The Farthest Point Map on the Regular Octahedron

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1 Introduction

1.1 Background

A classic recreational problem in mathematics poses the following kind of question: Given a point on the surface of box, what is the farthest point away in the intrinsic sense? The intrinsic sense means that distances between points on the surface are measured in terms of lengths of paths on the surface of the box and not in terms of the ambient 3-dimensional Euclidean distance. The solution to this problem usually involves unfolding the surface and pressing it into the plane, so that the shortest paths can be studied in terms of ordinary planar geometry.

In this paper we will study the same kind of question for the surface of the regular octahedron, and we will organize the results into a statement about dynamical systems. We begin with some generalities and then specialize to the case of the surface of a regular octahedron. Let $(X,d)$ be a compact metric space. The farthest point map, or farpoint map for short, associates to each point $x \in X$ the set $F_x \subset X$ of points $y \in X$ which maximize the distance function $y \rightarrow d(x,y)$. For instance, on the sphere, $F_x$ is always a single point, the point antipodal to $x$.

Several authors have studied this map extensively for general convex polyhedra. See [R1], [R2], [R3], [V1], [V2], [VZ], and [Z]. One focus has been

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on Steinhaus’s conjecture concerning the ubiquity of points \( x \) such that \( F_x \) is a single point. In a somewhat different direction, but still closely related to this work, several papers [AAH], [DFT] deal with structural questions concerning geodesics on platonic solids. More broadly, there is a large literature concerning polygonal billiards and straight line trajectories on translation surfaces.

From a dynamical systems point of view, it is nicer to have a map from points of \( X \) to points of \( X \) rather than from points of \( X \) to subsets of \( X \). So, let \( X' \subset X \) be the set of points \( x \in X \) such that \( F_x \) is just a single point. We then have a well defined map \( F : X' \to X \), which carries \( x \) to the unique point in \( X \) farthest from \( x \). To get a dynamical system, we define \( X^{(1)} = X' \). Inductively we let \( X^{(n+1)} \) be those points \( x \in X' \) such that \( F(x) \in X^{(n)} \). The full orbit is well defined on

\[
X^{(\infty)} = \bigcap_{n=1}^{\infty} X^{(n)}. \tag{1}
\]

In nice cases, \( X^{(\infty)} \) is large enough to still be interesting.

I learned about the study of the farpoint map on the regular octahedron from Peter Doyle. Originally, the question had been the subject of an undergraduate thesis by Doyle’s student, Annie Laurie Muahs-Pugh. Annie Laurie made partial progress on this problem while a student. I got interested in the problem one time while visiting Doyle, and ended up writing an extensive graphical user interface, called Spider’s Embrace [S3], which revealed essentially all the structure. In the intervening years, my PhD student Zili Wang wrote a thesis and a subsequent paper [W] which took Spider’s Embrace as inspiration. She generalized some of the results to the case of centrally symmetric octahedra having all equal cone angles.

The interested reader can play with Spider’s Embrace and see exactly what is going on, but I thought that it would be good to have rigorous statements and proofs of all the structural results illustrated in Spider’s Embrace. That is what this paper does.

### 1.2 Statement of Results

Henceforth \( X \) denotes the regular octahedron equipped with its intrinsic surface metric. Rather than think about the map \( F \), it is nicer to think about the composition

\[
f = FA = AF, \tag{2}
\]
where \( A : X \to X \) is the antipodal map. As our notation suggests, \( A \) and \( F \) commute. At first it might appear that in fact \( A = F \), so that \( f \) is the identity map, but this is not the case. Note that \( f^2 = F^2 \), so we are not really changing the problem much by studying \( f \) instead of \( F \).

The map \( f \) commutes with every isometry of \( X \), so it suffices to describe the action of \( F \) on a fundamental domain for the action of the isometry group. One sixth of a face of \( X \) serves as such a fundamental domain. After suitably scaling the metric and taking local coordinates, we can take for a fundamental domain the triangle \( T \) having vertices

\[
0, \quad 1, \quad \frac{1 + \omega}{2} = \frac{1}{4} + \frac{i\sqrt{3}}{4}. \quad (3)
\]

Here we are using complex notation. The number \( \omega = \exp(2\pi i/3) \) is a third root of unity. Figure 1.1 shows a picture of \( T \) and some other features which we describe presently. Figure 1.2 below shows how \( T \) (drawn blue) sits inside the face of \( X \) (drawn orange) that contains it.

