Build Boy's Surface

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1 Introduction

Boy's surface [**B**] is a famous immersed copy of the projective plane in $\boldsymbol{R}^3$. I have tried at various times to understand Boy's surface, but somehow the effort has never met with joy. From time to time I have been able to concentrate enough so that its crazy twists and turns fit together in my mind (or seemed to), but then rather quickly the image went away without a trace. Moira Chas recently gave me one of her beautiful wire-knit models [**C**] of Boy's surface, and this inspired me to try again. Now I have it straight. For a nice and succinct alternate description of Boy's surface, see [**K**].

These notes give a fairly conceptual description of Boy's surface that does not draw too much on three dimensional visualization. As I discuss in §7, the approach I take owes a big intellectual debt to a beautiful youtube video [**S**], produced by the Serbian Academy of Sciences, showing an airplane flying around and constructing Boy's surface. You could view these notes as an elaboration of the video.

I will explain how to build Boy's surface out of simple pieces. All the pieces except the last one are easy to make out of cardstock and tape, and the last piece is determined automatically from the others by a coning procedure. For the coning procedure to work most gracefully it is useful to think of Boy's surface as living in the 3-sphere, which I model as $\boldsymbol{R}^3 \cup \infty$.

I will include a kit that lets you print out and assemble all but the last piece. The kit works. Katherine Williams Booth made a model from it. Here is her photograph of her model.

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Figure 0: An assembled kit!

Here is an outline of these notes.

• In §2, I discuss the *octahedral graph* in some detail. This discussion gives a combinatorial explanation of why the boundary of the immersed surface in Figure 0 is a single loop. Of course, you can also just build the thing and trace the boundary around with your finger.

• In §3, I define the *octacross*. This piece is homeomorphic to the union of the 3 coordinate planes. It is the central piece in Figure 0.

• In §4, I define the *crossbridge*. This piece is homeomorphic to the union of 2 coordinate planes. The 3 crossbridges are the big hoops in Figure 0.

• In §5, I assemble most of Boy's surface: You attach 3 crossbridges to the octacross and then glue in 3 round disks. The disks are visible in Figure 0. The union M of these 7 pieces turns out to be an immersed Moebius band. This is what Figure 0 shows.

• In §6, I attach the final piece P, which I call *the pizza*. It turns out that the boundary ∂M is *cone-friendly*: Each ray through the origin intersects ∂M at most once. We set $P = \{tp|\ p \in \partial M,\ t \geq 1\}$. Here P is a topological disk we get by coning ∂M to ∞. The reason for the name is that ∂M has the combinatorial structure of an 18-gon, and correspondingly P is made from 18 *pizza slices*, each one a topological triangle. Boy's surface is $M \cup P$.

• In §7, I compare and contrast these notes with the youtube video [**S**]. I also make a connection to paper Moebius bands [**Sch**] and raise an optimization question related to the construction here.

• In §8, I include the kit which lets you make M, as in Figure 0.

• In §9, I present an alternate model using rectilinear pieces.

I thank Moira Chas, Peter Doyle, and Dan Margalit for helpful conversations about these notes. Finally, I thank Katherine Williams Booth for making the model in Figure 0.

2 The Octahedral Graph

Let G be the *octahedral graph*. This graph is the union of vertices and edges of the octahedron. A *perfect matching* for G is a union of 3 edges of G such that every vertex of G is contained in exactly one edge in the union. Figure 1 shows a perfect matching for G, drawn in a symmetric way.

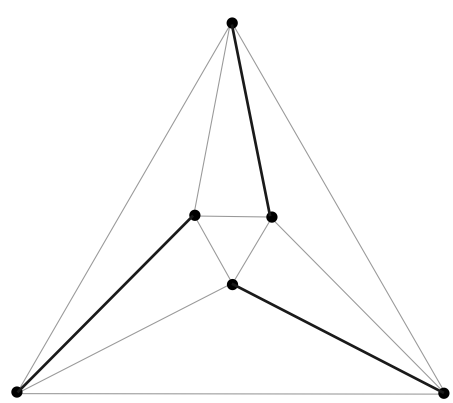


Figure 1: A perfect matching for the octahedral graph

We call the 3 edges in the matching *bridges*, and the remaining 9 edges *roads*. The reason for the name is that each bridge e connects the roads that are incident to vertices on either side of e. We can thicken e slightly and then replace e with three parallel copies. (I like to think of these parallel copies as lanes on the bridge.) This defines 3 paths of length 3, each of which has the form road-bridge-road. Figure 2 shows this.

