HW Exercises:

1. Let $\boldsymbol{E}$ be a Weierstrass elliptic curve. Prove that a point on $\boldsymbol{E}$ has order 3 if and only if it is an inflection point.

2. Let Λ_1 and Λ_2 be two lattices in $\boldsymbol{C}$. Prove that $\boldsymbol{C}/\Lambda_1$ and $\boldsymbol{C}/\Lambda_2$ are isomorphic groups. (Hint: They are all isomorphic to $\boldsymbol{R}^2/\boldsymbol{Z}^2$).

3. Let p be a prime number. Prove that there are p^2 points of order p in the group $\boldsymbol{C}/\Lambda$ and that these points form a subgroup. Conclude from Problem 1 that if $\boldsymbol{E}$ has a Weierstrass uniformization (meaning that $\boldsymbol{E}$ is isomorphic to $\boldsymbol{C}/\Lambda$ as a group) then $\boldsymbol{E}$ has 9 inflection points.

4. Let $a, b \in \boldsymbol{Q}$ and let $\boldsymbol{E}_K$ be the Weierstrass elliptic curve defined by $y^2 - (x^3 + ax + b)$ defined over the field K. Suppose K is a splitting field for some polynomial in $\boldsymbol{Q}[x]$. Prove that each Galois automorphism $\phi \in \text{Gal}(K, \boldsymbol{Q})$ indices a bijection $B_\phi : E_K \to E_K$ in such a way that $B_{\phi_1} \circ B_{\phi_2} = \phi_1 \circ \phi_2$.

5. Suppose that Λ is a lattice that is symmetric with respect to complex conjugation. That is, $\lambda \in \Lambda$ if and only if $\overline{\lambda} \in \Lambda$. Prove that the Weierstrass uniformization map $\Psi = (P, P')$ maps $\boldsymbol{C}/\Lambda$ onto an elliptic curve whose equation has the form $4y^2 - (x^3 + ax + b)$ where $a, b \in \boldsymbol{R}$. Moreover, prove that Ψ maps $\boldsymbol{R}/(\Lambda \cap \boldsymbol{R})$ into the set of real points of the elliptic curve.

6: The map $f(x, y) = (Ax, By)$ is a perfectly good transformation from $\boldsymbol{C}$ to $\boldsymbol{C}$. Here $A, B \in \boldsymbol{R}$ are positive numbers. (The positivity restriction is just to make the picture nicer.) Prove that f is a holomorphic map from $\boldsymbol{C}$ to $\boldsymbol{C}$ if and only if $A = B$.