The Robinson Tiles

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The purpose of these notes is to give an account of the Robinson tiles. These tiles were discovered by Raphael Robinson in 1978. They are the first simple example of an aperiodic tiling system. One place to read about the Robinson tiles in more detail is Charles Radin's book, *Miles of Tiles*.

1 Basic Definitions

There are several equivalent ways to define the Robinson tiles. In the way I'll define them, there are 7 square tiles. Figure 1 shows them. One of the tiles, the one with shading, is what I will call *the special tile*.

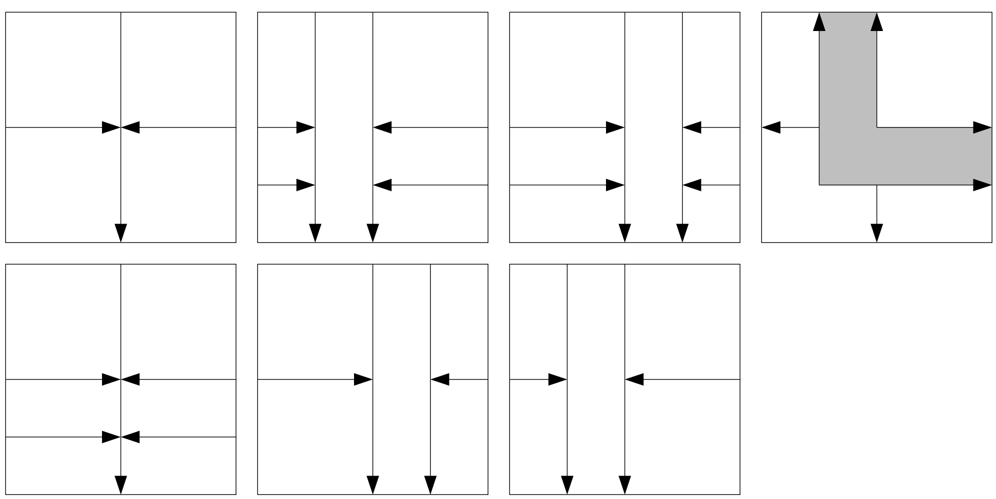

Figure 1

These tiles can be rotated in any way you like. The tiles can only fit together in such a way that the outgoing arrows exactly match the ingoing arrows across an edge. figure 2 shows some examples of legal and illegal matchings.

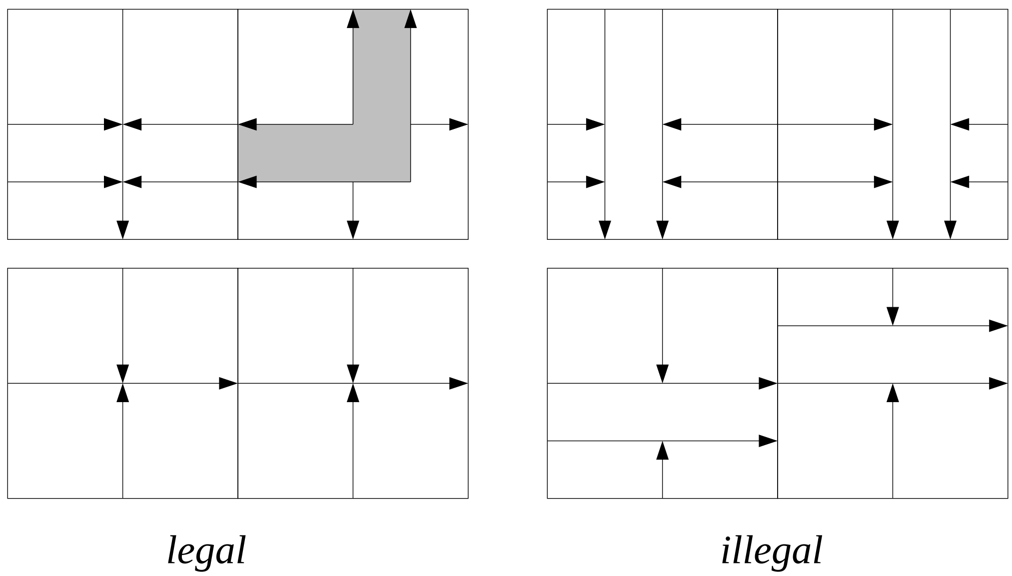

Figure 2

There are several more rules about how the tiles may be placed. Think of placing each Robinson tile so that its center lies at a point of the form (m, n), where m and n are integers. Then

1. Only the special tile can be placed on points (m, n) if both m and n are even.

2. The special tile cannot be placed on points of the form (m, n) where m is odd and n is even.

3. The special tile cannot be placed on points of the form (m, n) where m is even and n is odd.

4. The special tile can (but does not have to) be placed on points of the form (m, n) where both m and n are odd.

In this handout I will show that it is possible to tile the plane with robinson tiles, but not in a way that has any infinite symmetry. If you want to play with the Robinson tiles, you can copy Figure 3 multiple times and cut out the tiles.

