HW Assignment 6:

1: Formulate and prove a 3-dimensional version of Sperner's Lemma, using a tetrahedron that has been partitioned into smaller tetrahedra which meet face to face.

2: Let T be a triangulated triangle, as in Sperner's Lemma. Suppose that the verticas on the boundary of T has been labelled by numbers $1, 2, 3$ in such a way that, going around clockwise, there are more changes from 1 to 2 than there are from 2 to 1. Prove that any completion of the numbering of the vertices of T has at least one $(1, 2, 3)$-triangle.

3: Flesh out the following alternate proof of Sperner's Lemma. Given the classic setup of the result, consider the dual graph G of the triangulation (that has been labeled using the hypotheses of the result.) Start with an edge in G that crosses a $(1, 2)$ boundary edge. If there are no $(1, 2, 3)$ triangles then this edge can be continued in a unique way to a second edge of G. And so on, as in Figure 1. Consider all paths like this. Where is the contradiction?

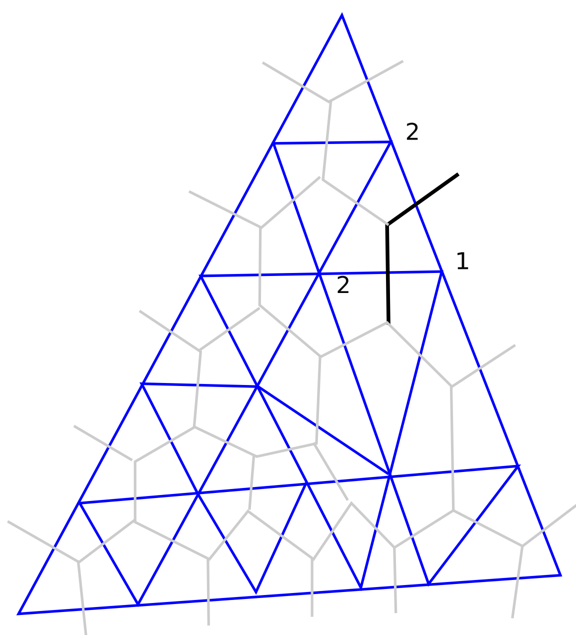

Figure 1: Exploring the dual graph.

4: In class, we considered electric networks in which all the edges had conductance 1. Consider a more general setup where each edge xy of a graph G is assigned a non-negative number C_{xy}. Given a function V on the vertices of G, define the current i_{xy} by the rule $i_{xy} = C_{xy}(V(x) - V(y))$. The higher the conductance C_{xy} the larger the current. With this setup, define the effective resistance of such a labeled graph and show that the effective resistance goes down when you lower the conductance of one of the edges. (Hint: If you don't want to fool around with these conductances, one approach is to first prove this for integer weights, then rational weights, and then take a limit. For integer weights, you can use multiple edges to reduce this problem to the original Rayleigh theorem.)

5: Let G be a graph and let a, b be distinct vertices of G. Let G' be a new graph obtained by replacing a single edge of G with a path of length 2. In otherwise, you get G' from G by inserting a new vertex in the middle of an edge. Prove that $R(G, a, b) \leq R(G', a, b)$. Here R denotes the effective resistance of the graph.

6: Prove that the resistance to ∞ is infinite for the regular hexagonal grid.

7: Let G be the infinite triangulation in which each vertex has 8 triangles going around a vertex. Prove that the resistance to ∞ in G is finite.

8: (Bonus problem) This problem is one of my favorites. Call a rectangle *good* if one of its side lengths is an integer. Suppose that R is a rectangle that has been partitioned into smaller good rectangles. Prove that R itself is good. This problem has several beautiful solutions, but without a clever trick it can be extremely difficult. It is fun to play with in any case, however.