![Figure 1.1: The domain \( T \), the curve \( J \) and the dynamics on \( T \).](image)

The polynomial \( x^3 - x^2 - 4x + 1 \) has 3 real roots, one being \( r \approx 0.239123 \). The almost-vertical curve \( J \) in Figure 1.1 is the graph of the function

\[
y = \frac{1}{\sqrt{3}} \left( 1 - x - (2 + x)(5 - 2x)(1 - 4x) \right)^{1/3}, \quad (4)
\]

plotted for the interval \( x \in [r, 1/4) \). The cube root we take is the positive branch. For the purpose of stating our result cleanly, we do not consider the top endpoint of \( J \), which is a vertex of \( T \), to be in \( J \).
Theorem 1.1 (Main) If \( p = (x + iy) \in T - J \) then \( F_p \) is a single point. If \( p \in T - J \) lies to the left of \( J \), then

\[
f(p) = \left( \frac{-xy - \sqrt{3}x + \sqrt{3}y^2 - y}{\sqrt{3}x + y - 2\sqrt{3}}, y \right) = \left( \frac{Ay + B_y}{Cy + D_y}, y \right).
\]

(5)

if \( p \in J \) lies to the right of \( J \), then

\[
f(p) = \left( \frac{-xy + 2\sqrt{3}x + \sqrt{3}y^2 - y}{\sqrt{3}x + y + \sqrt{3}}, y \right) = \left( \frac{Dy - By}{-Cy + Ay}, y \right).
\]

(6)

If \( p \) is the top vertex of \( T \) then \( f(p) = p \). If \( p \in J \) then \( F_p \) is the union of the two points given by the formulas above.

The random-looking formulas in the Main Theorem actually define very nice maps from a geometric point of view. Referring to Figure 1.2, the blue triangle is the fundamental domain \( T \) and the orange triangle corresponds to the face of \( X \) containing \( T \). The grey triangle is a reflected copy of the orange one. The map in Equation 5 maps the blue point to the white point (on the same horizontal line) and the map in Equation 6 maps the white point to the blue point. In particular, the two branches of \( f \) in \( T \), when analytically continued to have a common domain, are inverses of each other.

Figure 1.2 Geometric view of the maps.
The restriction of \( f \) to each maximal horizontal line segment \( \lambda \) of \( T - J \) is a linear fractional transformation having a unique fixed point in \( \lambda \). The fixed point, namely \( \lambda \cap \partial T \), is attracting. This fact, together with the rest of the Main Theorem, gives us the following dynamical consequence.

**Corollary 1.2** Let \( \partial_\infty T \) denote the union of the non-horizontal sides of \( T \).

1. \( X' \cap T = X^{(\infty)} \cap T = T - J \).
2. If \( p \in T - J \) then \( f(p) = p \) if and only if \( p \in \partial_\infty(T) \).
3. \( L_\infty(f) \cap T = \partial_\infty T \).

Figure 1.3 shows what the \( \omega \)-limit set \( L_\infty(f) \) looks like inside one of the 8 faces of \( X \). The picture is the same in each face.

There is one more picture we show. Figure 1.4 shows the image of the set \( J \) under 10 iterates of the dynamics. This picture illustrates how the dynamics moves points near \( J \) out to the boundary of \( T \). Let \( J_\ell \) and \( J_r \) be two copies of \( J \) which, so to speak, lie infinitesimally to the left and the right of \( J \). We will iterate the left branch of \( f \) on \( J_\ell \) and the right branch on \( J_r \). We have shaded in the regions between \( f^k(J_\ell) \) and \( f^k(J_r) \) for \( k = 1, ..., 10 \). One thing that this picture reveals is that points near the top vertex of \( T \) move, relatively speaking, very slowly out towards the edges under the map.
Here is the plan of the paper In §2 we give the proof modulo certain technical details which we take care of in subsequent chapters.

1.3 Acknowledgements

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2 The Proof in Broad Strokes

2.1 The Octahedral Plan

As in the introduction, $X$ denotes the regular octahedron equipped with its intrinsic metric. $X$ is locally Euclidean except for 6 cone points, each having cone angle $4\pi/3$. The space $X$ has 8 faces, each an equilateral triangle. Let $T$ be the fundamental domain discussed in the introduction. Let $\Delta_0$ denote the face of $X$ that contains $T$. We identify $\Delta_0$ with the equilateral triangle in the plane whose vertices are the cube roots of unity.

Figure 2.1 shows a union $P$ of triangles in the plane. The darkly shaded triangle is $T$ and the lightly shaded triangle that contains $T$ is $\Delta_0$. There is a locally isometric map

$$\Psi : P \to X$$

which simply wraps $P$ around $X$. We imagine $X$ sitting in the plane so that its bottom face precisely coincides with $\Delta_0$. We then cut $P$ out of the plane and fold $P$ around $X$ as if we were wrapping a gift. We have numbered the triangles to indicate their images under $\Psi$. Note that $\Psi$ maps the 6 outer faces labeled 7 to the face $\Delta_7$ on $X$ opposite $\Delta_0$. Each of the shaded triangles (labeled 7) also gets an auxiliary label ($k$) for $k = 0, ..., 5$. These auxiliary labels will be explained below. The thickly outlined triangle in Figure 2.1 plays a special role in our proof below.