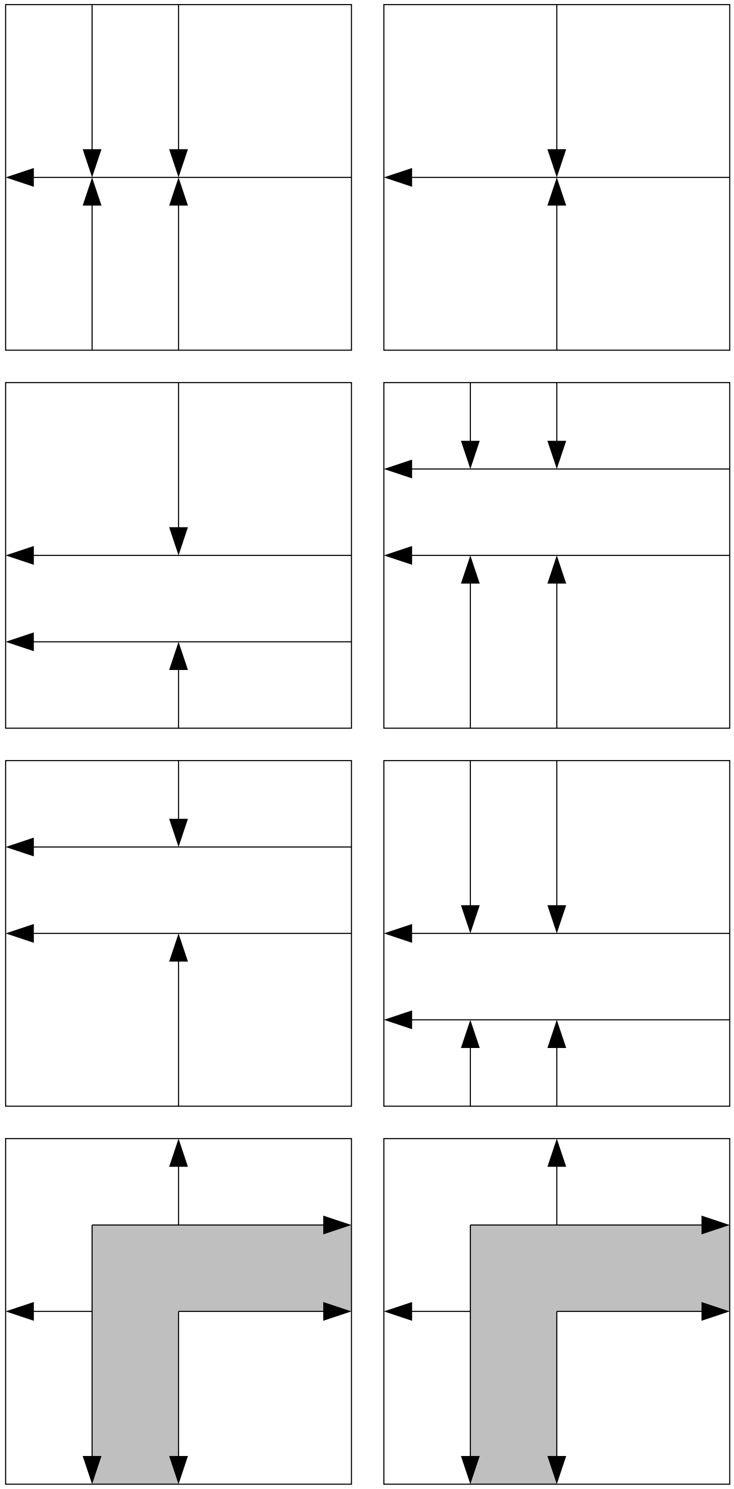

Figure 3

2 Existence of Tilings

The Robinson tilings are based on a hierarchical structure. Figure 4 shows 9 tiles joined together to form what I'll call a *level 1 supertile*.

