Existence of Bends on Paper Moebius Bands

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April 4, 2025

Let M be the *flat Moebius band*

$$M = [0,a] \times [0,b]/\sim, \qquad (t,0) \sim (a-t,b). \tag{1}$$

We are taking a rectangle and identifying opposite sides by the usual orientation reversing map. We suppress a, b from the notation. A *paper Moebius band* is a smooth isometric embedding $I : M \to \boldsymbol{R}^3$. That is, I is infinitely differentiable and the differential dI is an isometry. Let $\Omega = I(M)$.

A *bend* on Ω is a line segment that lies in the interior of Ω except for its endpoints, which lie in the boundary. The purpose of these note is to give an elementary and self-contained proof of the following classical result, which plays an important role in [**HW**] and [**S**].

Theorem 0.1 *There is a continuous partition of Ω into bends.*

We will deduce Theorem 0.1 from a subsidiary result, which we now describe. Let Ω^o be the interior of Ω. Let S^2 be the unit 2-sphere. The *Gauss map*, which is well defined and smooth on any simply-connected subset Ω^o, associates to each point $p \in \Omega^o$ a unit normal vector $n_p \in S^2$. Let dn_p be the differential of the Gauss map at p. Since the curvature Ω^o is 0 everywhere, dn_p has a nontrivial kernel. The point p has nonzero *mean curvature* if and only if dn_p has nontrivial image. Let $U \subset \Omega^o$ denote the subset having nonzero mean curvature. Theorem 0.1 is a quick consequence of the following result in differential geometry.

Lemma 0.2 *Each $p \in U$ lies in a unique bend γ. Furthermore, the interior of γ lies in U.*

Proof of Theorem 0.1: It follows immediately from Lemma 0.2 that U has a continuous partition into bends. The uniqueness implies the continuity. Let τ be a component of $\Omega - U$. If τ has empty interior then τ is a line segment, the limit of a sequence of bends. In this case τ is also a bend. Suppose τ has non-empty interior. The Gauss map is constant on τ and hence τ lies in a single plane. Two sides of τ, opposite sides, lie in $\partial\Omega$ and are straight line segments. The other two sides of τ, the other opposite sides, are bends. Thus τ is a planar trapezoid. But then we can extend our bend partition across τ by simply choosing any continuous family of segments on τ that interpolates between the two bends in its boundary. Doing this construction on all such components, we get our continuous partition of Ω into bends. ♠

On the bottom of p. 46 of [**HW**], Halpern and Weaver say that the result of Lemma 0.2 is well known. They cite the references [**CL**], [**HN**], and [**St**]. More precisely, Lemma 0.2 is a special case of the two essentially identical results, [**CL**, p. 314, Lemma 2] and [**HN**, §3, Lemma 2]. These results and proofs are done in a general multi-dimensional setting. I found these proofs quite difficult to read. What follows is an elementary proof of Lemma 0.2, tailored to the 2-dimensional case.

Let $U \subset \Omega^o$ as above. Let $p \to n_p$ be a local choice of the Gauss map. We can rotate and translate so that near the origin U is the graph of a function

$$F(x,y) = Cy^2 + \text{higher order terms}. \tag{2}$$

Here $C > 0$ is some constant. The normal vector at the origin is $n_0 = (0,0,1)$. The vector $v_0 = (1,0,0)$ lies in the kernel of dn_0. Let $w_0 = v_0 \times n_0 = (0,1,0)$. Let Π_0 be the plane spanned by w_0 and n_0. The image of $\Pi_0 \cap U$ under the Gauss map is (near n_0) a smooth regular curve tangent to w_0 at n_0.

Working locally, we have three smooth vectorfields:

$$p \to n_p, \qquad p \to v_p, \qquad p \to w_p = v_p \times n_p. \tag{3}$$

Here v_p is the kernel of dn_p and $\times$ denote the cross product. Let Π_p be the plane through p and spanned by w_p and n_p. From our analysis of the special case, and from symmetry, the image of $\Pi_p \cap U$ under the Gauss map is (near n_p) a smooth regular curve tangent to w_p at n_p. The *asymptotic curves* are the smooth curves everywhere tangent to the v vector field. Here is the first key point of the proof.