![Figure 2.1: The octahedral plan $P$.](image-url)
The 6 cone points of $X$ are grouped into 3 pairs of antipodal points. We use 3 colors to color these pairs: black, white, and grey. The vertices of the octahedral plan are colored according to this scheme. Thus, $\Psi$ maps all the white vertices to the union of the two white cone points of $X$, and likewise for the other colors.

Say that a distinguished line segment in $\mathbb{R}^2$ is one which has one endpoint in $\Delta_0$ and remains entirely inside the octahedral plan $P$. If $\gamma$ is a distinguished line segment, then $\Psi(\gamma)$ is a geodesic on $X$ connecting a point of $\Delta_0$ to some other point on $X$. We call $\Psi(\gamma)$ a distinguished geodesic segment on $X$.

In §3.2 we prove the following result.

**Lemma 2.1 (Octahedral Plan)** Let $p, q \in X$ be two points, with $p \in \Delta_0$. Every length minimizing geodesic segment connecting $p$ to $q$ is distinguished.

The Octahedral Plan Lemma allows us to deduce properties of $F_p$ just by looking at the picture in the plan $P$.

**Corollary 2.2** If $p$ is a cone point of $X$ then $F_p$ is the antipodal cone point.

**Proof:** It follows from [V1, Prop. 6] that if $p, q \in X$ maximize $d(p, q)$ on $X$ then both $p$ and $q$ are cone points. By the Octahedral Plan Lemma, we can figure out all possible distances between cone points by inspecting Figure 2.1. Doing this inspection we see that the maximum is uniquely realized for antipodal cone points, and the distance in question is 3. The corresponding distinguished segments correspond to the long diagonals of the rhombs made from two equilateral triangles joined across an edge. If $p$ is a cone point of $X$, then all points of $X$ are within 3 units of $p$ and only the antipodal cone point is exactly 3 units away. ♠

In §3.3 we prove the following result.

**Lemma 2.3** If $p$ is not a cone point then $F_p$ does not contain a cone point.

The last two results tell us that for the farpoint map, the cone points simply do not interact with the rest of $X$. The farpoint map carries each cone point to its antipodal point and each regular point to some collection of regular points. For now on, we will ignore the cone points.
2.2 Voronoi Decomposition

Recall that we are identifying the fundamental domain $T$ with a subset of a face of $X$ and also as a triangle in the plane. Let $T^o$ be the interior of $T$. Let $A_k$ be the triangle labeled 7 and $(k)$ in Figure 2.1. The triangle $A_0$ is thickly shaded in Figure 2.1 and, in Figure 2.2, multi-grey-shaded. There are isometries $I_0, ..., I_5$ which preserve the vertex coloring, which have the property that $I_j(A_j) = A_0$, and which satisfy the equation $\Psi \circ I_j = \Psi$ on $A_j$. Here $I_0$ is the identity. At the same time, $\Psi = \Psi \circ I_j^{-1}$ on $A_0$.

We define

$$p_j = I_j(p), \quad j = 0, ..., 5.$$  \hspace{1cm} (8)

Figure 2.2: The Voronoi cells intersected with $A_0$. 

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Let \( \hat{H}_p \) be the (solid) hexagon with vertices \( p_0, \ldots, p_5 \). In §4.3 we prove

**Lemma 2.4** \( \hat{H}_p \) is a strictly convex hexagon when \( p \in T^o \).

Given \( q \in \hat{H}_p \) let

\[
\mu_p(q) = \min_{j \in \{0, \ldots, 5\}} |q - p_j|.
\]  (9)

We say that a minimal index for \( q \) is an index \( k \) such that \( \mu_p(q) = |q - p_k| \).

The \( k \)th Voronoi cell \( C_k \) is the set of points \( q \in \hat{H}_p \) having \( k \) as one of their minimal indices. That is, \( \mu_p(q) = |q - p_k| \). The list \( C_0, \ldots, C_k \) is the Voronoi decomposition of \( \hat{H}_p \). The Voronoi cells are convex polygons. Their edges are contained in the union of bisectors defined by pairs of vertices of \( \hat{H}_p \). See Figure 2.2. The yellow vertices in Figure 2.2 highlight some nice extra structure of the bisectors which we note but do not explicitly use.

Say that an interior vertex of \( \hat{V}_p \) is a vertex of the Voronoi cell that does not lie in the boundary of \( \hat{H}_p \).

