Also, we haven't come any closer to saying what $e^{i\theta}$ means.

3 Powers of e: third pass

Construction of f: Now we're going to construct the function f advertized above. Define

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \tag{2}$$

Here $2! = 2 \times 1$ and $3! = 3 \times 2 \times 1$, and so on. This is supposed to be an infinite sum. The sum is finite for any value of x, because $n!$ is eventually much larger than $|x|^n$, no matter what the choice of x is.

Property 1: It's pretty clear that $f(0) = 1$.

Property 2: If you think of f as an "infinite polynomial" then you might guess that you can compute $f'(x)$ just by differentiating term-by-term. This is actually true, but it takes some work to prove. I won't justify this step. But, assuming that term-by-term differentiation gives the right answer, you can see right away that $f'(x) = f(x)$.

Property 3: This is kind of a sneaky argument. Notice that $f(x) > 0$ when $x > 0$ because, in that case, $f(x)$ is a sum of positive numbers. Suppose that there is some value u where $f(u) = 0$. Choose u to be as large as possible, so that $f(x) > 0$ for all $x > u$. The proof that $f(a+b) = f(a)f(b)$, given above, works as long as $a, b > u$. But then, by continuity, $f(u)f(-u) = 1$. (The point is that $-u > u$.) This is impossible if $f(u) = 0$.

Complex Exponentiation: Notice that the formula for the function f given in Equation 2 works even if you plug in complex numbers. So, now we can say what it means to raise e to a complex power. It means

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \tag{3}$$

Grouping Real and Imaginary Parts: Now set $z = i\theta$, and group real and imaginary terms.

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + ... = C(\theta) + iS(\theta),$$

where

$$C(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...,$$

$$S(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}...,$$

We used the fact that the successive powers of i are $1, i, -1, -i$ repeating.

All we have to do is prove that $C(\theta) = \cos(\theta)$ and $S(\theta) = \sin(\theta)$ and we're done.

4 Recognizing C and S

Elementary Properties: Notice that

- $C(0) = \cos(0) = 1$.
- $S(0) = \sin(0) = 0$.
- $C'(\theta) = -S(\theta)$ and $\cos'(\theta) = -\sin(\theta)$.
- $S'(\theta) = C(\theta)$ and $\sin'(\theta) = -\cos(\theta)$.

So, C and S do have some things in common with cosine and sine. That's a good start.

The Pythagorean Identity: Consider the function $g(\theta) = C^2(\theta) + S^2(\theta)$. Using the product rule, and the facts above, we compute

$$g'(\theta) = -2C(\theta)S(\theta) + 2S(\theta)C(\theta) = 0.$$

So, the function $g(\theta)$ is constant. Also $g(0) = 1^1 + 0^2 = 1$. That means that $g(\theta) = 1$ for all θ. We've just proved the identity $C^2(\theta) + S^2(\theta) = 1$. This is another thing that C and S have in common with cosine and sine.

The End of the Proof: Let's interpret the Pythagorean identity geometrically. Consider the curve $p(\theta) = (C(\theta), S(\theta))$. The x-coordinate is described by $C(\theta)$ and the y-coordinate is described by $S(\theta)$. From the Pythagorean identity, we know that $p(\theta)$ lies on the unit circle $x^2 + y^2 = 1$ for all values of θ. Also, the velocity vector of $p(\theta)$ is given by

$$p'(\theta) = (-S(\theta), C(\theta)).$$

Note that $p'(\theta)$ has length 1. This means that our curve has speed 1 (if we think of θ as time.) But then both curves $(C(\theta), S(\theta))$ and $(\cos(\theta), \sin(\theta))$ trace out the unit circle starting at the point $(1, 0)$ and moving at the same speed (and going in the same direction). So, this means that the two curves do exactly the same thing: They are the same curve. So, $C(\theta) = \cos(\theta)$ and $S(\theta) = \sin(\theta)$.

A Fine Point: There is a stupid fine point. Why don't the two curves trace out the circle in opposite directions? Well, they have the same velocity at $\theta = 0$, namely $(0, 1)$, so they are both going the same direction, namely counterclockwise.