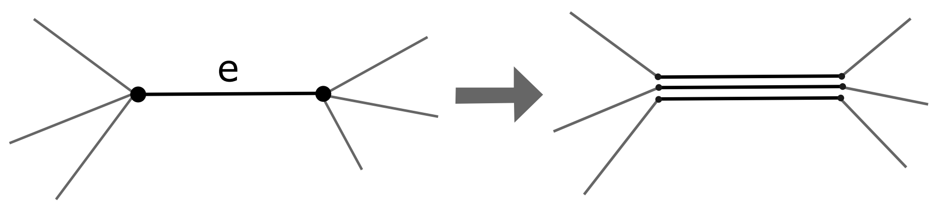


Figure 2: A bridge between incident roads

Figure 3 shows what happens when we do this construction simultaneously for all three bridges. The result is a length 18 circuit that alternates between roads and bridges. I have added numbers and directions to the roads to help you trace out the circuit.

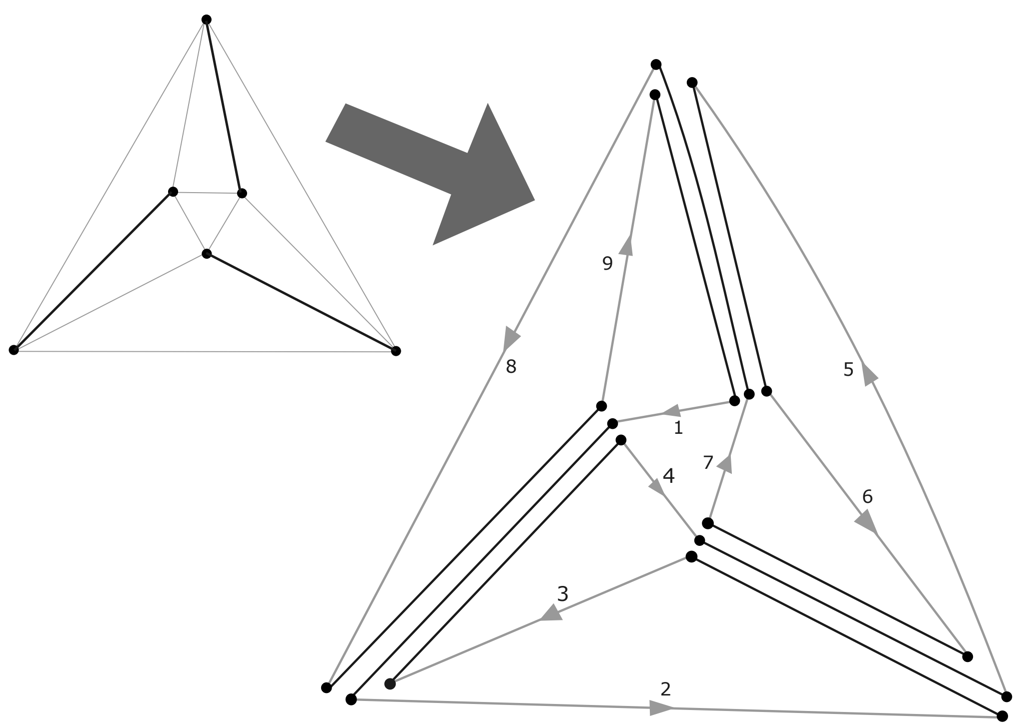


Figure 3: A circuit of length 18

Figure 4 is the same as Figure 3, but with different labels. The labels correspond to the usual choice of octahedron having vertices $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. Geometrically, you are staring at the octahedron along the axis $X = Y = Z$. The vertex $(\pm 1, 0, 0)$ is labeled by $(\pm, 0, 0)$. etc. We label the roads by the signs of the coordinates of points on them.

