Countable and Uncountable Sets

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The purpose of this handout is to explain the notions of *countable* and *uncountable* sets.

1 Basic Definitions

A map f between sets S_1 and S_2 is called a *bijection* if f is one-to-one and onto. In other words

- If $f(a) = f(b)$ then $a = b$. This holds for all $a, b \in S_1$.
- For each $b \in S_2$, there is some a in S_1 such that $f(a) = b$.

We write $S_1 \sim S_2$ if there is a bijection $f : S_1 \to S_2$. We say that S_1 and S_2 are *equivalent* or *have the same cardinality* if $S_1 \sim S_2$. This notion of equivalence has several basic properties:

1. $S \sim S$ for any set S. The identity map serves as a bijection from S to itself.
2. If $S_1 \sim S_2$ then $S_2 \sim S_1$. If $f : S_1 \to S_2$ is a bijection then the inverse map f^{-1} is a bijection from S_2 to S_1.
3. If $S_1 \sim S_2$ and $S_2 \sim S_3$ then $S_1 \sim S_3$. This boils down to the fact that the composition of two bijections is also a bijection.

These three properties make $\sim$ into an *equivalence relation*.

Let $\boldsymbol{N} = \{1, 2, 3...\}$ denote the natural numbers. A set S is called *countable* is $S \sim T$ for some $T \subset \boldsymbol{N}$. Here is a basic result about countable sets.

Lemma 1.1 *If S is both countable and infinite, then there is a bijection between S and $\boldsymbol{N}$ itself.*

Proof: For any $s \in S$, we let $f(s)$ denote the value of k such that s is the kth smallest element of S. This map is well defined for any s, because there are only finitely many natural numbers between 1 and s. It is impossible for two different elements of S to both be the kth smallest element of S. Hence f is one-to-one. Also, since S is infinite, f is onto. ♠

Lemma 1.2 *If S is countable and $S' \subset S$, then S is also countable.*

Proof: Since S is countable, there is a bijection $f : S \to \boldsymbol{N}$. But then $f(S') = \boldsymbol{N}'$ is a subset of $\boldsymbol{N}$, and f is a bijection between S' and $\boldsymbol{N}'$. ♠

A set is called *uncountable* if it is not countable. One of the things I will do below is show the existence of uncountable sets.

Lemma 1.3 *If $S' \subset S$ and S' is uncountable, then so is S.*

Proof: This is an immediate consequence of the previous result. If S is countable, then so is S'. But S' is uncountable. So, S is uncountable as well. ♠

2 Examples of Countable Sets

Finite sets are countable sets. In this section, I'll concentrate on examples of countably infinite sets.

2.1 The Integers

The integers $\boldsymbol{Z}$ form a countable set. A bijection from $\boldsymbol{Z}$ to $\boldsymbol{N}$ is given by $f(k) + 2k$ if $k \geq 0$ and $f(k) = 2(-k) + 1$ if $k < 0$. So, f maps $0, 1, 2, 3...$ to $0, 2, 4, 6...$ and f maps $-1, -2, -3, -4...$ to $1, 3, 5, 7....$

2.2 The Rational Numbers

I'll give a different argument than the one I gave in class. Let L_q denote the finite list of all rational numbers between $-q$ and q that have denominator at most q. There are at most $q(2q+1)$ elements of L_q. We can make the list $L_1, L_2, L_3, ...$ and throw out repeaters. This makes a list of all the rational numbers. As above, we define $f(p/q)$ to be the value of k such that p/q is the kth fraction on our list.

2.3 The Algebraic Numbers

A real number x is called *algebraic* if x is the root of a polynomial equation $c_0 + c_1x + ... + c_nx^n$ where all the c's are integers. For instance, $\sqrt{2}$ is an algebraic integer because it is a root of the equation $x^2 - 2 = 0$. To show that the set of algebraic numbers is countable, let L_k denote the set of algebraic numbers that satisfy polynomials of the form $c_0 + c_1x + ... + c_nx^n$ where $n < k$ and $\max(|c_j|) < k$. Note that there are at most k^k polynomials of this form, and each one has at most k roots. Hence L_k is a finite set having at most k^{k+1} elements. As above, we make our list L_1, L_2, L_3 of all algebraic numbers and weed out repeaters.

2.4 Countable Unions of Countable Sets

Lemma 2.1 *Suppose that $S_1, S_2, ... \subset T$ are disjoint countable sets. Then $S = \bigcup_i S_i$ is a countable set.*

Proof: There are bijections $f_i : S_i \to \boldsymbol{N}$ for each i. Let L_k denote the set of elements $s \in S$ such that s lies in some S_i for $i < k$, and $f_i(s) < k$. Note that L_k is a finite set. It has at most k^2 members. The list $L_1, L_2, L_3...$ contains every element of S. Weeding out repeaters, as above, we see that we have listed all the elements of S. Hence S is countable. ♠

The same result holds even if the sets S_i are not disjoint. In the general case, we would define

$$S'_k = S_k - \bigcup_{i=1}^{k-1} S_i,$$

and apply the above argument to the sets $S'_1, S'_2....$ The point is that S'_i is countable, the various S' sets are disjoint, and $\bigcup_i S_i = \bigcup_i S'_i$.

