A Geometric View of a Continuous Nowhere Differentiable Function

Richard Evan Schwartz*

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This note gives a geometric proof of the following theorem.

Theorem 0.1 *There exists a continuous and nowhere differentiable function defined on the unit interval.*

This famous result appears in the 1872 paper [**W**] by Karl Weierstrass, and it is often a topic in a first course in real analysis. See e.g. [**A**, §5.4]. At the end of this note we give formulas for Weierstrass's original examples. For a classic treatment of them, see G. H. Hardy's 1916 paper [**H**]. The examples we give are the coordinate functions for the standard parametrization of the Koch snowflake [**K**]. See [**F**] or [**M**] for modern treatments of the snowflake. After giving the short proof, I will venture just a tiny bit into the vast area of math related to Theorem 0.1, and also I'll suggest a few projects for students. I thank Peter Doyle, Dan Margalit, and Valentin Ovsienko for helpful comments about this note, and I thank the National Science Foundation for their support.

Subdivision and Refinement: Let $I_0 = [0,1]$ be the unit interval. Let $\mathcal{I}_n$ be the partition of I_0 into 2^n intervals of length 2^{-n}, with the intervals ordered left to right. We get $\mathcal{I}_{n+1}$ by subdividing each interval of $\mathcal{I}_n$ in half.

The *refinement* of an obtuse isosceles triangle T is the collection of two smaller and similar triangles $T_1, T_2 \subset T$ whose long sides respectively match the two short sides of T. Let T_0 be the triangle having long side length 1 and short side length $1/\sqrt{3}$. We start with $\mathcal{T}_0 = \{T_0\}$ and then inductively get

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$\mathcal{T}_{n+1}$ by refining each triangle of $\mathcal{T}_n$. So, $\mathcal{T}_n$ is a collection of 2^n triangles, all similar to T_0, having long side length $(\sqrt{3})^{-n}$. We order these triangles so that the first triangle is leftmost and consecutive triangles share a vertex. Figure 1 shows this process. Note that a continuous curve seems to emerge.