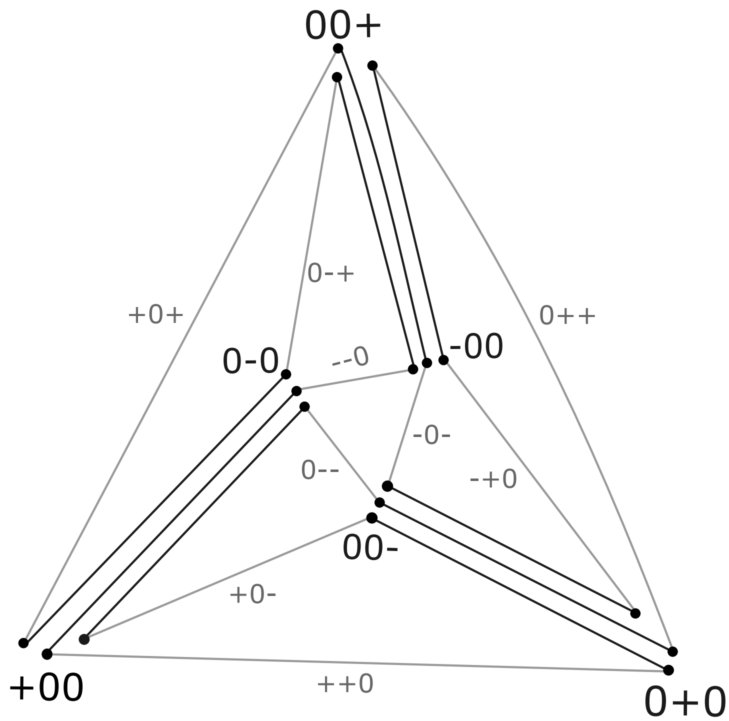


Figure 4: Same path as in Figure 3, with different labels

All our constructions below are invariant under the order 3 rotation about the axis $X = Y = Z$, namely the map $(X, Y, Z) \to (Y, Z, X)$. Likewise, all our constructions are invariant under the order 2 rotations that respectively fix the lines $X + Y = 0$ and $Y + Z = 0$ and $Z + X = 0$. The first of these rotations is given by the map $(X, Y, Z) \to (-Y, -X, -Z)$. The labelings in Figure 4 reflect these symmetries.

Referring to Figure 0, the 18 vertices in Figure 4 correspond to the 18 points of non-smoothness on the surface boundary – i.e., the 18 points where the octacross attaches to the crossbridges. Most of the tape appears around these points.

3 The Octacross

We first mention some notation which we will use repeatedly.

- Given a set A_{XY}, we get A_{YZ} and A_{ZX} by cycling the coordinates.
- $D_r(p)$ denotes the closed disk of radius r centered at p.
- A^o and ∂A respectively denote the interior and boundary of A.

As in Figure 5, define

$$O_{XY} = \Big([-1,1]^2 - \bigcup D^o_{2/3}(\pm 1, \pm 1)\Big) \times \{0\}. \quad (1)$$

The union is taken over the 4 disks we get by taking all sign choices.

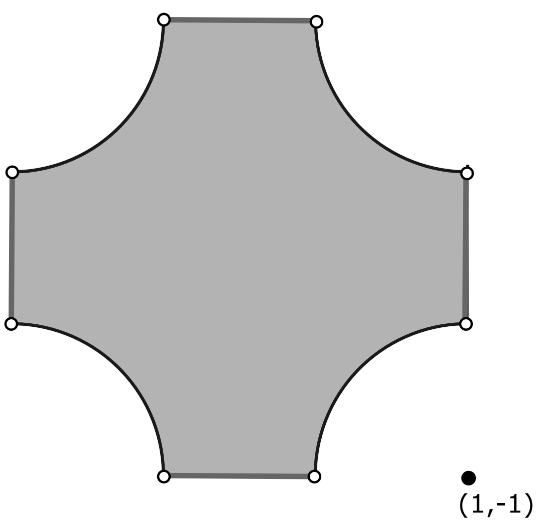


Figure 5: The set O_{XY}

The set O_{XY} is an octagon made from 8 sides, with curved and straight sides alternating. The *octacross* is

$$O_{XYZ} = O_{XY} \cup O_{YZ} \cup O_{ZX}. \quad (2)$$

The octacross has 24 edges, 12 curved and 12 straight. We call the curved edges *roads*. The 12 straight edges cross each other in pairs, making 6 *crosses*, two per coordinate axis. If we were to crush these crosses down to points, the resulting graph would be the octahedral graph.

4 The Crossbridges

Let

$$A_{XY} = \Big(D_{4/3}(1,-1) - D^o_{2/3}(1,-1)\Big) \times \{0\},$$

$$B_{XY} = \partial D_1(1,-1) \times [-1/3, 1/3]. \tag{3}$$

A_{XY} and B_{XY} are annuli which intersect in the unit circle $\partial D_1(1,-1) \times \{0\}$. One of the *crossbridges* is

$$Q_{XY} = \{(x,y,t) \in A_{XY} \cup B_{XY} | \ \max(|x|,|y|) \geq 1\}. \tag{4}$$

We are chopping one quadrant out of $A_{XY} \cup B_{XY}$ to create Q_{XY}. Figure 6 shows the projection of Q_{XY} into the XY plane.