3 Examples of Uncountable Sets

3.1 The Set of Binary Sequences

Let S denote the set of infinite binary sequences. Here is Cantor's famous proof that S is an uncountable set. Suppose that $f : S \to \boldsymbol{N}$ is a bijection. We form a new binary sequence A by declaring that the nth digit of A is the opposite of the nth digit of $f^{-1}(n)$. The idea here is that $f^{-1}(n)$ is some binary sequence and we can look at its nth digit and reverse it.

Supposedly, there is some N such that $f(A) = N$. But then the Nth digit of $A = f^{-1}(N)$ is the opposite of the Nth digit of A, and this is a contradiction.

3.2 The Real Numbers

Let $\boldsymbol{R}$ denote the reals. Let $\boldsymbol{R}'$ denote the set of real numbers, between 0 and 1, having decimal expansions that only involve 3s and 7s. (This set $\boldsymbol{R}'$ is an example of what is called a *Cantor set*.) There is a bijection between $\boldsymbol{R}'$ and the set S of infinite binary sequences. For instance, the sequence 0101001... is mapped to .3737337.... Hence $\boldsymbol{R}'$ is uncountable. But then Lemma 1.3 says that $\boldsymbol{R}$ is uncountable as well.

3.3 The Transcendental Numbers

A real number x is called *transcendental* if x is not an algebraic number. Let $\boldsymbol{A}$ denote the set of algebraic numbers and let $\boldsymbol{T}$ denote the set of transcendental numbers. Note that $\boldsymbol{R} = \boldsymbol{A} \cup \boldsymbol{T}$ and $\boldsymbol{A}$ is countable. If $\boldsymbol{T}$ were countable then $\boldsymbol{R}$ would be the union of two countable sets. Since $\boldsymbol{R}$ is uncountable, $\boldsymbol{R}$ is not the union of two countable sets. Hence $\boldsymbol{T}$ is uncountable.

The upshot of this argument is that *there are many more transcendental numbers than algebraic numbers*.

3.4 Tail Ends of Binary Sequences

Let T denote the set of binary sequences. We say that two binary sequences A_1 and A_2 are *equivalent* if they have the same tail end. For instance 1001111... and 111111... are equivalent.

Lemma 3.1 *For any binary sequence A, there are only countably many binary sequences equivalent to A.*

Proof: Let L_n denote the set of sequences that differ from A only in the first n digits. Then L_n is a finite set with at most 2^n elements. Now we list $L_1, L_2, L_3....$ This gives a list of all the binary sequences equivalent to A. The rest of the proof is as above. ♠

Say that a *tail end* is the collection of sequences all equivalent to a given one. Note that T is the union of tail ends. Each tail end it a countable set, and T is uncountable. Hence, there are uncountably many tail ends, by Lemma 2.1.

3.5 The Penrose Tiles

To each Penrose tiling P we can associate a tail end $\tau(P)$. Recall that there is an infinite sequence $P = P_0, P_1, P_2, ...$ where P_n is the *parent* of P_{n-1}. In other words, P_n is obtained from P_{n-1} by the *grouping* process discussed in class.

We say that the nth digit of $\tau(P)$ is a 0 if x is contained in a kite of P_n and a 1 if x is contained in a dart of P_n. We might need to move x slightly to avoid choosing a point that lies right on a crack. If we replace x by x', then only the initial part of the sequence changes. So, $\tau(P)$ is well defined.

By using the subdivision operation, we can produce a Penrose tiling P that has any $\tau(P)$ we like. Hence, there are uncountably many different Penrose tilings.

4 A Heirarchy of Infinite Sets

For any set S let 2^S denote the set of subsets of S.

Lemma 4.1 *There is no bijection between S and 2^S.*

Proof: This is really a generalization of Cantor's proof, given above. Suppose that there really is a bijection $f : S \to 2^S$. We create a new set A as follows. We say that A contains the element $s \in S$ if and only if s is not a member of $f(s)$. This makes sense, because $f(s)$ is a subset of S.

Since A is a subset of S, we have $A = f(a)$ for some $a \in S$. If $a \in A$ then $a \in f(a)$. But then, by definition, a is not a member of A. On the other hand, if $a \notin A$, then $a \in f(a)$. But, again, this is a contradiction. The only way out of the contradiction is to realize that there can be no bijection f. ♠

We can start with $S_0 = \boldsymbol{N}$, and recursively define $S_n = 2^{S_{n-1}}$. That is, S_n is the set of subsets of S_{n-1}. Then, the sets $S_0, S_1, S_2, \ldots$ form an infinite heirarchy of sets, each one so much larger than the previous one that there is no bijection between it and the previous one.

The fun doesn't stop there. We can define

$$\Sigma_0 = \bigcup_{n=0}^{\infty} S_n.$$

Then, there is no bijection between Σ_0 and S_n for any n. The set Σ_0 is larger than all of the sets previously defined. One can now define $\Sigma_n = 2^{\Sigma_{n-1}}$. And so on.