Hadamard's Theorem

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1 The Result and Proof Outline

The purpose of these notes is to prove the following theorem.

Theorem 1.1 (Hadamard) *Let M_1 and M_2 be simply connected, complete Riemannian manifolds having constant sectional curvature -1. Then M_1 and M_2 are isometric.*

I'll outline the proof in this section and then fill in the details in subsequent sections. Choose points $p_j \in M_j$. We isometrically identify the tangent space $T_{p_1}M_1$ with the tangent space $T_{p_2}M_2$, and we call the common tangent space T. Here T is just n-dimensional Euclidean space. We have the exponential map $E_j : T \to M_j$ which has the following properties.

- $E_j(0) = p_j$.
- E_j maps lines through the origin to geodesics rays of M_j.
- dE_j is an isometry at 0.

We will prove the following lemma below.

Lemma 1.2 (Nonsingular) *dE_j is nonsingular at every point of T.*

The Nonsingular Lemma only uses the fact that M_j has nonpositive curvature. The proof comes down to the statement that Jacobi fields do not vanish in nonpositive curvature, and to Gauss's lemma about dE preserving a certain orthogonal splitting.

The Nonsingular Lemma as the following corollary.

Lemma 1.3 *E_j is a diffeomorphism.*

Proof: For the proof we set $M = M_j$, etc. Since dE is nonsingular, E is a local diffeomorphism. The fact that M is complete means that every point of $q \in M$ can be connected to p by a geodesic. But then this geodesic is in the image of f. Hence E is a surjective local diffeomorphism. It just remains to prove that E is injective.

Suppose that $E(p) = E(q)$. Consider the image $\alpha_0 = E(S_0)$, where S_0 is the straight line segment connecting p to q. Since M is simply connected, there is a homotopy of loops α_t, based at $E(p)$ which connects α_0 to the constant loop α_1. Since E is a local diffeomorphism, we can find a preimage S_t such that $E(S_t) = \alpha_t$. Moreover, we can take S_t so that its endpoints do not move. But α_1 is a single point and hence $E(S_1)$ is a single point. This contradicts the fact that E is a local diffeomorphism. ♠

Consider the map $g = E_2 \circ E_1^{-1} : M_1 \to M_2$. The map g is also a diffeomorphism. Below we will prove

Lemma 1.4 (Local Isometry) *dg is an isometry at each point.*

The Local Isometry Lemma also boils down to Jacobi fields.

Since g is a global diffeomorphism and an dg is an isometry at each point, g is a global isometry.

2 Jacobi Fields

Let M be a metric of constant negative curvature -1. Let $\langle , \rangle$ be the metric and $[,]$ the Lie bracket and $D_X Y$ be the covariant derivative of Y in the direction of X. Here D is the Levi-Civita connection. Here X and Y are vector fields.

The basic things we have are

- $D_X Y - D_Y X = [X, Y]$. This is the symmetry of the connection.
- $R(V, W)X = D_V(D_W X) - D_W(D_V X) - D_{[V,W]} X$. This just a definition.
- $R(V, W, X, Y) = \langle R(V, W)X, Y\rangle$. This is just a definition.
- $R(X, Y, X, Y) = -1$ when $\{X, Y\}$ is an orthonormal set of vectors.

There are other equalities which I'll mention as we go along.

Consider the map $E : T \to M$ defined above. We consider the restriction of E to a 2-dimensional subspace $\boldsymbol{R}^2 \subset T$. We write $E = E(s,t)$. Holding $s = s_0$ fixed, the function $E(s_0, t)$ is a unit speed geodesic. The vectorfield

$$J(t) = \frac{dE}{ds}(s_0, t) \tag{1}$$

is a Jacobi field along the geodesic $E(s_0, t)$. We take $s_0 = 0$ to do our analysis, though the same statements apply for any value of s_0.

Let $\gamma(t) = E(0,t)$. Let $T(t)$ denote the unit velocity vector field along γ. As a convention we write

$$\frac{DV}{ds} = D_J V, \qquad \frac{DV}{dt} = D_T V. \tag{2}$$

Here is the equation for Jacobi fields

Lemma 2.1

$$\frac{D}{Dt}\frac{D}{dt}J = -R(T,J)T.$$

Proof: We compute

$$\frac{D}{Dt}\frac{D}{dt}J = \frac{D}{Dt}\frac{D}{dt}\frac{dE}{ds} =_1 \frac{D}{Dt}\frac{D}{ds}\frac{dE}{dt} =_2$$

$$\frac{D}{Ds}\frac{D}{dt}T - R(T,J)T =_3 -R(T,J)T.$$

Equality 1 comes from the fact that

$$\frac{D}{dt}\frac{dE}{ds} - \frac{D}{dt}\frac{dE}{ds} - = D_T J - D_J T = [J,T] = 0.$$

The Lie bracket vanishes because these vector fields are the image of the commuting fields d/dx and d/dy under a smooth map. Equality 2 is the definition of the curvature tensor. Equatity 3 comes from the fact that $D_T T = 0$ because T is the velocity field of a geodesic. ♠

3 Local Diffeomorphism

We keep using the same notation as above. The following lemma also works in nonpositive curvature but we will make it easy and use the constant negative curvature condition.