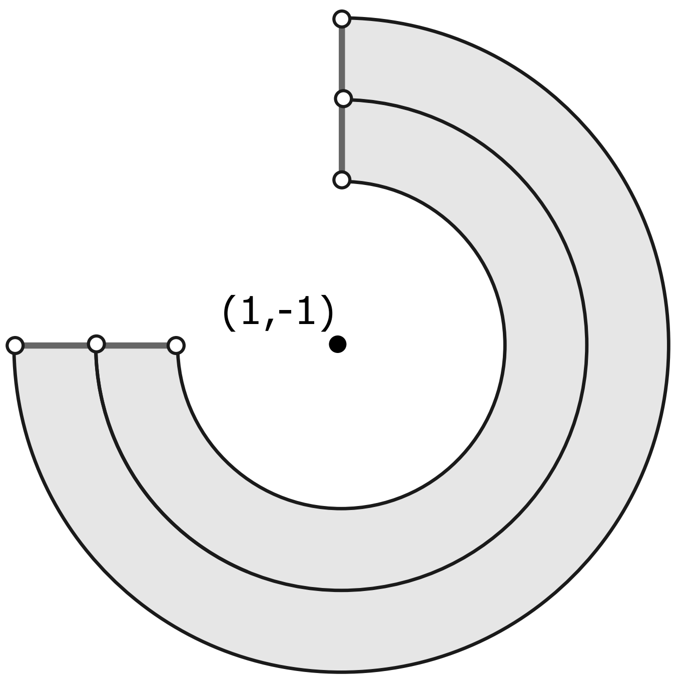


Figure 6: The XY-projection of Q_{XY}.

Q_{XY} is made by sweeping out a cross along three-quarters of a circle. The boundary of Q_{XY} consists of 4 curved edges and 4 straight edges. We call the curved edges *bridges*. The straight edges in the boundary cross in pairs, making 2 crosses. These crosses match 2 of the crosses of the octacross.

The other crossbridges are Q_{YZ} and Q_{ZX}.

5 Assembly

Figure 7 shows the XY-projection of the octacross O_{XYZ} and the crossbridge Q_{XY}. This picture is a bit like a head with one hooped earring.

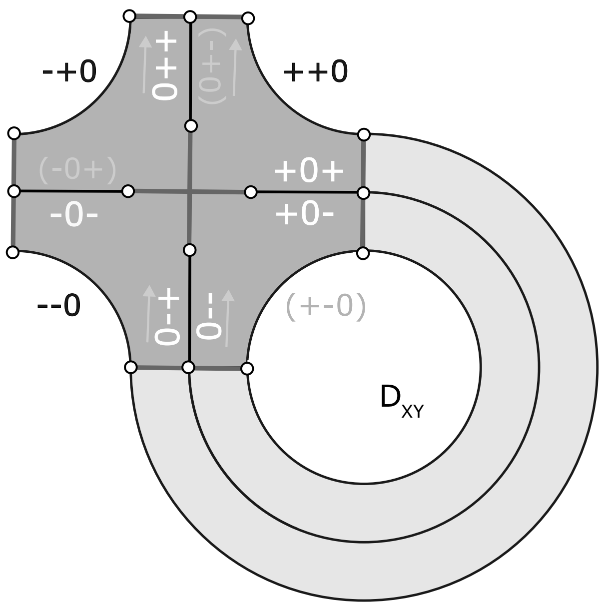


Figure 7: The XY-projection of $O_{XYZ} \cup Q_{XY}$.

The set

$$H = O_{XYZ} \cup Q_{XY} \cup Q_{YZ} \cup Q_{ZX} \qquad (5)$$

is like a head with 3 hooped earrings. Again, see Figure 0. As Figure 7 suggests, ∂H contains 3 circles, each made from a road-bridge pair, one per coordinate plane. We glue in the disks D_{XY} and D_{YZ} and D_{ZX} bounded by these circles. That is, we define

$$M = O_{XYZ} \cup Q_{XY} \cup Q_{YZ} \cup Q_{ZX} \cup D_{XY} \cup D_{YZ} \cup D_{ZX}. \qquad (6)$$

∂M has $9 = 12 - 3$ roads and $9 = 12 - 3$ bridges. Together these make an embedded loop, a combinatorial-length 18 circuit in which the roads and bridges alternate. Combinatorially, this is the same circuit as the one in Figure 4. For instance, looking at Figure 7 we can see that the roads $(++0)$, $(--0)$ and $(+0+)$, $(0-+)$ and $(+0-)$, $(0--)$ connect across the shown crossbridge. This matches Figure 4. (Our vertical labels are meant to be read from bottom to top, as the arrows indicate.)

Now I show that M is an immersed Moebius band. Let M_{XY} denote the intersection of M with the XY-plane. Here M_{XY} is a topological rectangle with alternating straight and curved edges. See Figure 8. The closure of $M - M_{XY} - M_{YZ} - M_{ZX}$ is the union of the B-parts of the crossbridges, B_{XY} and B_{YZ} and B_{ZX}. Each of these pieces is also a topological rectangle, with alternating straight and curved edges. We have a cycle of 6 pieces, with adjacent pieces sharing a common straight edge:

$$\begin{array}{ccccc} M_{XY} & \leftrightarrow & B_{YZ} & \leftrightarrow & M_{ZX} \\ \updownarrow & & & & \updownarrow \\ B_{ZX} & \leftrightarrow & M_{YZ} & \leftrightarrow & B_{XY} \end{array}$$