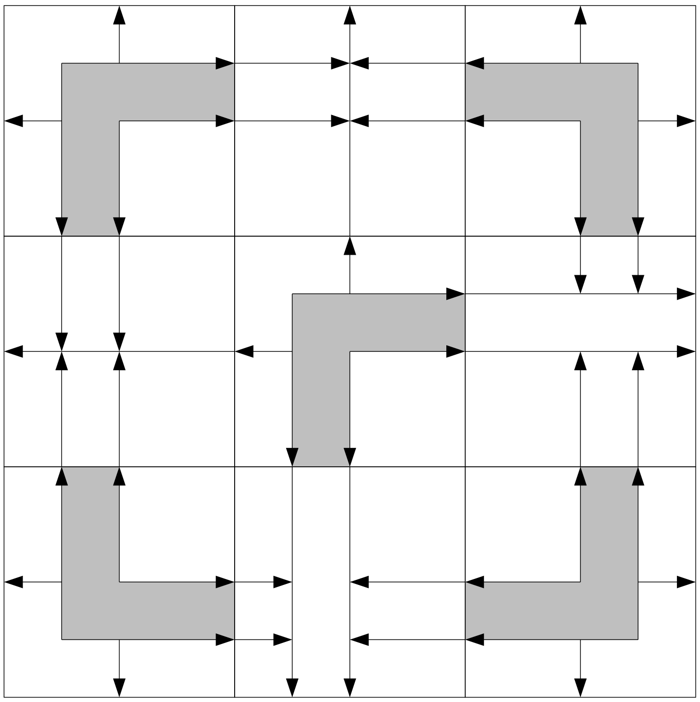

Figure 4: level 1 supertile

The 4 corners of the supertile are all "aligned", and one sees a kind of loop made out of a thick band. The structure of the level 1 supertile is very similar to the structure of the special tile. One can see that the supertile is kind of an extension of the special tile in the center.

Figure 5 shows what I'll call a *level 2 supertile*. It is made as follows:

- Place 4 level 1 supertiles in such a way that they are all aligned, and there is room for a single tile in the very center.
- Place a special tile in the center.
- Extend the branches of the central special tile in the only possible, using other Robinson tiles.

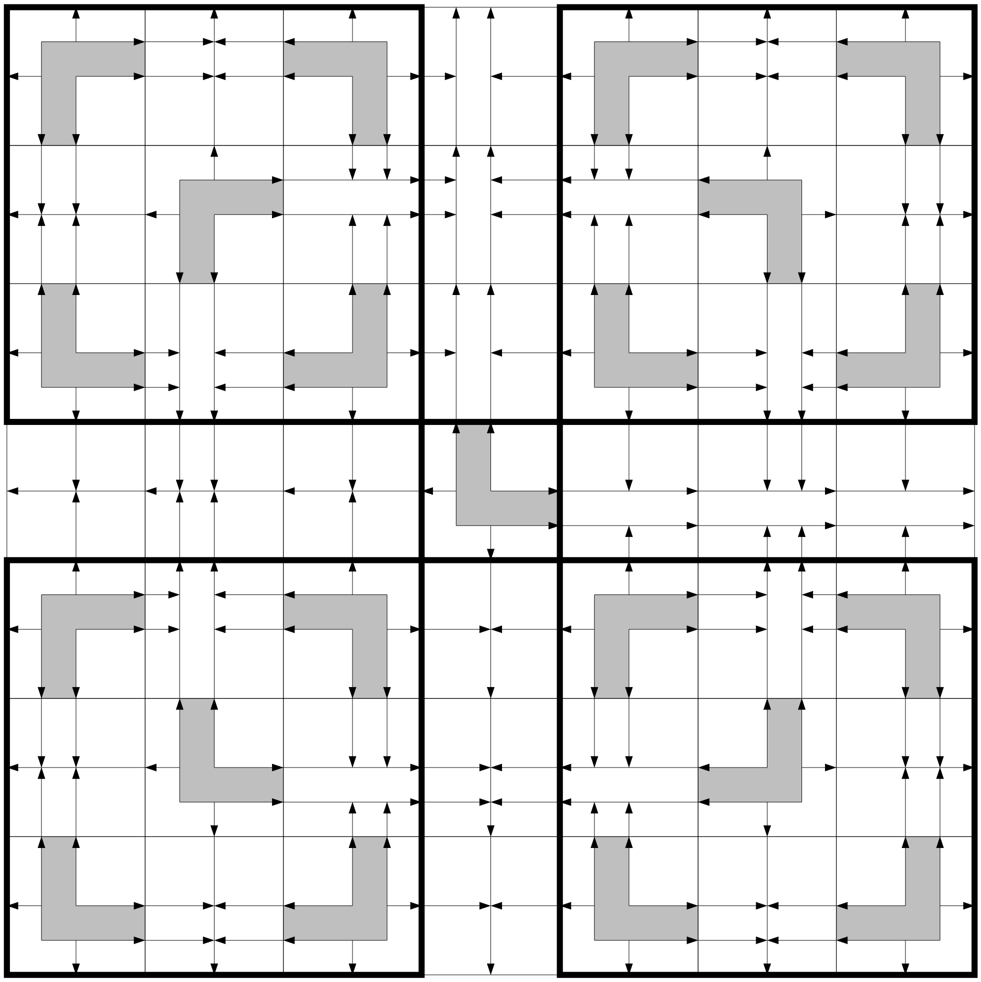

Figure 5: level 2 supertile

In general, one can build a level n super tile by repeating the above method, using 4 level $n-1$ supertiles.