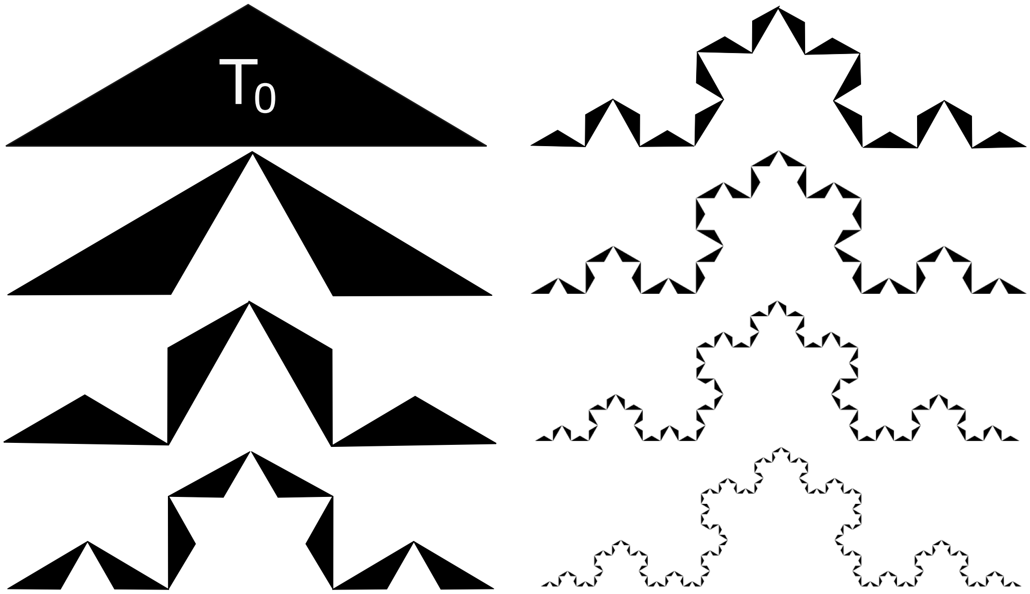

Figure 1: The first 8 steps of the triangle refinement process

The Koch Snowflake Curve: There is an order-preserving bijection between the intervals of $\mathcal{I}_n$ and the triangles of $\mathcal{T}_n$. Two intervals of $\mathcal{I}_n$ share a vertex if and only if the corresponding triangles of $\mathcal{T}_n$ do, and an interval of $\mathcal{I}_{n+1}$ is contained in an interval of $\mathcal{I}_n$ if and only if the corresponding triangle of $\mathcal{T}_{n+1}$ is contained in the corresponding triangle of $\mathcal{T}_n$.

We now define a map $K : I_0 \to \boldsymbol{R}^2$. Every $t \in [0,1]$ can be written as $t = \bigcap I_n$, where I_n is one of the intervals of $\mathcal{I}_n$. This description is unique unless t is a common endpoint of two consecutive partition intervals. In this case, there is a second nested intersection $t = \bigcap J_n$, and t is the common endpoint of I_n and J_n once n is sufficiently large. We define $K(t) = \bigcap T_n$, the corresponding triangle intersection. This defines a unique point because we have a nested intersection of compact sets whose diameters tend to 0. When t is the common vertex of two consecutive partition intervals, the two corresponding triangle intersections define the same point. In short, K is a well-defined map. The image $K(I_0)$ is the *Koch snowflake curve*.

Lemma 0.2 *K is continuous.*

Proof: Let $\epsilon > 0$ be given. Choose n such that $2 \times (\sqrt{3})^{-n} < \epsilon$ and let $\delta = 2^{-n}$. If $s, t \in I_0$ satisfy $|s-t| < \delta$ then either s and t lie in the same interval of $\mathcal{I}_n$ or in consecutive ones. So, $K(s)$ and $K(t)$ lie in the same triangle of $\mathcal{T}_n$ or in consecutive triangles. Either way, $\|K(s) - K(t)\| < \epsilon$. ♠

We write $K(t) = (X(t), Y(t))$. Since K is continuous, so are X and Y.

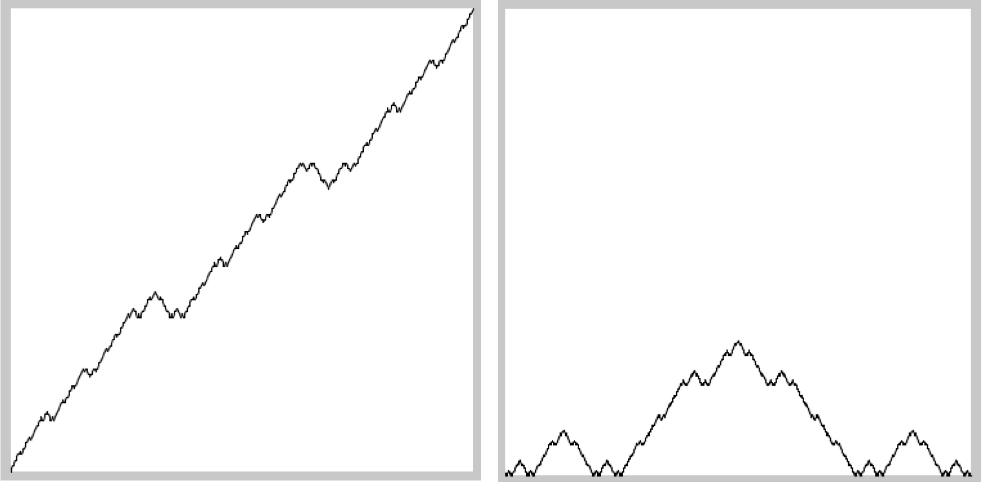

Figure 2: The graphs of the coordinate functions X and Y

Lemma 0.3 *X and Y are nowhere differentiable.*

Proof: Let $|J|$ denote the length of an interval J. Suppose X is differentiable at $t \in I_0$. Then there is a constant C such that $|X(I)| \leq C|I|$ for all sufficiently small intervals I containing t. When such an I belongs to $\mathcal{I}_n$ we have $|X(I)| \leq C|I| = C/2^n$. Hence $K(I) \subset \Sigma$, a vertical strip of width $C/2^n$.