**Lemma 2.5 (Vertex)** All points of \( F_p \) have the form \( \Psi \circ I_j^{-1}(q) \) where \( q \in \hat{H}_p \) is an interior vertex which maximizes \( \mu_p \) and \( j \) is some index.

**Proof:** No point in the triangles labeled 0, 1, 2, 3 in the octahedral plan can belong to \( \Psi^{-1}(F_p) \) for any \( p \in T \). They are clearly too close. At the same time, every point of the union of triangles labeled 4, 5, 6, 7 in the octahedral plan is isometric, via some map \( I_j \), to a point in the hexagon \( \hat{H}_p \). So, all points of \( F_p \) have the form form \( \Psi \circ I_j^{-1}(q) \) for some \( q \in \hat{H}_p \). We also observe that

\[
|q - p_j| = |I_j^{-1}(q) - p|.
\]

Hence, the minimum distance between \( p \) and points in \( \Psi^{-1}(F_p) \) is exactly \( \mu_p(q) \). But this minimum distance is exactly the distance between \( p \) and points of \( F_p \). Hence all points of \( F_p \) have the form \( \Psi \circ I_j(q) \) where \( q \in \hat{H}_p \) maximizes \( \mu_p \).

If \( q \in \hat{H}_p \) does not belong to at least 3 Voronoi cells then we can find a nearby point \( q' \) with \( \mu_p(q') > \mu_p(q) \). From this we conclude that our \( \mu_p \) maximizers must be interior vertices. ♣

**Remark:** For the reader who knows about the cut locus, we remark that the Vertex Lemma can also be deduced from \([R2, Lemma 3]\).
2.3 Structural Stability and Vertex Competition

Assume for ease of exposition that \( p \in T^o \). Looking at Figure 2.2 we see that there are 4 interior vertices, all of which lie in the interior of \( A_0 \). These are the points

\[
(012), \quad (025), \quad (235), \quad (345). \tag{10}
\]

Here \((ijk)\) is the vertex \( q \) whose minimal indices are \( \{i, j, k\} \). In §4 we prove algebraically that this combinatorial pattern persists for all \( p \in T^o \). In particular, when \( p \in T^o \) all the interior vertices lie in the interior of \( A_0 \).

Since \( \Psi = \Psi \circ I^{-1}_j \) on \( A_0 \), the Vertex Lemma says that \( F_p \) consists of those points \( \Psi(q) \) where \( q \) is an interior vertex maximizing \( \mu_p \). Moreover, \( q \) must be amongst the 4 points in Equation 10.

In §5.1 we prove the positivity of a certain two polynomials \( F_1 \) and \( F_2 \) and thereby establish the equations

\[
|p_0 - (025)|^2 - |p_0 - (12)|^2 > 0, \quad |p_3 - (235)|^2 - |p_3 - (345)|^2 > 0. \tag{11}
\]

(Squaring makes the algebra nicer.) This is meant to hold for all \( p \in T^o \).

These equations say that \((012)\) and \((345)\) do not maximize \( \mu_p \). Hence

\[
F_p \subset \Psi(025) \cup \Psi(235), \quad \forall p \in T^o. \tag{12}
\]

Finally, we find the polynomial \( F_3 \) that vanishes if and only if

\[
|p_2 - (025)|^2 - |p_2 - (235)|^2 = 0
\]

and we show that \( F_3 \) vanishes precisely on the curve \( J \) from the Main Theorem when restricted to \( T^o \). Keeping track of the signs of \( F_3 \), we deduce that

- \( F_p = \Psi(025) \) when \( p \) lies to the left of the curve \( J \).
- \( F_p = \Psi(235) \) when \( p \) lies to the right of \( J \).
- \( F_p = \Psi(025) \cup \Psi(235) \) when \( p \in J \).

We then explicitly derive Equations 5 and 6 from these facts. We give details of the calculation in §5.2. All the statements in the Main Theorem, at least for points in \( T^o \), follow from these two equations.

To complete the proof of the Main Theorem, it only remains to treat the cases when \( p \in \partial T \), the boundary of the fundamental domain. Our structural stability result no longer holds, but in §5.2 we will see these boundary cases as limits of the interior cases.
3 The Octahedral Plan

3.1 The Developing Map

In this chapter we prove the Octahedral Plan Lemma and also Lemma 2.3. We first discuss a well-known circle of ideas surrounding flat cone surfaces. Let $T$ denote the equilateral tiling of the plane which contains the tiles of the octahedral plan.

Let $p \in \Delta_0$ and let $q \in X$ be some other point. Let $\alpha \subset X$ be a length minimizing geodesic connecting $p$ to $q$. Because the cone points of $X$ have positive curvature, $\alpha$ does not contain a cone point in its interior. Otherwise, we could take a short cut around the side of the cone point and produce a shorter path connecting $p$ to $q$.