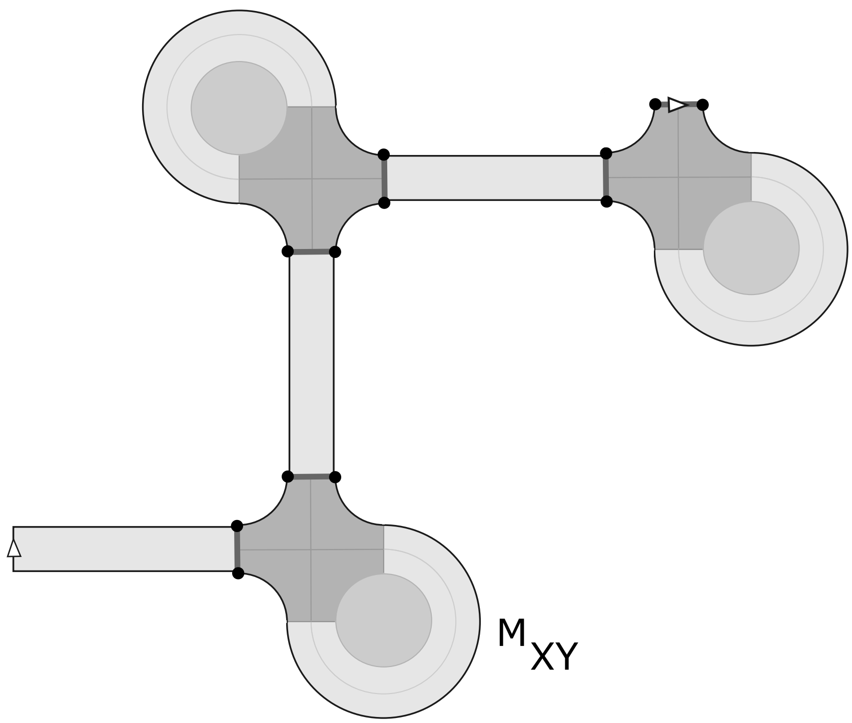


Figure 8: M_{XY} and the Moebius band M^*.

We treat each piece as a space in itself, and then form the identification space by gluing along the common straight boundary edges. This gives an abstract Moebius band M^* that is decomposed into 6 topological rectangles. Figure 8 shows a fundamental domain for M^*. The arrowed sides are meant to be glued. The obvious map $M^* \to M$ is an isometric immersion.

6 Adding the Pizza

We work in the 3-sphere $S^3 = \boldsymbol{R}^3 \cup \infty$. Let M be our immersed Moebius band, as in Equation 6. As we mentioned in the introduction, ∂M is cone-friendly, meaning that each ray through the origin intersects ∂M at most once. We define the *pizza* as the cone of ∂M to ∞:

$$P = \{tp|\ p \in \partial M,\ t \geq 1\} \cup \{\infty\}. \tag{7}$$

We have $M \cap P = \partial M$. The union $M \cup P$ is Boy's surface. We get Boy's surface by attaching an embedded disk to an immersed Moebius band along the topological circle $\partial M = M \cap P$. Hence $M \cup P$ is an immersed projective plane. $M \cup P$ has the 6-fold symmetry discussed at the end of §2.

We now look more closely at P, with a view towards making the cone-friendliness of ∂M more transparent. P is made from 18 topological triangles, the *pizza slices*, one per edge of ∂M.