Lemma 3.1 *$J(t) \neq 0$ for all $t > 0$.*

Proof: Let $f(t) = \langle J(t), J(t) \rangle$. We have $f(0) = 0$ We compute

$$f'(t) = 2\left\langle \frac{DJ}{dt}, J \right\rangle.$$

Since $J(0) = 0$, we have $f'(0) = 0$.

We compute

$$f''(t) = \frac{d^2}{dt^2}\langle J, J\rangle =_1 2\frac{d}{dt}\left\langle \frac{DJ}{dt}, J \right\rangle =_2$$

$$2\left\|\frac{dJ}{dt}\right\|^2 - \left\langle J, \frac{D^2J}{dt^2}\right\rangle =_3$$

$$2\left\|\frac{DJ}{dt}\right\|^2 - 2R(T, J, T, J) =_4$$

$$2\left\|\frac{DJ}{dt}\right\|^2 + 2 \geq 2.$$

Equalities 1 and 2 come from the compatibility of the connection and the metric. The last inequality comes from the constant negative curvature condition.

In summary, $f(0) = 0$ and $f'(0) = 0$ and $f''(t) > 0$ for all t. This shows that $f(t) > 0$ for all $t > 0$. ♠

Remark: In nonpositive curvature, we just get $\|J'(t), J'(t)\| \geq 0$, and we would need to make the further argument that $J'(t)$ is not identically 0. This comes from the fact the Jacobi equation is linear and in the relevant basis there is a whole $(2n)$-dimensional vector space of nontrivial solutions.

The next result is known as Gauss's Lemma. It is one of the millions of things due to Gauss, at least in the two dimensional case.

Lemma 3.2 $\langle J, T\rangle = 0$ *for all* t.

Proof: Let $g(t) = \langle J(t), T(t)\rangle$. We have

$$g'(t) = \left\langle \frac{DJ}{dt}, T\right\rangle + \left\langle \frac{DT}{dt}, J\right\rangle.$$

The second term vanishes and DJ/dt is perpendicular to T at 0 because $dE|_0$ is an isometry. Hence $g'(0) = 0$. A calculation like the one above shows that

$$g''(t) = R(T, J, T, T) = 0.$$

The second equality comes from the fact that the curvature tensor is antisymmetric in the last two slots. So, $g(0) = 0$ and $g'(0) = 0$ and $g''(t) = 0$ for all t. Hence $g(t) = 0$ for all t. ♠

Our arguments apply to any 2-plane in E. We conclude two things.

- dE respects the orthogonal splitting coming from the polar coordinate system on the vector space T.
- dE is nonzero on the vectors tangent to the spheres in the polar coordinate system.

These two properties show that dE is everywhere nonsingular.

3.1 Local Isometry

Let $J(t)$ be a Jacobi field perpendicular to γ. I'll show that $\|J(t)\| = \|J'(0)\| \sinh(t)$, independent of whether we are in M_1 or M_2. That is, the Jacobi fields grow at the same rate, a rate which only depends on the constant curvature condition. This situation, together with Gauss's Lemma, forces dg to be an isometry at each point.

Let $\{P_k\}$ denote a parallel orthonormal frame along γ, all of whose vectors are perpendicular to T. The following lemma only works in constant curvature. It is responsible for the clean growth formula we get.

Lemma 3.3 *$R(T,P_k,T,P_m) = -1$ if $m=k$ and otherwise 0.*

Proof: When $k=m$ we are just compute the sectional curvature of some plane. When $m \neq k$ we let $V = (P_k + P_m)/\sqrt{2}$. Then V is a unit field perpendicular to T and

$$1 = 2R(T,V,T,V) =$$

$$2R(T,P_k,T,P_k) + R(T,P_m,T,P_m,T) + R(T,P_k,T,P_m) + R(T,P_m,T,P_k)$$

Hence

$$R(T,P_k,T,P_m) + R(T,P_m,T,P_k) = 0. \tag{3}$$

But these last two terms are equal, by the general symmetry

$$(R(A,B,C,D) = R(C,D,A,B).$$

Hence, each of the terms in Equation 3 is 0. ♠

Let J be a Jacobi field perpendicular to T. We can write $J = \sum a_k P_k$. We compute

$$\frac{D^2 J}{dt^2} = \sum a_k'' P_k$$

Therefore

$$\left\langle \frac{D^2 J}{dt^2}, P_m \right\rangle = a_m''. \tag{4}$$

At the same time

$$-R(T,J,T,P_m) = a_m \tag{5}$$

by the lemma above. Hence, the Jacobi equation gives $a_m'' = a_m$. we normalize so that $\|J(0)\| = 0$ we find that $a_k(t) = a_k'(0)\sinh(t)$. Therefore $\|J(t)\| = \|J'(0)\|\sinh(t)$ as claimed.