HW Assignment 5:

1: In class I mentioned that there is a good analogy between discrete objects and vector calculus. This problem lets you work one of these out. Let G be a planar graph, in which every vertex has degree at least 3. Let $E(G)$ be the set of edges of G and let $F(G)$ denote the set of faces of G which are not the outermost face. Let ω be an assignment of a direction and a number to each edge of G. Let ∂G denote the outermost face of G. Orient ∂G counterclockwise and then define

$$\int_{\partial G} \omega = \sum_{e \in \partial G} \omega(e) \cdot e,$$

Here $\omega(e) \cdot e$ equals the number associated to e if the direction points counterclockwise and otherwise equals minus the number. We now define a function $d\omega$ on $F(G)$ as follows: $d\omega(\sigma)$ is the sum of the ω values of the edges going around σ, where the contribution of each edge is counted positively if the edge direction aligns with the counterclockwise orientation around σ and otherwise negatively. Next, define

$$\int_G d\omega = \sum_{\sigma \in F(G)} d\omega(\sigma).$$

Here we are summing over all the faces except the outer one. Prove "Green's Theorem":

$$\int_{\partial G} \omega = \int_G d\omega.$$

Hint: First try this for one of those "ladder graphs" made out of squares that have been the subject of several HW problems.

2: Let Z_3 denote the standard tiling of $\boldsymbol{R}^3$ by unit cubes. Let G be a graph made by taking a connected union of finitely many cubes of Z_3 and then considering the vertices and edges involved in the union. Let $E(G)$ denote the set of edges of G and let $F(G)$ denote the set of 2-dimensional faces of the cubes involved. Define a *surface with boundary* Σ to be a finite union of squares in $F(G)$ such that each square in Σ meets at most one other square across an edge. Then $\partial\Sigma$ is just the edges which only belong to one square. Formulate and prove a version of Stokes' Theorem along the lines of Problem 1 using this setup. The final result should be

$$\int_{\partial \Sigma} d\omega = \int_{\Sigma} d\omega.$$

3: Formulate and prove a version of Max Flow Min Cut in a situation where the single source s is replaced by a finite union $s_1, ..., s_m$ of sources and the single sink t is replaced by a finite union $t_1, ..., t_n$ of sinks.

4: In class I explained how Hall's Theorem follows from Max Flow Min Cut. Compare the Max Flow Min Cut *proof* as applied to the graph associated to Hall's Theorem with the direct proof involving M-alternating paths. They should basically be the same thing.

5: Give an example of a finite directed graph with source and sink, together with capacities on the edges, for which there are infinitely many maximum feasible flows.

6: We can identify a flow with a point in $\boldsymbol{R}^n$, where n is the number of edges of the graph just by listing out the values of the flow in some order. Show that the set of maximal feasible flows, when so considered, is a convex subset of $\boldsymbol{R}^n$. Problem 5 tells you that this subset can be more than just a single point.

7: Let G be a graph. A labeling f of the vertices of G by real numbers is *harmonic* at the vertex v if the value at v is the average of the values at the neighbors of v. Such a labeling defines a flow on G as follows. Given an edge of G having endpoints a and b. We orient e from the vertex with the higher value to the vertex with the lower value, and we assign $|f(a) - f(b)|$ to the edge. Prove that this flow is conservative at a vertex v if and only if f is harmonic at v. (This problem is really just a matter of unravelling the definitions.)

8: An infinite graph is called *hyperbolic* if there exists a nonzero and bounded positive labeling of its vertices which is harmonic except at one vertex and which tends to zero on any sequence of vertices whose distance to the non-harmonic vertex tends to ∞. The corresponding flow has one source and no sinks. Show that the graph made from the integers by connecting consecutive sides is not hyperbolic and give an example of a hyperbolic graph.