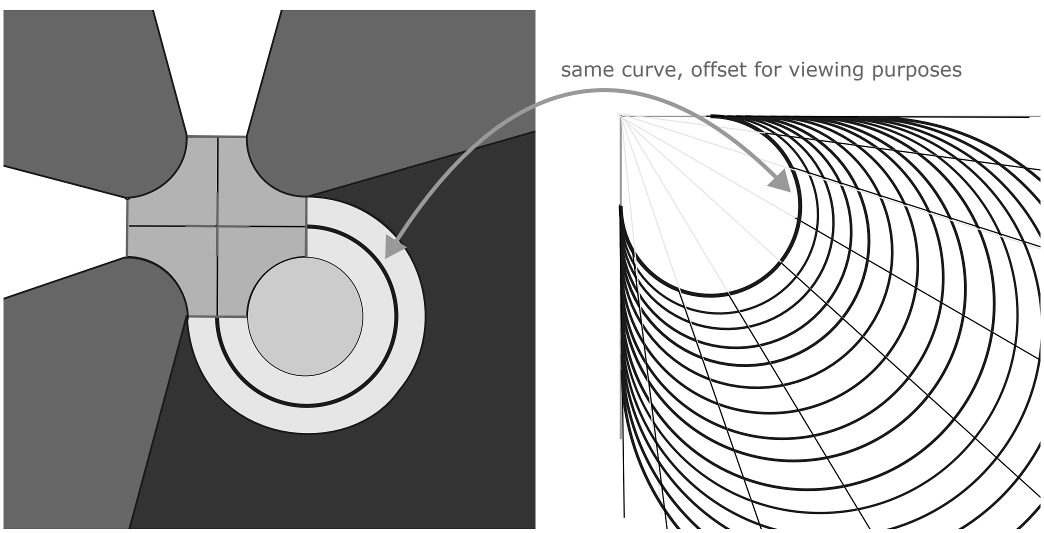


Figure 9: The 4 slices in the XY plane and a hint of a non-planar slice

Of these, 12 lie in coordinate planes and 6 do not. Each of the 6 non-planar ones lies in an elliptical cone, and each cone lies in a different orthant. Figure 9 shows the 4 pizza slices in the XY-plane. Figure 9 also shows the non-planar slice contained in the $+-+$ orthant. We are taking the double foliation of this slice by (upward rising) rays and (3/4)-circles and projecting it down into the XY-plane. The other pizza slices are obtained from these 5 by the symmetry discussed at the end of §2.

7 Discussion

The Airplane Movie: The airplane flies around and produces a set H' which is the same as H (from Equation 5) topologically and close geometrically. The movie then fills in the three disks to make a set M' which is likewise close to M. Finally, the movie explicitly caps off M' with embedded (curvy) disk $P' \subset \boldsymbol{R}^3$. These notes differ from the movie in several ways.

- The airplane builds H' in a different order than we build H. If you follow along the corridor in Figure 8, bottom to top, you are making the airplane's path.
- The movie does not explain why M' is a Moebius band or why $\partial M'$ is connected. (In general, nothing is explained or defined or discussed in the movie; everything is exhibited.)
- The disks P' and P differ in several ways. First, P' lives in $\boldsymbol{R}^3$. This breaks the 6-fold symmetry discussed at the end of §2. Second, P' seems to be smoother than P. The cone point at ∞ for P is rather complicated.

I don't mean to suggest that these unexplained things are deficiencies in the movie. The movie is amazing, and it has different aims than these notes.

Intrinsic Flatness: The abstract Moebius band M^* is a developable surface, as indicated by Figure 8. The universal cover $\widetilde{M}^*$ of M^* is an infinite topological strip. The symmetries discussed at the end of §2 all correspond to symmetries of $\widetilde{M}^*$. For instance, the order 3 isometry of $\boldsymbol{R}^3$ which cycles the coordinates lifts to give a glide reflection of $\widetilde{M}^*$. The cube of this glide reflection is the generator of the deck transformation group.

When we think of ∂M^* as a curve in the boundary of an intrinsically flat Moebius band, it has total geodesic curvature 9π. In particular, ∂M^* is not a paper Moebius band in the sense of [**Sch**]. However, the situation here does raise an optimization question. How much can you decrease the total geodesic curvature by applying an ambient homeomorphism of $\boldsymbol{R}^3$ which retains the intrinsic flatness of M^* and also the cone-friendliness of ∂M?

It is also worth mentioning that the intersection $P \cap \boldsymbol{R}^3$ is also intrinsically flat. However, the geodesic curvature of ∂P, with respect to P, is rather complicated on the non-planar pizza slices. So, $M \cup P$ has a strange intrinsic geometry along parts of $M \cap P$.

8 Build Your Own

Here is our kit for building M. The kit is set up in such a way that you don't need to glue in 3 round disks. These come pre-attached.

Step 0: On cardstock, print 3 copies each of the next 2 pages.

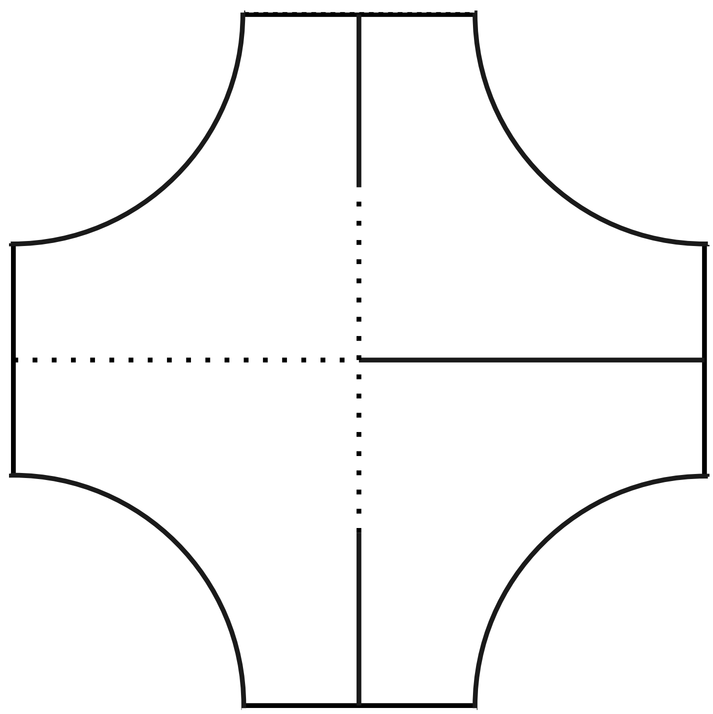
Step 1A: Cut out all 3 copies of the piece which matches Figure 5. On the first copy, cut along all solid lines. On the second copy, cut only along the long solid line. Cut the third copy in half horizontally, then cut on the vertical dotted lines. I found it useful to extend the cuts just a tiny bit, to make it easier to fit the pieces together all the way.