One possibility for $\alpha$ is that it lies in a single edge of one of the faces of $X$. We consider this case completely understood and henceforth in this section we assume that $\alpha$ does not have this property.

Assuming that $\alpha$ is not contained in a single face edge, there is a unique sequence $\Delta_0, \Delta_{i_1}, \Delta_{i_2}, \ldots$ of tiles containing successive segments of $\alpha$. We call this the triangle sequence. Corresponding to the triangle sequence, we have the following objects:

- A line segment $\alpha' \subset \mathbb{R}^2$ having the same length as $\alpha$ and connecting $p$ to some other point $q' \in \mathbb{R}^2$.
- A sequence $\Delta'_0, \Delta'_{i_1}, \Delta'_{i_2}, \ldots$ of equilateral triangles of $T$ which contain the corresponding segments of $\alpha'$.
- A locally isometric map $\psi$ from $\bigcup \Delta'_{i_j}$ to $\bigcup \Delta_{i_j}$ which maps each $\Delta'_{i_j}$ isometrically to $\Delta_{i_j}$ and which maps $\alpha'$ isometrically to $\alpha$. In case the $\Delta'$ triangles lie in the octahedral plan, $\psi$ agrees with the map $\Psi$ defined above.

We get the triangles in the plane by developing $X$ out according to the triangle sequence. The 0th triangle in the sequence is $\Delta_0$, which we have identified already with a particular triangle in the plane. We set $\Delta'_0 = \Delta_0$ and we then roll $X$ from $\Delta'_0$ onto $\Delta'_{i_1}$ in such a way that $\Delta_{i_1}$ sits precisely on top of $\Delta'_{i_1}$. And so on. This procedure is commonly called the developing map.
3.2 Proof of the Octahedral Plan Lemma

Let \( p \in T \) and \( q \in X \). We show that a length minimizing geodesic segment \( \alpha \) connecting \( p \) to \( q \) must be distinguished. Let \( i_0, i_1, i_2, \ldots \) be the associated triangle sequence. We decompose \( X \) into layers according to Figure 2.1:

\[
\begin{align*}
0. & \quad \Delta_0 \\
1. & \quad \Delta_1 \cup \Delta_2 \cup \Delta_3 \\
2. & \quad \Delta_4 \cup \Delta_5 \cup \Delta_6 \\
3. & \quad \Delta_7
\end{align*}
\]  

(13)

Corresponding to the triangle sequence, there is a sequence \( j_0, j_1, j_2, \ldots \) of integers such that \( \Delta_{i_k} \) is a triangle in layer \( j_k \). We call this the layer sequence. If the layer sequence is any of 0, 1 or 0, 1, 2 or 0, 1, 2, 3 then the corresponding \( \Delta' \) triangles lie in the octahedral plan and \( \alpha \) is distinguished. We just have to rule out other possible layer sequences. For the pictures we draw below, the unshaded triangles belong to the octahedral plan and the shaded triangles do not. Also, \( \Delta'_k \) will always denote a triangle in the plan that is labeled \( k \). There are 6 cases, according to the length of the layer sequence.

Cases 1 and 2: The layer sequences must be 0 and 0, 1 in these cases.

Case 3: Suppose that the triangle sequence has length 3. A geodesic segment cannot exit a triangle and return to it immediately thereafter, two triangles in the same layer are not adjacent, and a triangle in Layer 3 is not adjacent to a triangle in Layer 1. These considerations rule out the layer sequences 0, 1, \( k \) for \( k = 0, 1, 3 \). Hence, the layer sequence is 0, 1, 2.

Case 4: Suppose the triangle sequence has length 4. From Case 3, the only possibilities are 0, 1, 2, 3 or 0, 1, 2, 1. We just have to rule out 0, 1, 2, 1. In this case we can, by symmetry, assume that the triangle sequence is 0, 2, 5, 1. Figure 3.1 shows the corresponding triangles in the plane. The triangles \( \Delta'_1, \Delta'_2, \Delta'_5 \) all share a common vertex \( v \), colored yellow in Figure 3.1.

\[\begin{align*}
\text{Figure 3.1: Comparing two points.}
\end{align*}\]
Moreover, \( \psi(q') = \Psi(q'') \), where \( q'' \in \Delta'' \) is the point such that \( q' \) and \( q'' \) are vertices of an equilateral triangle \( \tau \) centered at \( v \). The triangle \( \tau \) is colored red in Figure 3.1. Because the blue line in Figure 3.1 separates \( p \) from the third vertex of \( \tau \), we have \( \|p - q''\| < \|p - q'\| \). (Equality occurs only when both \( q \) and this vertex are on the blue line, but in this case the triangle sequence is 0, 2, 5.) Notice that there is a shorter geodesic segment in \( X \) connecting \( p \) to \( q \), namely \( \Psi(pq'') \). This is a contradiction.