Continuing in this way, you can tile ever larger portions of the plane with Robinson tiles. Let's call these partial tilings $T_1, T_2, T_3, \ldots$ By translating the picture around, you can guarantee that the union of these partial tilings covers the entire plane. Taking a limit, you get a tiling of the whole plane by Robinson tiles.

3 Forcing Supertiles

In this section we will show that any Robinson tiling must contain supertiles of all levels. This is key step in the proof that a Robinson tiling cannot have a infinite symmetry group.

3.1 Forcing Level 1

To get started on this proof, we first observe that, in any Robinson tiling, a special tile pointing (with its shaded arms) in some direction forces a level 1 supertile extending in that direction. Figure 6 shows what we have in mind. The proof of this fact is a simple exercise in trial-and-error.

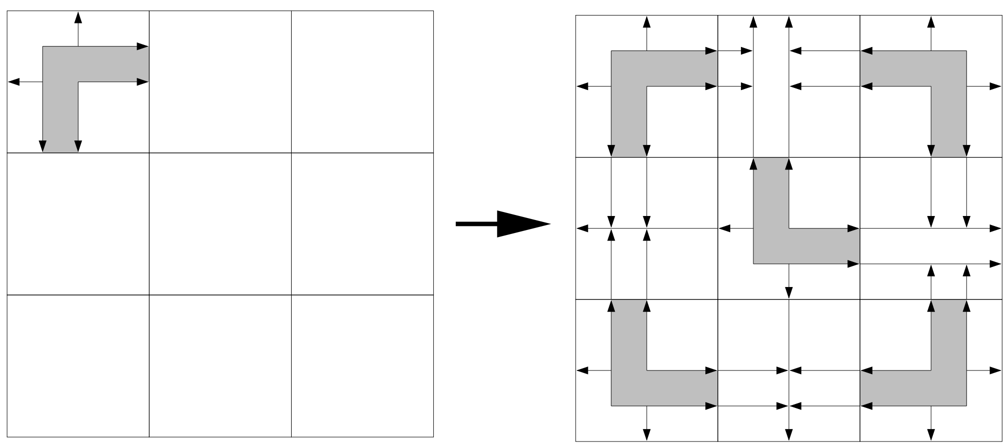

Figure 6

Thus, T must contain level 1 supertiles. Moreover, any special tile in a Robinson tiling forces a level 1 supertile in one of the 4 possible directions. In some sense, all the special tiles in a Robinson tiling are magnetized. Each one influences the alignment of others.

3.2 Forcing Level 2

Now we turn our attention to the way in which level 1 supertiles force level 2 supertiles. Each level 1 supertile has one of 4 orientations. We will show that a level 1 supertile pointing in some direction (say soutwest) with its thick arms forces a level 2 supertile extending in that direction. The main step amounts to ruling out a *misalignment* of level 1 supertiles, as shown in Figure 7.