Let T be the triangle of $\mathcal{T}_n$ corresponding to I. We have $S(T) = T_0$, where S is a map of $\boldsymbol{R}^2$, a *similarity*, which expands distances by a factor of $(\sqrt{3})^n$. Because the refinement process is compatible with such maps, we have $K(I_0) = S(K(I))$. Hence $K(I_0) \subset S(\Sigma)$, a strip of width $C(\sqrt{3}/2)^n$. We can make this width as small as we like by increasing n. Hence $K(I_0)$ is contained in an arbitrarily thin strip, a contradiction.

The same proof works for Y, with *horizontal* replacing *vertical*. ♠

Generalizations and Deeper Waters: Geometrically-minded readers may appreciate that we get a contradiction in Lemma 0.3 provided we choose n so that $C(\sqrt{3}/2)^n < \sqrt{3}/12$, half the height of the initial triangle T_0. Analytically-minded readers may appreciate that the "arbitrarily thin strip" conclusion implies that $K(I_0)$ is contained in a single line, an even more stark contradiction. Also, the proof more generally shows that $L \circ K$ is nowhere differentiable when $L : \boldsymbol{R}^2 \to \boldsymbol{R}$ is any linear projection.

We could have started with a different obtuse isosceles triangle. Figure 3 shows a nice choice. In general, these curves are called *Cesàro fractals*.

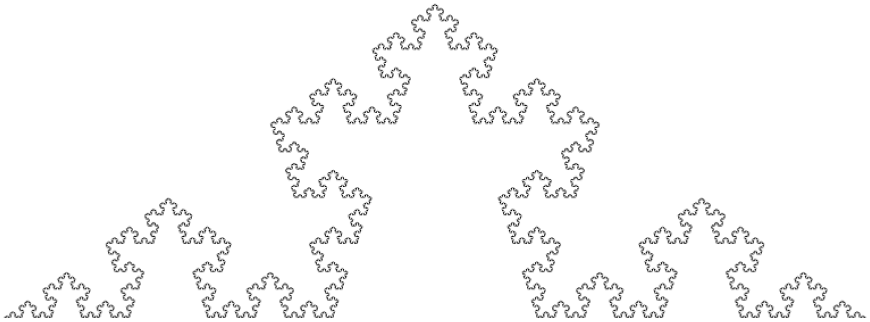

Figure 3: A Cesàro fractal with hints of the regular pentagon

The example in Figure 3 is "bushier" than the example in Figure 1. This is because the triangle shape is less obtuse. The refinement process cuts out less area. The *Hausdorff dimension* [**F**] of these curves coincides with the much simpler *fractal dimension* [**M**] and equals

$$D = \log(2)/\log(r) \in (1,2). \tag{1}$$

Here $r \in (\sqrt{2},2)$ is the long-to-short side length ratio of the triangles. We have $D = \log(4)/\log(3)$ for the Koch example. What about for Figure 3?

Let $\alpha = 1/D \in (1/2,1)$. The proof of Lemma 0.2 shows that K and its coordinate functions are *α-Hölder Continuous*:

$$\|K(s) - K(t)\| \leq C|s-t|^{\alpha} \tag{2}$$

for some constant $C > 0$ and all $s,t \in I_0$. The same kind of equation is meant to hold for X and Y. The smaller α is, the more "locally stretchy" the maps. Differentiability corresponds to $\alpha = 1$. The proof of Lemma 0.3 shows that X and Y are nowhere β-Hölder continuous for any $\beta > \alpha$. The proofs above are quite sharp, and they work for all the Cesàro fractals.