Lemma 0.3 *The asymptotic curves are line segments.*

Proof: Let γ be an asymptotic curve. By construction, the Gauss map is constant along γ. About each point in γ there is a small neighborhood V which is partitioned into asymptotic curves that transversely intersect each plane Π_p when $p \in \gamma \cap V$. Hence the image of V under the Gauss map equals the image of $\Pi_p \cap V$ under the Gauss map. This latter image is a smooth regular curve tangent to w_p at n_p. Since this is true for all $p \in \gamma \cap V$ and since n_p is constant along γ we see that w_p is constant along γ. Hence v_p is constant along γ. Hence γ is a line segment. ♠

The nonzero mean curvature implies that γ is the unique line segment through any of its interior points. To finish the proof of Lemma 0.2, we just have to rule out the possibility that γ reaches ∂U before it reaches $\partial\Omega$. Assume for the sake of contradiction that this happens. We normalize as in Equation 2.

We now allow ourselves the liberty of dilating our surface. This dilation preserves all the properties we have discussed above. By focusing on a point of γ sufficiently close to ∂U and dilating, we arrange the following:

- A neighborhood V of Ω^o is the graph of a function over the disk of radius 3 centered at the origin.
- The normal to V at $(0,0,0)$ is $(0,0,1)$.
- $\gamma \subset V$ contains the arc connecting $(0,0,0)$ to $(3,0,0)$, but $(0,0,0) \not\in U$.
- Given $p \in V$ let p' be the projection of p to the XY-plane. We have $|p_1' - p_2'| > (2/3)|p_1 - p_2|$ for all $p_1, p_2 \in V$.

For $a \in (0,3)$ and at $(a,0,0)$ we have

$$v_a = (1,0,0), \qquad w_a = (0,1,0), \qquad n_a = (0,0,1).$$

Let Π_a be the plane $\{X = a\}$. Near $(a,0,0)$, the intersection $U_a = U \cap \Pi_a$ is a smooth curve tangent to w_a at $(a,0,0)$.

Let $\zeta = (1,0,0)$. Fix $\delta > 0$. By continuity and compactness, the asymptotic curves through points of U_1 sufficiently near ζ contain line segments connecting points on U_2 to points on U_δ. Call these *connectors*. There exists a canonical map $\Phi_\delta : U_1 \to U_\delta$ defined in a neighborhood of ζ: The points $q \in U_1$ and $\Phi_\delta(q) \in U_\delta$ lie in the same connector.

Lemma 0.4 *Φ_δ expands distances by less than a factor of* 3.

Proof: For each $X \subset V$ let $X' \subset \boldsymbol{R}^2$ denote the projection of X. Let ℓ_1 and ℓ_2 be two connectors through U_1. Let $b_j = \ell_j \cap U_\delta$ and $a_j = \ell_j \cap U_1$. We have

$$\frac{|b_2 - b_1|}{|a_2 - a_1|} < \frac{3}{2}\frac{|b_2' - b_1'|}{|a_2 - a_1|} \leq \frac{3}{2}\frac{|b_2' - b_1'|}{|a_2' - a_1'|} \leq^* \frac{3}{2} \times 2 = 3.$$

The first two inequalities come from the properties of projection on V arranged above. Here is the explanation of the starred inequality. The line segments ℓ_1' and ℓ_2' have slopes less than $1/100$ in absolute value provided that we take U_1 small enough, because these two segments are disjoint from the x-axis, and intersect a small neighborhood of $(1, 0)$, and extend at least $1/2$ away from this small neighborhood in either direction. Geometrically, a_1', b_1' and a_2', b_2' are the endpoints of nearly parallel line segments. This gives the starred inequality easily. ♠

Fix $\epsilon > 0$. The mean curvature along U_δ supposedly tends to 0 as we let $\delta \to 0$. If we choose δ sufficiently small then the Gauss map expands distances along U_δ in a neighborhood of $(\delta, 0, 0)$ by a factor of less than ϵ. Combining Lemma 0.4 and the fact that $n_q = n_{\Phi_\delta(q)}$, we see that the Gauss map expands distances by at most a factor of 3ϵ along U_1 in a small neighborhood of ζ. Since ϵ is arbitrary, $w_1 \in \ker(dn_\zeta)$. But $v_1 \in \ker(dn_\zeta)$ by definition. Hence dn_ζ is the trivial map. The contradicts the fact that $\zeta \in U$. This completes the proof of Lemma 0.2.

References:

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