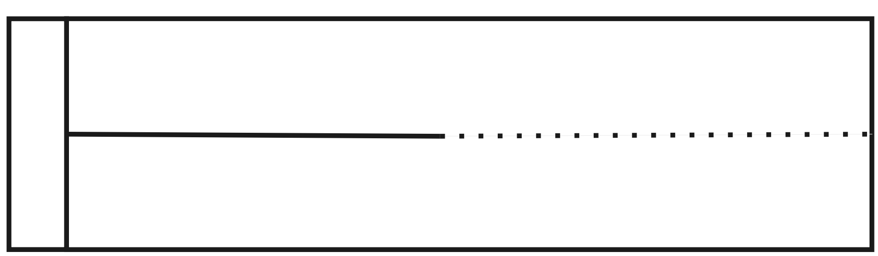
Step 1B: Take the two whole central pieces and slide them together along the cut slits. Tape together. Now slot in the two halves of the central piece to finish the octacross. Tape everything together as needed to make it firm.

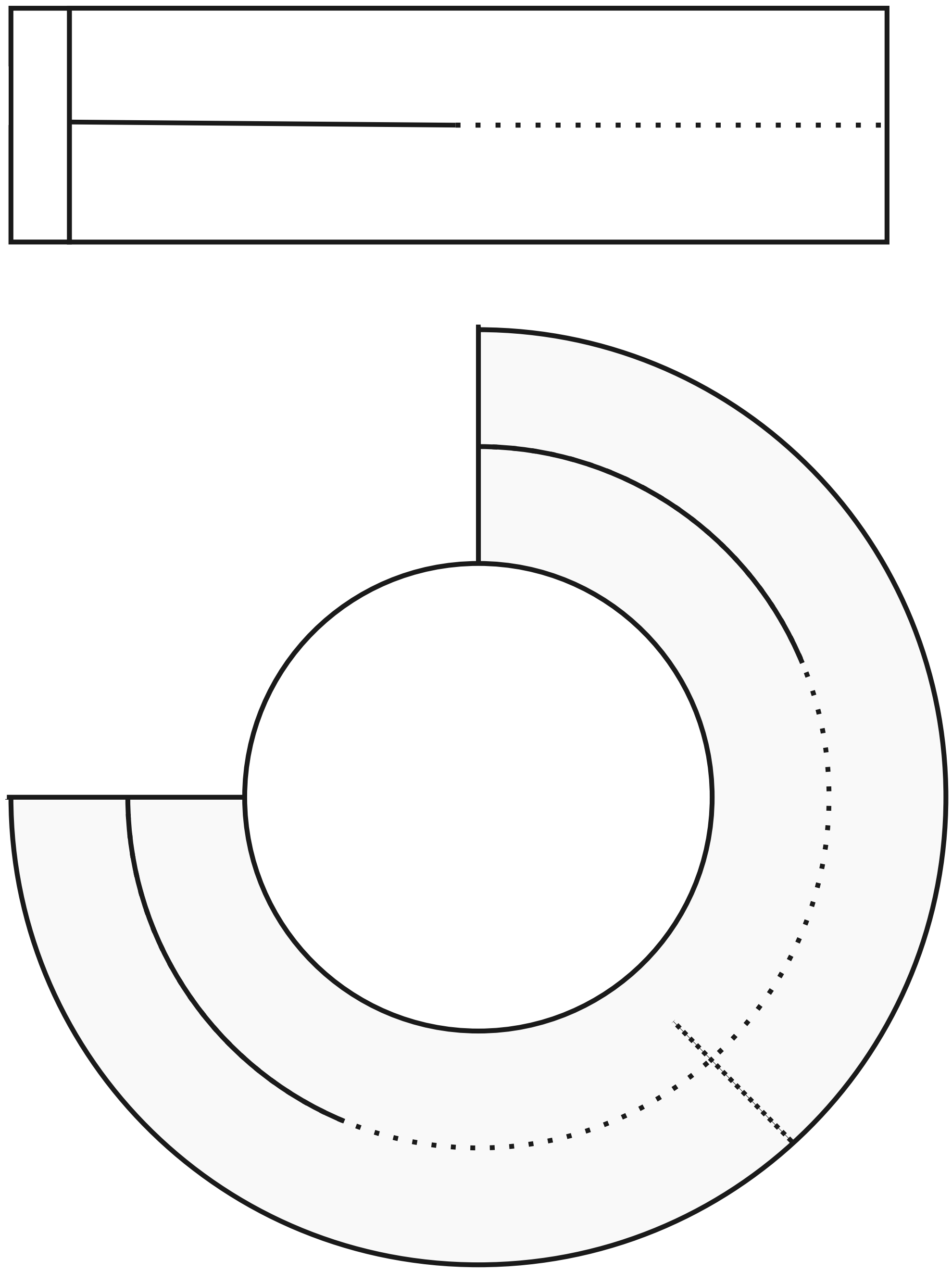
Step 2A: Cut out the 6 strips and then cut along the dotted lines. Be sure to include the thin rectangles on the left side of each strip. I will explain these in the next step. Cut out the 3 pieces made from a C-shaped annulus and the central disk, then cut along all dotted curves. (Note: Don't separate the central disk from the annulus.) The short crosscut on each annulus is just an access cut, to help you make the longer curved cut.

