The Cauchy-Binet Theorem

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The Cauchy-Binet theorem is one of the steps in the proof of the Matrix Tree Theorem. Here I'll give a proof.

Let A be an $n \times N$ matrix and let B be an $N \times n$ matrix. Here $n < N$. The matrix AB is an $n \times n$ matrix. Given any subset $S \subset \{1, ..., N\}$ having n-elements, form the two $n \times n$ matrices A_S and B_S, obtained by just using the rows (or columns) indexed by the set S. Define

$$f(A,B) = \det(AB), \qquad g(A,B) = \sum_S \det(A_S)\det(B_S). \tag{1}$$

The sum ranges over all choices of S. The Cauchy-Binet theorem is that $f(A,B) = g(A,B)$ for all choices of matrices.

Think of A and B each as n-tuples of vectors in $\boldsymbol{R}^N$. We get these vectors by listing out the rows of A and the columns of B. So, we can write

$$f(A,B) = f(A_1, ..., A_n, B_1, ..., B_n), \tag{2}$$

and likewise for g.

The idea of the proof is to check that the values of f and g change in the same way when the list $A_1, ..., A_n$ and the list $B_1, ..., B_n$ are changed just one vector at a time. All the properties we list come from well-known properties of the dot product and the determinant.

- If A_i is replaced by λA_i then $f(A,B)$ is replaced by $\lambda f(A,B)$.
- If B_i is replaced by λB_i then $f(A,B)$ is replaced by $\lambda f(A,B)$.
- If A_i is replaced by λA_i then $g(A,B)$ is replaced by $\lambda g(A,B)$.
- If B_i is replaced by λB_i then $g(A,B)$ is replaced by $\lambda g(A,B)$.

Now consider the analogous operation of addition. Let A' denote the list obtained from A by changing the vector A_i to A'_i. Likewise define A'' and B' and B''. We only change things in the one position. Suppose $A_i = A'_i + A''_i$ and $B_i = B'_i + B''_i$. Then

- $f(A, B) = f(A', B) + f(A'', B)$.
- $f(A, B) = f(A, B') + f(A, B'')$.
- $g(A, B) = g(A', B) + g(A'', B)$.
- $g(A, B) = g(A, B') + g(A, B'')$.

In view of the fact that f and g transform exactly the same way under all the operations above, it suffices to consider the case when all the vectors are amongst the standard basis vectors. If $A_i = A_j$ for some pair of indices, then $\det(A_S) = 0$ for all S and also $\det(AB) = 0$ because AB has a repeated row. The same goes if $B_i = B_j$ for some pair of indices. So, we can assume that no two vectors of A are the same and no two vectors of B are the same. Call the associated matrices *special*.

In short, it suffices to prove the Cauchy-Binet theorem when A and B are special matrices. So, A and B are both matrices with n ones and the rest zeros. The rows of A are linearly independent and the columns of B are linearly independent. In this situation, there are unique sets S_A and S_B of n elements such that $\det(A_{S_A}) = 1$ and $\det(B_{S_B}) = 1$. For all other sets we get zero. So $g(A, B) = 1$ if $S_A = S_B$ and otherwise $g(A, B) = 0$. When $S_A = S_B$, the matrix AB is the identity so $f(A, B) = 1$. Otherwise, AB has at most $n-1$ nonzero entries. Hence $f(A, B) = 0$. So, in all cases $f(A, B) = g(A, B)$.

This completes the proof.