With a little care, one can define the refinement process for any obtuse triangle. See Figure 4.

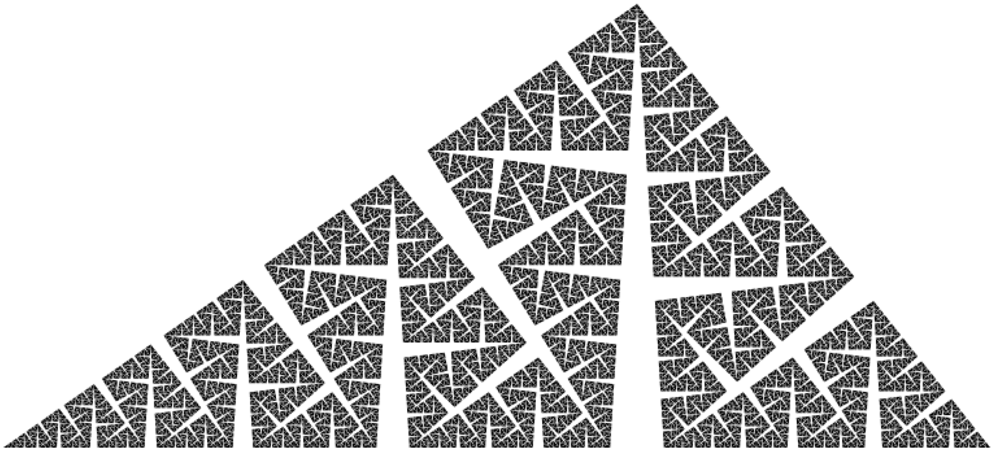

Figure 4: A variant based on a non-isosceles triangle

Proofs very much like the ones above show that in all these cases the corresponding coordinate functions are continuous but nowhere differentiable. You might enjoy working out (or estimating) the Hausdorff dimension and the exponent of Hölder continuity in the non-isosceles case. This seems like a much harder challenge, due to the more complicated scaling rules.

You can further generalize the process so that it works for any triangle with a unique longest side. If you start with a right triangle, the resulting curve fills the entire initial triangle in a satisfying way. When the triangle is acute, the image curve also fills the initial triangle but it is rather messy. Figure 5 shows what you can get when you use acute triangles and plot them with a semi-transparent color. A nice project would be to investigate the interference patterns you see in the acute case.

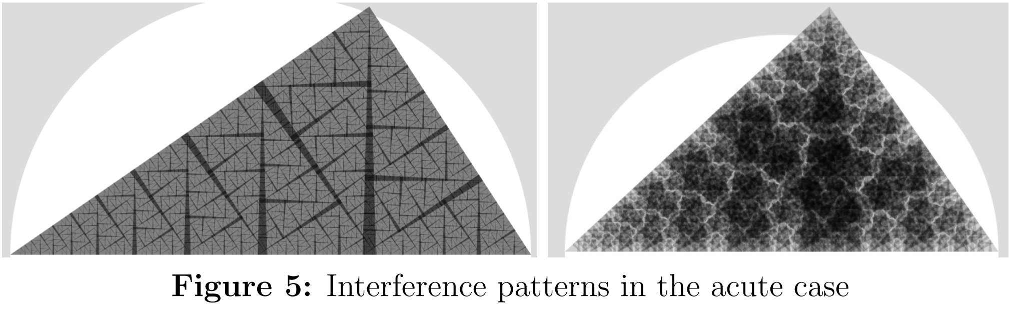

Figure 5: Interference patterns in the acute case

The approach above also works for the Sierpinski triangle. To test your understanding, prove that the Sierpinski triangle is the image of a continuous curve with nowhere differentiable coordinate functions. See Figure 6.