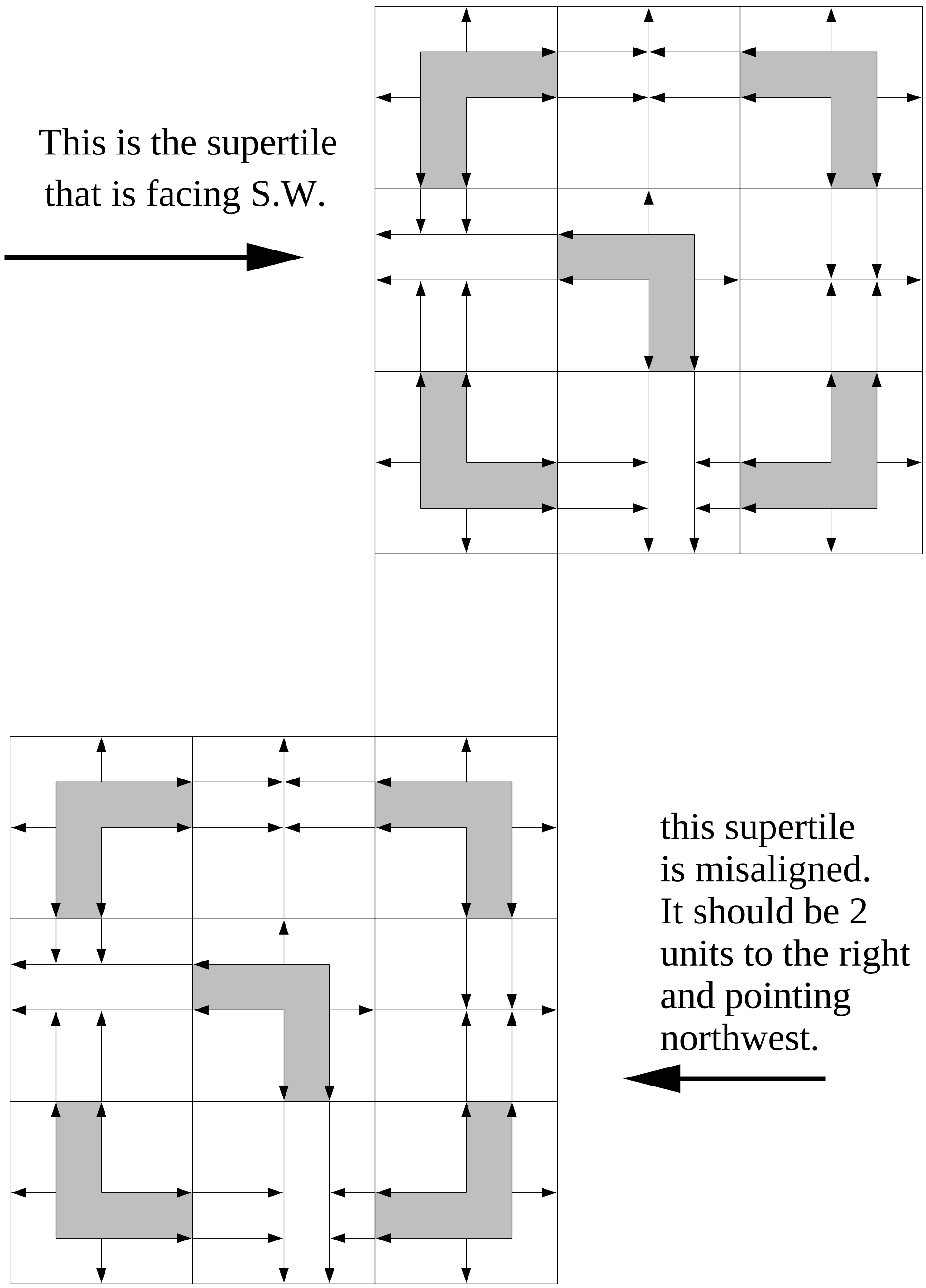
This is the supertile
that is facing S.W.
this supertile
is misaligned.
It should be 2
units to the right
and pointing
northwest.

Figure 7

In case there is a misalignment like this, we would have 3 supertiles arranged as in Figure 8.