**Case 5:** Suppose that the triangle sequence has length 5. Then the layer sequence must be 0, 1, 2, 3, 2. By symmetry, we can assume that the triangle sequence is either 0, 1, 5, 7, 4 or 0, 1, 5, 7, 6. We treat these cases in turn.

**Case 5A:** Our argument refers to Figure 3.2. In the first case, there is a vertex \( w \) common to \( \Delta_4', \Delta_5', \Delta_7' \), and a corresponding equilateral triangle \( \tau v \) centered at \( w \) such that \( q' \in \Delta_4' \) and \( q'' \in \Delta_4'' \) are both vertices of \( \tau \).

![Figure 3.2](image)

**Figure 3.2:** The triangle sequence 0, 1, 5, 7, 4.

Here, again, \( \psi(q') = \Psi(q'') \). Once again we have \( \|p - q''\| < \|p - q'\| \) because the blue line in Figure 3.2 separates the third vertex of \( \tau \) from \( p \).

**Case 5B:** Our argument refers to Figure 3.3 below. Now we have the triangle sequence 0, 1, 5, 7, 6. The yellow vertex \( x \) in Figure 3.3 is a vertex of \( \Delta_6'' \). Let \( \rho \) be the rotation by \( 2\pi/3 \) counterclockwise about \( x \). We can see that \( \rho(\Delta_6') = \Delta_6'' \). Because \( \rho \) is an isometry of \( T \) which preserves the vertex coloring, we have that \( \Psi \circ \rho = \psi \) on \( \Delta_6' \). Setting \( q'' = \rho(q') \) we have \( \psi(q') = \Psi(q'') \). The points \( q' \) and \( q'' \) lie on an equilateral triangle \( \tau \) centered at \( x \). The blue line, the altitude of \( \tau \) through the third vertex of \( \tau \), is disjoint...
from the interior of $\Delta_0$. For this reason, $\|p - q''\| < \|p - q'\|$ and we get the same contradiction as above.

![Figure 3.3: The triangle sequence 0, 1, 5, 7, 6.](image-url)

**Case 6+:** Any triangle sequence of length at least 6 would contain a triangle sequence of length 5. We have just ruled this out. Hence, the only triangle sequences associated to length minimizing geodesic segments starting in $\Delta_0$ are the ones which correspond to the octahedral plan. Hence, $\alpha$ is distinguished.

This completes the proof of the Octahedral Plan Lemma.

### 3.3 Dealing with the Cone Points

In this section we prove Lemma 2.3: If $p$ is not a cone point of $X$ then $F_p$ does not contain a cone point of $X$. It suffices to assume that $p \in T$ the fundamental domain. The cone points of $X$ are represented by the vertices of the triangles in the octahedral plan. The 3 vertices of $\Delta_0$, certainly do not belong to $F_p$. The remaining vertices belong to the triangles labeled 7. Call these the *candidate* vertices.
When $p$ lies in the interior of $T$, the candidate vertices shown at left in Figure 3.4 are the unique ones closest to $p$. Call these *preferred*. The bisector defined by a preferred vertex and some other candidate vertex of the same color is such that $p$ lies on the same side of the bisector as the preferred vertex. This justifies our claim about the preferred vertices.

The colored segments indicate how we can vary a given preferred vertex $q$ away from $p$ in such a way that the new point $q'$ is strictly farther from $p$. By continuity, none of the other representatives of $q'$ is closer. Hence $\Psi(q) \not\in F_p$.

The argument works the same way when $p$ lies in an edge of $T$, except when $p$ lies on the edge between triangles 0 and 1. In this case, there are 2 preferred grey vertices and two preferred black vertices. Here we proceed as above, using two simultaneous variations. This gives us a pair of new points which represent the same point in $X$ and which are farther from $p$. ♠
4 Structural Stability

4.1 Overview

The goal of this chapter is to prove the structural stability of the Voronoi decomposition, as discussed in §2.3. Let \( \hat{H}_p \) and \( \hat{V}_p \) be as in §2.2. We will show that as \( p \) varies in \( T^o \), the interior of the fundamental domain, the combinatorial structure of \( \hat{V}_p \) does not change and the intersection of \( \hat{V}_p \) with \( A_0 \) does not change.

