The Poincare Lemma

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The purpose of these notes is to explain the proof of Poincare's lemma from the book in somewhat less compressed form.

1 The Main Result

A domain $U \subset \boldsymbol{R}^n$ is *star shaped with respect to* $p \in \boldsymbol{R}^n$ if, for each $q \in U$, the entire segment $\overline{pq}$ lies in U. We say that U is star-shaped if U is star-shaped with respect to some point. Here is the main result. Recall that $\Omega^r(U)$ is the space of smooth r-forms on U.

Lemma 1.1 (Poincare) *Let U be an open star-shaped subset of $\boldsymbol{R}^n$ and let $\omega \in \Omega^r(U)$. Suppose that $d\omega = 0$. Then there is some $\alpha \in \Omega^{r-1}(U)$ such that $d\alpha = \omega$.*

The proof in the book is stated in terms of convex domains, but it really just uses the star-shaped property.

2 The Algebra Behind the Result

By symmetry, it suffices to consider the case when U is star-shaped with respect to the origin. We define $\widehat{U} \subset \boldsymbol{R}^n \times \boldsymbol{R}$ to be the set of points (u,t) such that $tu \in U$. Note that $\widehat{U}$ is an open set which contains $U \times [0,1]$.

There is a map $F : \widehat{U} \to U$ given by

$$F(u,t) = ut. \tag{1}$$

There are also two maps $g_0, g_1 : U \to \widehat{U}$ given by

$$g_i(u) = (u, i). \tag{2}$$

Notice that

- $F \circ g_1(u) = F(u, 1) = u$. Hence $F \circ g_1$ is the identity map. This means that $g_1^* \circ F^* = (F \circ g_1)^*$ is the identity on differential forms.

- $F \circ g_0(u) = F(u, 0) = 0$. Therefore, $F \circ g_0$ is the constant map. Since $D(F \circ g_0) = 0$, this means that $g_0^* \circ F^* = (F \circ g_0)^*$ is the 0-map.

Note that both g_0^* and g_1^* are maps from $\Omega^r(\widehat{U})$ to $\Omega^r(U)$. The main step in the proof is do construct a map $J : \Omega^{r+1}(\widehat{U}) \to \Omega^r(U)$ with the property that

$$Jd - dJ = \pm(g_1^* - g_0^*). \tag{3}$$

The sign depends on r in a way that we don't care about. Equation 3 makes sense, because all maps go from $\Omega^r(\widehat{U})$ into $\Omega^r(U)$.

Let's see what Equation 3 gives us. We start with $\omega \in \Omega^r(U)$ such that $d\omega = 0$. We then define

$$\widehat{\omega} = F^*(\omega). \tag{4}$$

Note that

$$d\widehat{\omega} = dF^*(\omega) = F^* d\omega = F^*(0) = 0. \tag{5}$$

Hence $Jd\widehat{\omega} = 0$. We compute

$$dJ(\widehat{\omega}) =$$

$$0 + dJ(\widehat{\omega}) =$$

$$-Jd\widehat{\omega} + dJ\widehat{\omega} =$$

$$\mp(g_1^*(\widehat{\omega}) - g_0^*(\widehat{\omega})) =$$

$$\mp(g_1^* F^* \omega - g_0^* F^* \omega) =$$

$$\mp(\omega - 0) = \mp\omega.$$

We take

$$\alpha = \mp J(\widehat{\omega}). \tag{6}$$

If we pick the sign right then $d\alpha = \omega$.

3 Construction of the Main Map

Now we construct the map $J : \Omega^{r+1}(\widehat{U})$ to $\Omega^r(U)$. It is convenient to set $t = x_{n+1}$.

Any form $\eta \in \Omega^{r+1}(\widehat{U})$ can be written in the standard basis:

$$\eta = \sum_K c_K \; dx_K, \tag{7}$$

The sum take place over increasing multi-indexes of length $r + 1$. We can write $\eta = \eta_1 + \eta_2$, where η_1 is the sum over the multi-indices which do not involve $n+1$ and η_2 is the sum over the multi-indices which do involve $n+1$.

We have

$$\eta_1 = \sum_I a_I dx_I, \qquad \eta_2 = \sum_J b_J dx_J \wedge dt. \tag{8}$$

The first sum is taken over multi-indices of length $r+1$ which involve $n+1$. The second sum is taken over multi-indices J of length r which do not involve $n+1$. Define

$$J(\eta) = \sum_J B_J dx_J, \qquad B_j(p) = \int_0^1 b_j(p,t) \; dt \tag{9}$$

Notice that J really is a linear map from $\Omega^{r+1}(\widehat{U})$ to $\Omega^r(U)$. Note also that $J(\eta_1) = 0$.

4 The Calculations

Both sides of Equation 3 respect sums. So, it suffices to prove Equation 3 for a form $\eta = a dx_I$. There are 2 cases, depending on whether I involves $n+1$.

4.1 Case 1

Suppose that I does not involve $n+1$. Then $J\eta = 0$ and so

$$dJ\eta = 0. \tag{10}$$

On the other hand

$$d\eta = \frac{\partial a}{\partial t} dt \wedge dx_I + \beta,$$

where β only has terms which do not involve dt. Since $J(\beta) = 0$, we have

$$Jd\eta = (-1)^r A dx_I, \qquad A(p) = \int_0^1 \frac{\partial a(p,t)}{\partial t}\ dt = a(p,1) - a(p,0). \tag{11}$$

Note that $g_1^*(dx_i) = dx_i$ for $i = 1, ..., n$. For this reason

$$g_0^*(\eta) = a(\cdot, 0) dx_I, \qquad g_1^*(\eta) = a(\cdot, 1) dx_I. \tag{12}$$

Equation 3 follows in this case from Equations 10, 11, and 12.

4.2 Case 2

Now suppose that $\eta = a dx_I$ where I involves $n+1$. We can write

$$\eta = a\ dx_J \wedge dt,$$

where J is obtained from I by dropping the $n+1$.

We compute

$$J\eta = Ax_J, \qquad A(p) = \int_0^1 a(p,t)\ dt.$$

Therefore,

$$dJ\eta = dA \wedge dx_J = \sum_{i=1}^n \frac{\partial A}{\partial x_i} dx_i \wedge dx_J. \tag{13}$$

Differentiating under the integral sign, we get

$$\frac{\partial A}{\partial x_i}(p) = \int_0^1 \frac{\partial a}{\partial x_i}(p,t). \tag{14}$$

$$d\eta = \sum_{i=1}^n \frac{\partial a}{\partial x_i} \wedge dX_J \wedge dt.$$

We never differentiate by t because every term of η involves dt. Finally

$$Jd\eta = \sum_{i=1}^n \frac{\partial A}{\partial x_i} \wedge dX_J = dJ\eta.$$

In this case $dJ(\eta) - Jd(\eta) = 0$. On the other hand $g_j^*(dt) = 0$ for $j = 0, 1$. So, Equation 3 again holds.