Step 2B: Slot two of the strips into one of the C shaped annuli along the cut slits, then tape together as needed. Finally, tape the access cut shut. This makes one crossbridge together with the central disk already attached. Now repeat for the other two crossbridges. One technical note: The extra thin rectangle on the end of the strips is something like a tab. The lengths of the strips are not precisely the same as the length of half the core curve, and this extra tab is supposed to absorb the mismatch.

Step 3: Join each crossbridge-plus-disk to the octacross as discussed in §5. Tape together as needed. The disk part of the crossbridge-plus-disk should fit together with the relevant curved edge of the octacross. Note: if you want the disks to appear just as in Figure 0, you have to take care about the orientations when you do this step. As with Ikea instructions, these instructions allow you to get the orientations subtly wrong.







9 A Rectilinear Model

Figure 10 shows a version of Figure 7 using alternate rectilinear models for the octacross, crossbridge, and disk.

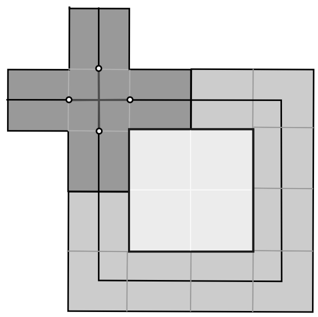


Figure 10: Rectilinear models for the octacross, crossbridge, and disk.

The rectilinear pieces have especially nice thickenings:

- The octacross sits inside a union of 7 cubes.
- Each crossbridge sits inside a C-shaped chain of 9 cubes.
- Each square disk sits inside a union of $4 = 2 \times 2$ cubes.

The union Ω of these 7 pieces is a topological ball made from 46 cubes. Letting $M^\square$ denote our rectilinear model for M, we have $M^\square \subset \Omega$ and $\partial M^\square \subset \partial\Omega$. We give an explicit description. Let Q_r denote the square of side-length r centered at $(3, -3)$. Scaling appropriately, we have

$$\Omega = \Omega_{XY} \cup \Omega_{YZ} \cup \Omega_{ZX}, \qquad M^\square = M^\square_{XY} \cup M^\square_{YZ} \cup M^\square_{ZX}, \tag{8}$$

$$\Omega_{XY} = Q_4 \times [-1, 1], \qquad M^\square_{XY} = (Q_4 \times \{0\}) \cup (\partial Q_3 \times [-1, 1]).$$

Ω is the union of three $8 \times 8 \times 2$ slabs, whose intersection is the $2 \times 2 \times 2$ cube centered at the origin. Ω looks like a cubical head with 3 big ears.

This analysis points out some flexibility in how we add the final disk to cap off Boy's surface. The complement $S^3 - \Omega$ is also a ball, and $\partial M^\square$ is an embedded loop on the boundary. So, topologically speaking, it is easy to extend $\partial M^\square$ to an embedded disk in $S^3 - \Omega$. The coning construction, which works in this model as well, does it in a canonical way.

Ω is tiled by $368 = 46 \times 8$ unit cubes whose intersection with $M^\square$ is a union of 1, 2, or 3 adjacent faces. Thus, you could build $(\Omega, M^\square)$ by taking 368 cubes, suitably painting their faces, and sticking them together.

10 References

[**B**] W. Boy *Über die Curvatura integra und die Topologie geschlossener Flächen*, Math. Ann. Vol 57 (1903), 151–184.

[**C**] M. Chas, Instagram,
www.instagram.com/reel/C-n6ExptzhI/?igsh=MTl0NGd4ZTg0YThveg

[**K**] R. Kirby *What is ... Boy's surface*, Notices of the A.M.S, Vol 54, Number 10. (2007)

[**S**] Serbian Academy of Sciences, Project Ziva Mathematica, Youtube Video
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[**Sch**] R. E. Schwartz, *The Optimal Paper Moebius Band*, Annals of Mathematics, 2025.