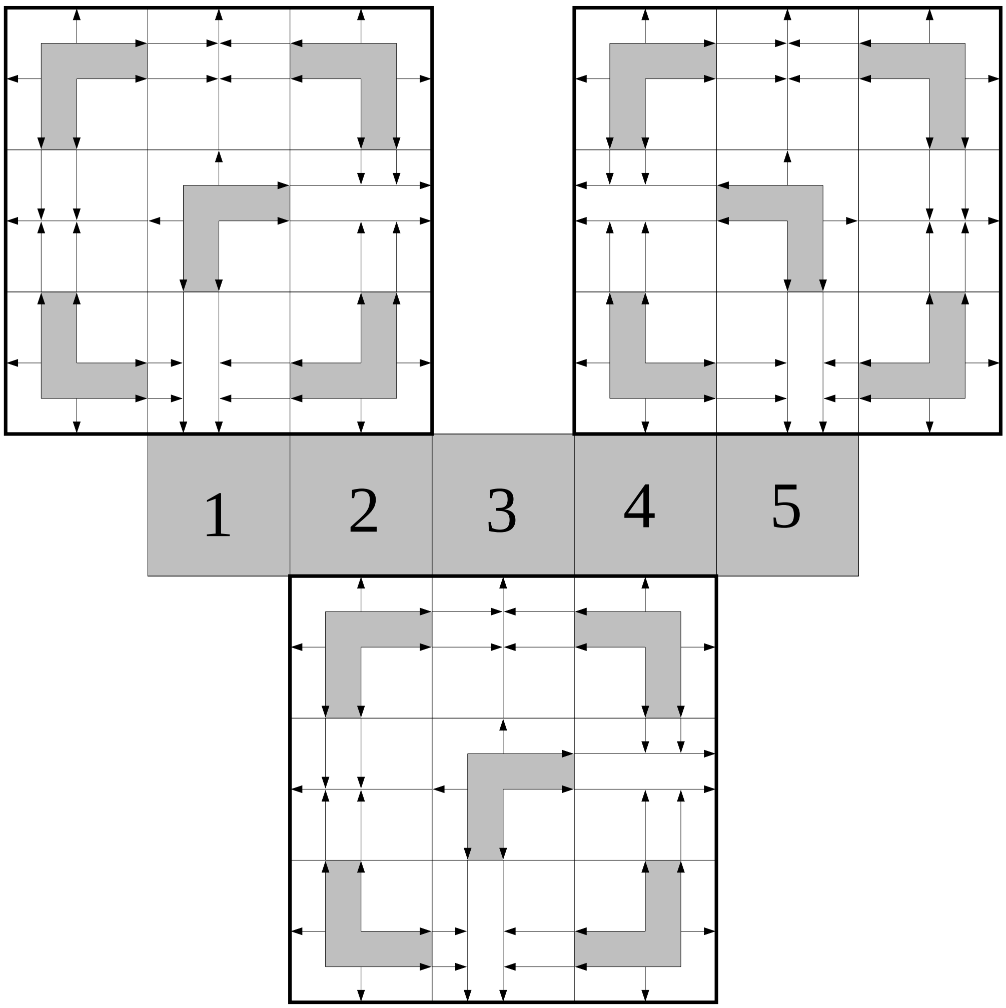

Figure 8

Consider the shaded row of square tiles. Due to the all the outward arrows above and below the shaded row, we see that all the tiles in this row point in the same direction, either to the left or to the right. However, given the S.E. pointing thick arm above square 1, there is no east-pointing tile that can be placed in square 1. Likewise, there is no west-pointing tile that can be placed in square 5. This contradiction rules out the misalignment. Hence, our SW pointing supertile forces a second supertile as shown in Figure 9.

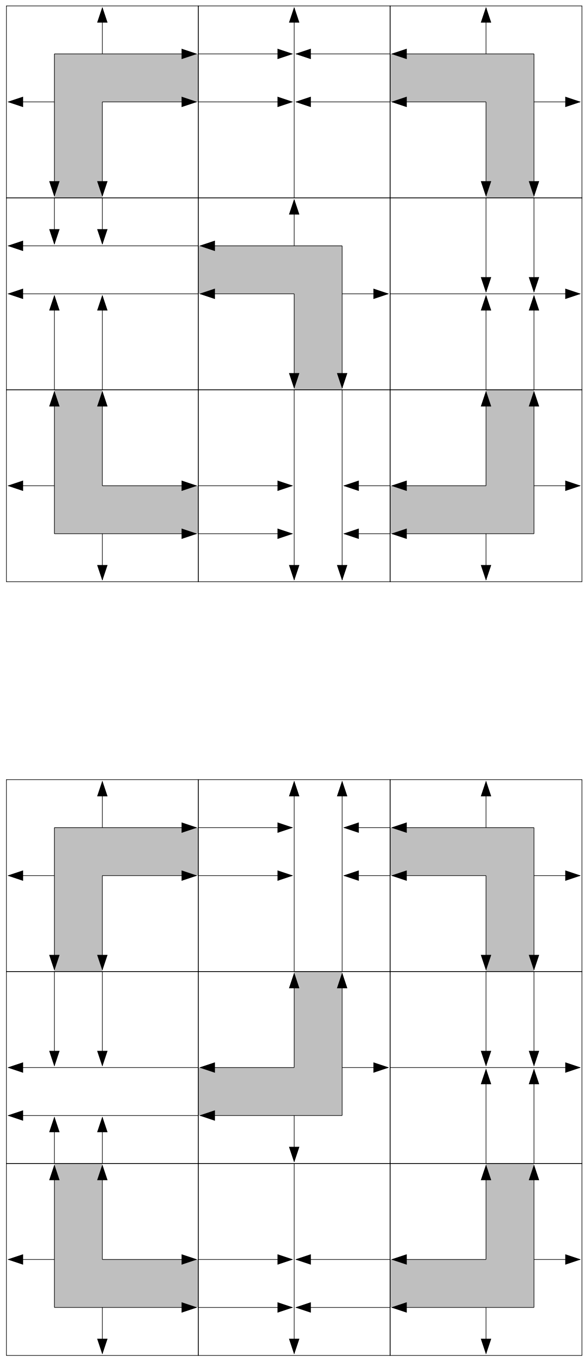

Figure 9

Continuing this argument, we see that each level 1 supertile forces the existence of 4 aligned level 1 tiles, as shown in Figure 10.

