Moebius Maps Preserve Circles

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December 3, 2023

Moebius transformations are maps of the form

\[ z \rightarrow \frac{az + b}{cz + d} \]

where \(a, b, c, d \in \mathbb{C}\) are complex numbers such that \(ad - cb \neq 0\). These form a group \(G\) and act on the Riemann sphere \(\mathbb{C} \cup \infty\) as homeomorphisms. A circle in \(\mathbb{C} \cup \infty\) is either a circle of \(\mathbb{C}\) or else the union of a line with \(\infty\). We call the latter extended lines. In particular, \(C_0 = \mathbb{R} \cup \infty\) is an extended line. In these notes I’ll give a strange proof that Moebius transformations map circles to circles. The proof is based on 4 properties.

1. If \(\gamma\) is a loop in \(\mathbb{C} \cup \infty\) which is not an extended line, then there is some circle \(D\) such that \(\gamma \cap D\) contains at least 3 points.

2. For any circle \(C\), there is some \(T_C \in G\) such that \(T(C_0) = C\).

3. If \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\) are two triples of distinct points on \(C_0\), then \(\exists R \in G\) such that \(R(C_0) = C_0\) and \(R(a_i) = b_i\) for \(i = 1, 2, 3\).

4. \(R \in G\) is determined by where it takes 3 distinct points of \(C_0\).

Main Argument: Let \(M \in G\) and let \(C\) be a circle. Let \(\gamma = M(C)\). If \(\gamma\) is an extended line, we are done. Otherwise let \(D\) be the circle from Property 1. Let \(L = T_D \circ M \circ T_C^{-1}\). By construction, \(L(C_0) \cap C_0\) contains 3 points \(b_1, b_2, b_3\). Let \(a_i = L^{-1}(b_i)\) for \(i = 1, 2, 3\). Let \(R \in G\) be given by Property 2. Then \(R\) and \(L\) agree on \(a_1, a_2, a_3\). But then, by Property 3, \(R = L\), which forces \(L(C_0) = C_0\). but then \(\gamma = D\) and \(\gamma\) is a circle.
Property 1: $\gamma$ has 3 non-collinear points. Every 3 non-collinear points $a, b, c \in \gamma$ lie in the circle $D$ of radius $|x - a|$ centered at $x$, where $x$ is the intersection of the perpendicular bisectors of the segments $\overline{ab}$ and $\overline{bc}$.

Property 2: Using similarities, we reduce to the case when $C$ is the unit circle. The Moebius transformation $T(z) = (z + i)/(z - i)$ evidently maps $C_0$ into $C$, and the upper halfplane outside the unit disk, and the lower halfplane inside the unit disk. Since $T$ is a homeomorphism, we must have $T(C_0) = C$.

Property 3: By the group property, it suffices to consider the case when $(b_1, b_2, b_3) = (0, 1, \infty)$. The map

$$T(z) = -\frac{(a_2 - a_3)(a_1 - z)}{(a_1 - a_2)(a_3 - z)}$$

is a Moebius transformation and has all the properties.

Property 4: Using Property 3, and the group property, it suffices to show that a Moebius transformation is the identity provided that it fixes $(0, 1, \infty)$. Starting with $T(z) = (az + b)/(cz + d)$, and plugging $T(0) = 0$ gives $b = 0$. Plugging in $T(1) = 1$ gives $a = c + d$. Plugging in $T(\infty) = \infty$ gives $c = 0$. We’re left with $T(z) = z$. 