4.2 A Polynomial Map

We will be considering a number of functions on \( T^o \), the interior of the fundamental domain. It is more convenient sometimes to deal with functions on the open unit square \((0,1)^2\). So, we explain a convenient map from \((0,1)^2\) to \( T^o \). Before defining the rather complicated formula we explain the simple idea behind it. The map \((x,y) \rightarrow (x,xy)\) maps the square \([0,1]^2\) to the triangle with vertices \((0,0)\), \((1,0)\) and \((1,1)\). Our map is the composition of this map with a suitable linear transformation. We define

\[
(x, y) = \phi(a, b) = \left( a + \frac{1}{4}(1-a)b, \frac{\sqrt{3}}{4}(1-a)b \right). \tag{14}
\]

Given a polynomial \( \psi(x,y) \), we just plug in the formula for \( x \) and \( y \) above and we get a function on \((0,1)^2\). The map \( \phi \) is a homeomorphism which is also a polynomial map.

4.3 Formulas for the Vertices

Here we use Equation 14 to parametrize the vertices of \( \hat{H}_p \) by \( (a,b) \in (0,1)^2 \). Using Equation 14, we have

\[
p_0 = \left( a + \frac{b}{4} - \frac{ab}{4} \right) + i \left( \frac{\sqrt{3}}{4}b(1-a) \right). \tag{15}
\]

Taking \( \sqrt{-3} \) to be the root with positive imaginary part, we define

\[
T(i_1,j_1,i_2,j_2;z) = w(i_1,j_1)z + w(i_2,j_2), \quad w(i,j) = \frac{i + j\sqrt{-3}}{2}. \tag{16}
\]

We have
1. \( p_1 = T(-1, +1, +3, -1; p) \).
2. \( p_2 = T(-1, -1, +9, +1; p) \).
3. \( p_3 = T(+2, +0, +9, +3; p) \).
4. \( p_4 = T(-1, +1, +3, +5; p) \).
5. \( p_5 = T(-1, -1, +0, +4; p) \).

The spurious (+) signs are introduced to make the typesetting nice.

**Proof of Lemma 2.4:** We want to prove that \( \hat{H}_p \) is a strictly convex hexagon when \( p \in T^o \), the interior of \( T \). Our argument refers to Figure 2.2. Note that for \( j = 2, 3, 4, 5 \) the blue triangle containing \( p_j \) lies entirely to one side of any line connecting points in adjacent blue triangles. This shows that \( \hat{H}_p \) is locally strictly convex at \( p_j \) for \( j = 2, 3, 4, 5 \). For \( j = 1 \) we note that the line \( p_1 p_2 \) has positive slope and hence \( p_0 \) always lies above this line. This establishes that \( \hat{H}_p \) is locally strictly convex at \( p_0 \). To treat the local convexity at \( p_0 \) we resort to an explicit calculation using the formulas above:

\[
2 \text{ area}(p_5, p_0, p_1) = \text{Im}[(p_0 - p_5)(p_0 - p_1)] = \\
\frac{3\sqrt{3}}{8} (1 - a)(8 - 4a - 6b + 2ab + b^2 - ab^2).
\]

This polynomial is positive on \((0, 1)^2\), which means that \( p_5, p_0, p_1 \) are never collinear. Since \( \hat{H}_p \) is strictly locally convex at \( p_0 \) for the parameter choice plotted in Figure 2.2, it remains so for all parameter choices. ♠

### 4.4 Nice Polynomials

Call a polynomial \( u(a, b) \) **nice** if it does not vanish on \((0, 1)^2\). Our proof relies on the list below of 26 polynomials. We let \( \mathcal{U} \) be the set of polynomials which factor into polynomials on this list. Every polynomial in \( \mathcal{U} \) is nice provided that the the 26 listed polynomials are nice.

Now we explain a general positivity criterion which we call **positive dominance**. If \( v(a, b) = \sum c_{i,j} a^i b^j \) we define the partial sums

\[
C_{ij} = \sum_{\alpha \leq i} \sum_{\beta \leq j} c_{\alpha, \beta}.
\]
We show in [S1], [S2] that if $C_{i,j} > 0$ for all $i, j$ then $v > 0$ on $(0, 1)^2$. All the polynomials on the list below satisfy the positive dominance criterion and hence are positive on $(0, 1)^2$. Hence they are nice.