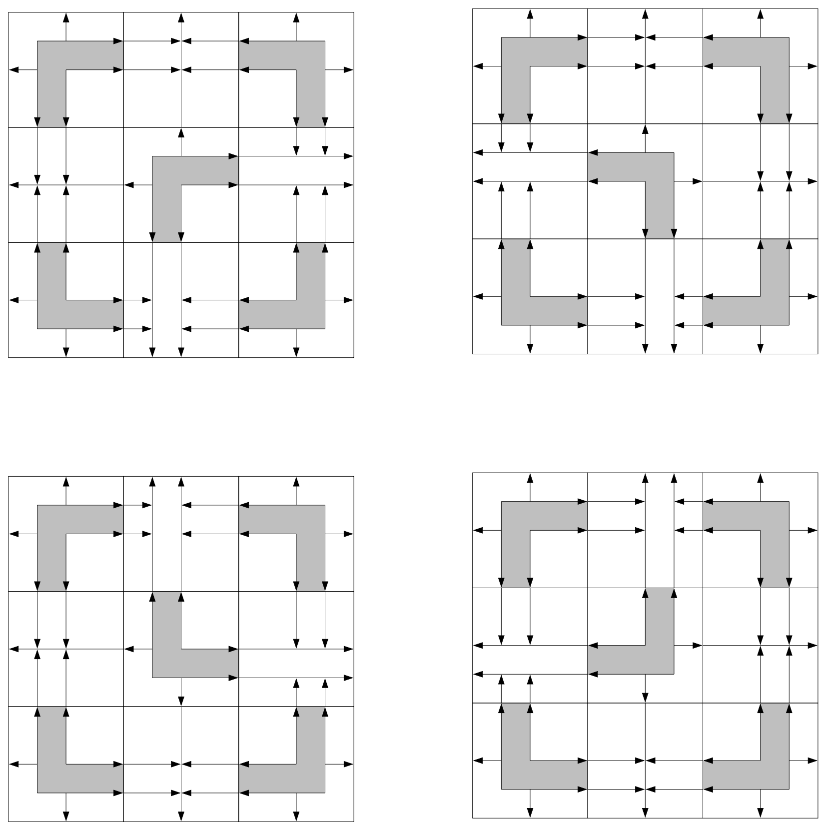

Figure 10

Now we focus on what tiles can be placed in the very center of Figure 10. If we do not use a special tile, then we produce a row of tiles all pointing in the same direction and we get the same kind of contradiction that we discussed above. Thus, the central tile is a special tile. From here, it is easy to see that the rest of the tiles must complete to make the arms of a level 2 supertile.

3.3 Higher Level Supertiles

Essentially the same argument as we gave for level 1 $\rightarrow$ 2 shows that each level n supertile in a Robinson tiling forces a level $n+1$ supertile.

4 Pairs of Supertiles

Lemma 4.1 *Suppose that X_1 and X_2 are two level n supertiles contained in the same Robinson tiling. Then the center of X_1 is at least 2^n units away from the center of X_2.*

Proof: Let C_1 denote the central cross of X_1 and let C_2 denote the central cross of X_2. The left half of Figure 10 shows the central cross of a level 2 supertile, and the general picture is similar.

