HW Assignment 1:

1: Prove that if a finite graph has an Eulerian circuit, then all the vertices of the graph have even degree. (This is the proof-practice problem.)

2: Describe the tree whose Prufer code is 111222333...(100)(100)(100). The labels are $1, 2, ..., 300$.

3: Suppose that a tree T has m internal nodes with degrees $d_1, ..., d_m$, and k leaves. Prove that $d_1 + ... + d_m = k + 2m - 2$. Hint: start pulling off the leaves.

4: Let T be a finite tree. The vertices of T can be colored black and white in such a way that every edge is incident to one black vertex and one white vertex. Suppose that T has the same number of black vertices as white vertices. Prove that T has at least one black leaf and at least one white leaf. This is Problem 2.2.13a in West's book.

5: Let G_n be the graph shown in Figure 1 for the case $n = 6$. In general, G_n is a 3-valent graph with a ring of m θ graphs joined in a loop. Prove that G_n has $2n8^n$ spanning trees. This is problem 2.2.6 in West's book.

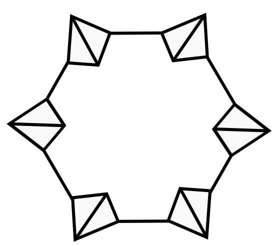

Figure 1: The graph G_6.

6: Let G_n be the graph shown in Figure 2. Let τ_n denote the number of spanning trees of G_n. Prove that τ_n satisfies the recurrence relation $\tau_n = 4\tau_{n-1} - \tau_{n-2}$. (For this to work for all $n \geq 2$ we have to set $\tau_0 = 1$.) This is problem 2.2.15 in West's book.

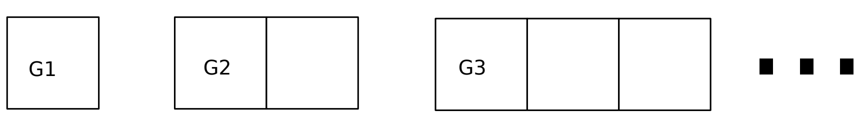

Figure 2: The graphs $G_1, G_2, G_3, ...$

7: Prym's algorithm produces a minimum spanning tree on a connected weighted graph as follows: Supposing that some connected subtree T has already been constructed, pick any vertex v of T which is incident to vertices not in T. Augment T by adding the edge which joins v to the vertex not in T using the lowest-weight edge. (Break ties arbitrarily.) Continue until all edges are reached. Prove that Prym's algorithm produces a minimum weight spanning tree of G. To make your life easier, you can assume that the edges of G have distinct weights. This is Problem 2.3.10 in West's book.

8: Consider the set of all decompositions of a regular N-gon into triangles by drawing diagonals of the polygon. Figure 3 shows (when $N = 7$) an example of an *elementary move* in which one such triangulation is converted into another by switching the diagonal in a single quadrilateral. Prove, for any N, that any two such triangulations of the regular N-con can be transformed into each other by a finite number of elementary moves.

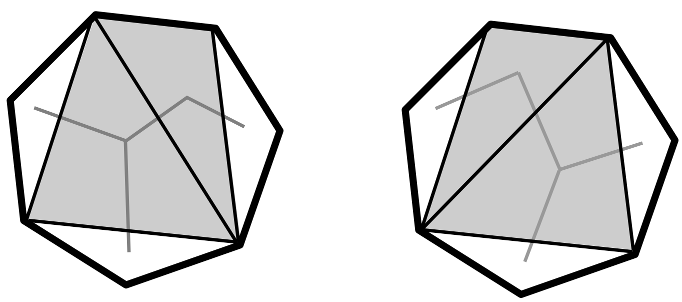

Figure 3: An elementary move.

Incidentaly, Figure 3 also shows how such a triangulation corresponds to a tree with $N - 2$ vertices.

Extra Credit: Referring to Problem 8, you can associate a vector in $\boldsymbol{R}^N$ to each triangulated N-gon. The point is $(t_1, ..., t_N)$, where t_k is the number of triangles touching vertex k. If the polygons in Figure 3 are labeled so that the bottom vertex is labelled 1 and the labelings increase counterclockwise, then the first polygon corresponds to the vector $(1, 4, 1, 2, 3, 1, 3)$ and the second one corresponds to the vector $(1, 3, 1, 3, 2, 1, 4)$. Let P_N denote the set of all vectors arrived at this way. Prove that P_N is the vertex set of a convex polytope in $\boldsymbol{R}^N$ and that the elementary moves correspond to the edges of this polytope.