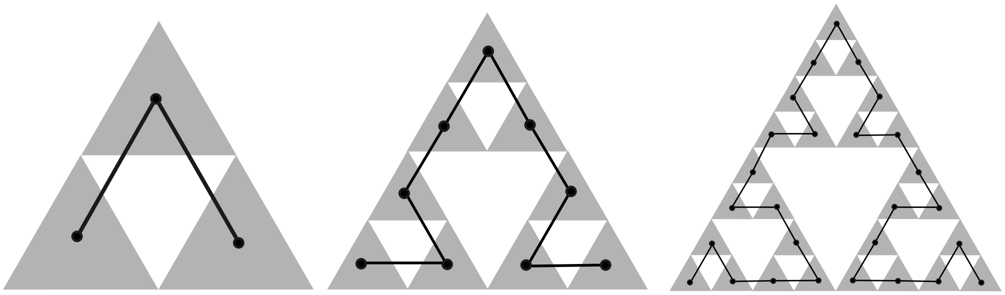

Figure 6: Ordering the triangles in a Sierpinski partition

Figure 7 shows a purely one-dimensional construction based on the idea of recursively refining a directed interval. The construction lies entirely in the unit interval I_0 but we have separated out the intervals to make the construction more clear. The refinement in the picture replaces one directed interval by four overlapping ones of half the size. The underlying binary "seed" is 1101. The same proofs as above show that the resulting function A_{1101} is $\frac{1}{2}$-Hölder continuous but nowhere β-Hölder continuous for $\beta > \frac{1}{2}$.

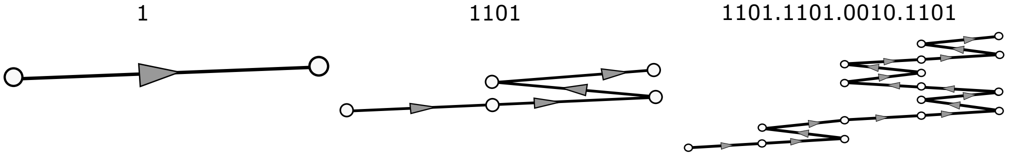

Figure 7: Recursive refinement of directed intervals

Many other seeds also produce continuous nowhere differentiable functions this way. Curiously, $F = (A_{1101}, A_{1011})$ maps I_0 onto a right isosceles triangle, but in a fundamentally different way than the right-isosceles Cesàro curve does. The map (A_{110101}, A_{110011}) also has a solid image. What is it? Does $(A_{110101}, A_{101011}, A_{101101})$ fill a solid region in $\boldsymbol{R}^3$?

To bring this note full circle, let me ask one last question. You can numerically investigate the Fourier series for the various examples above. How do they compare with Weierstrass's original examples, given below?

$$W_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \quad a \in (0,1), \quad b \in 2\boldsymbol{Z}+1, \quad ab > 1 + \frac{3\pi}{2}. \tag{3}$$

References:

[**A**] S. Abbott, *Understanding Analysis*, 2nd ed., Springer Undergraduate Mathematics Series (2018).

[**F**] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 3rd ed., John Wiley & Sons (2014).

[**H**] G. H. Hardy, *Weierstrass's non-differentiable function*, Trans. Amer. Math. Soc., Vol. 17 (1916), pp. 301–325.

[**K**] H. von Koch, *Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire*, Arkiv för Matematik, Astronomi och Fysik, Band 1 (1904), 681-702.

[**M**] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman & Co., New York (1982).

[**W**] K. Weierstrass, *Über continuirliche Functionen eines reellen Arguments, die für keinen Werth desselben einen bestimmten Differentialquotienten besitzen*, *Sitzungsber. Königl. Preuss. Akad. Wiss. Berlin* (1872); reprinted in *Math. Werke*, Vol. 2, Mayer & Müller, Berlin (1895), pp. 71–74.