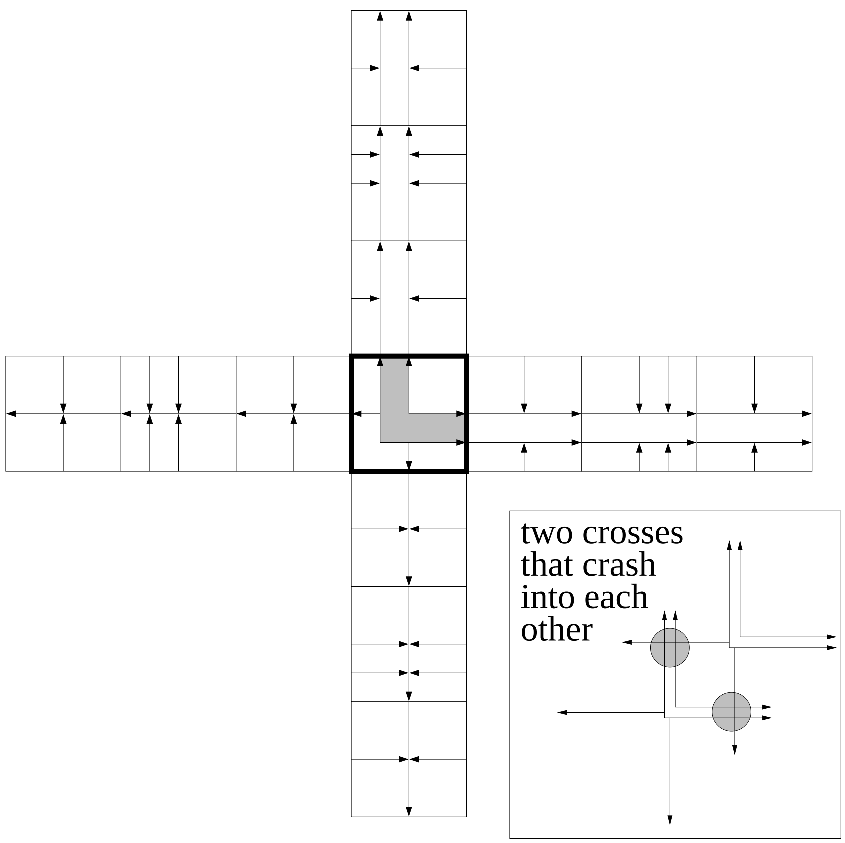

Figure 10: central cross

If X_1 and X_2 are within 2^n units of each other, then one of the arms of C_1 would crash through one of the arms of C_2, and *vice versa*, as shown in the right half of Figure 10. ♠

5 No Infinite Symmetry

Let T be a tiling of the plane by Robinson tiles. Let G denote the symmetry group of T. That is, each element $g \in G$ is an isometry of the plane such that $g(T) = T$. The purpose of this section is to show that G must be a finite group.

Lemma 5.1 *If G is an infinite group, then G contains a translation.*

Proof: Since T is made up of square tiles, every element of G preserves the coordinate axes of $\boldsymbol{R}^2$. Thus, every finite order element of G is one of several types:

1. A rotation by $\pi/2$ about some point.
2. A rotation by π around some point.
3. A reflection through a line of $\boldsymbol{R}^2$. The line must parallel to one of 4 directions, because the reflection preserves the coordinate axes.

If G has infinitely many reflections, then G contains two elements g_1 and g_2 that are reflections in parallel lines. But then $g_1 g_2$ is a translation along the lines perpendicular to the lines of reflection.

If G only has finitely many reflections, then G has infinitely many rotations. If some element g is rotation by $\pi/2$, then the square g^2 is rotation by π. Therefore, G contains infinitely many rotations by π. If g_1 and g_2 are two such rotations, then $g_1 g_2$ is again a translation. ♠

Suppose that G is an infinite group. Then G has a translation g, by the previous result. Choose n to be much larger than the translation length of g, and let X be a level n supertile contained in T. Then $g(X)$ is also a level n supertile. However, X and $g(X)$ are too close together, and we contradict Lemma 4.1. We can say more about G.

Lemma 5.2 *The elements of G have a common fixed point.*

Proof: Let $g_1, ..., g_n$ be the elements of G and let x be any point in the plane. Then the center of mass $\frac{1}{n}(g_1(x) + ... + g_n(x))$ is fixed by all elements of G. ♠

Since all elements of G preserve the coordinate axes, there are only 8 possible elements fixing the same point. Hence G has at most 8 elements.

6 Uncountably Many Robinson Tilings

Here we will show that there are uncountably many distinct Robinson tilings. By *distinct* we mean that the one tiling is not the translate of the other. The argument we give is similar to what we did for the Penrose tilings.

Let's say that a *robust* Robinson tiling is one created by the methods of §2, in which all 4 directions of supertiles are used infinitely often. A robust Robinson tiling has the following *robustness property*: For any pair of points x and x' in the plane, there is some supertile in the tiling that eventually contains both x and x'. The point is that the supertiles of the robust tiling grow outward in a kind of "spiralling" fashion that eventually engulfs the entire plane. Figure 11 shows a caricature of this

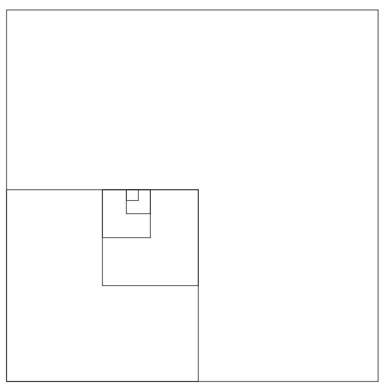

Figure 11: central cross

Let T be a robust Robinson tiling. Given any point $x \in \boldsymbol{R}^2$, we can assign an infinite sequence $\tau(x)$. The nth term in $\tau(x)$ records the direction of the level n supertile containing x. Eventually, this sequence is well defined. The robustness property guarantees that the sequences $\tau(x)$ and $\tau(x')$ have the same tail end. Hence, there is a well-defined tail end $\tau(T)$ associated to T.

Two robust Robinson tilings have the same tail end if and only if they are equivalent by a translation. Furthermore, there are uncountably many different tail ends. Therefore, there are uncountably many distinct Robinson tilings.