1. $1 - a$
2. $a$
3. $1 + a$
4. $1 - 2a + a^2$
5. $4 - b$
6. $1 - b$
7. $b$
8. $8 + 4a + b - ab$
9. $10 - 4a - b + ab$
10. $2a + b - ab$
11. $5 + a - b - ab$
12. $2 + a - b - ab$
13. $4 - a - b + ab$
14. $8 - 2a + b - ab$
15. $4 + 2a - b + ab$
16. $4 - 4a + 4b + 2ab + b^2 - ab^2$
17. $8 + 4a + 2b - 2ab - b^2 + ab^2$
18. $16 - 4a^2 - 8b + 6ab + 2a^2b + b^2 - a^2b^2$
19. $8a + 4a^2 + 6b + 8ab - 2a^2b + 3b^2 - 4ab^2 + a^2b^2$
20. $4 + 16a + 4a^2 + 4b - 2ab - 2a^2b + b^2 - 2ab^2 + a^2b^2$
21. $28 - 20a + 4a^2 - 2b + 4ab - 2a^2b + b^2 - 2ab^2 + a^2b^2$
22. $8 + 8a - 4a^2 + 2b - 4ab + 2a^2b - b^2 + 2ab^2 - a^2b^2$
23. $28 + 16a + 4a^2 - 2b + 4ab - 2a^2b + b^2 - 2ab^2 + a^2b^2$
24. $4a - 4a^2 + 6b + 4ab + 2a^2b + 3b^2 - 2ab^2 - a^2b^2$
25. $24a + 4b - 8ab + 4a^2b + 4b^2 - 2ab^2 - 2a^2b^2 + b^3 - 2ab^3 + a^2b^3$
26. $24a + 8b - 16ab - 4a^2b + 2b^2 - 4ab^2 + 2a^2b^2 - b^3 + 2ab^3 - a^2b^3$

Remark: One can treat all these polynomials in an elementary way. Polynomials 1-15 are positive on $(0, 1)^2$ simply because $ab < \min(a, b)$ and $a^2 < a$. Polynomial 16 can be written as $4(1 - a) + 4b + 2ab + b^2(1 - a)$, a sum of positive terms. The remaining polynomials have similar treatments but we leave this to the interested reader. As a nifty experimental confirmation of the positivity, one can do a contour plot of the minimum of the 26 functions in Mathematica and inspect the result.
4.5 Stability of the Voronoi Decomposition

Referring to Figure 2.3, the triple points in the Voronoi decomposition of \( \hat{H}_p \) correspond to the triples 012, 025, 235, and 345. Thus, for instance, one of the triple points is equidistant from \( p_0, p_1, p_2 \), and these are the closest vertices of \( \hat{H}_p \) to the given point. One finds the edges of the Voronoi decomposition by taking 2-element subsets of these triples.

Say that a quad is a quadruple of indices \( ijk\ell \) such that that some point is equidistant from \( p_i, p_j, p_k, p_\ell \). In this case, the 4 points are cocircular. This is defined relative to the initial choice of \( p = p_0 \).

The combinatorial type of the Voronoi decomposition can change only if there is some choice of \( p \) in the interior of \( T \) for which some quad exists and is obtained by adding some index to one of the triples above. For each of the 4 triples \( ijk \) above there are 3 quads we need to consider, namely \( ijk\ell \) for \( \ell \notin \{i, j, k\} \). This gives us a total of 12 quads, but there is some redundancy. For instance 0125 = 0251. When we put the quads in increasing order and weed out redundancy we get 9 quads:

\[
0123, \quad 0124, \quad 0125, \quad 0235, \quad 0245, \quad 0345, \quad 1235, \quad 1345, \quad 2345.
\]

If no such quad arises for and choice of \( p \in T^o \) then Item 1 above holds.

One could probably settle this geometrically, but we will give an algebraic proof. We first describe the isometries the points \( p_1, \ldots, p_5 \) as functions of \( p_0 \).

Given 4 distinct points \( z_1, z_2, z_3, z_4 \), consider the quantity

\[
\chi(z_1, z_2, z_3, z_4) = \Im((z_1 - z_2)(z_3 - z_4)(z_1 - z_3)(z_2 - z_4))
\]

This quantity is a positive real multiple of the imaginary part of the cross ratio of the 4 points, and hence vanishes if and only if these points are cocircular.

We write \( p_0 = x + iy \), then define the 9 functions

\[
T_{ijkl} = \chi(p_i, p_j, p_k, p_\ell) \circ \phi,
\]

Where \( \phi \) is as in Equation 14. This gives functions in the variables \((a, b)\). Using Mathematica [Wo] we check that all the polynomials belong to \( \mathcal{U} \) and hence do not vanish on \((0, 1)^2\). This proves that the Voronoi decomposition is stable.

Here record of which polynomials on our list of 26 appear in our factorization of the 9 polynomials above.
We also remark on some boundary behavior. This behavior is easily seen from the polynomials involved in the factorization. It will come in useful when we analyze the boundary cases of our result.

- When $b = 0$ and $a \in (0, 1)$, functions 3 and 9 vanish identically and the rest never vanish. Function 3 corresponds to (0125) and function 9 corresponds to (2345).

- When $b = 1$ and $a \in (0, 1)$, function 4 vanishes identically and the rest never vanish. Function 4 corresponds to (0235).

- When $b \in (0, 1)$ and $a = 0$ function 4 vanishes identically and the rest never vanish. Again, function 4 corresponds to (0235).

### 4.6 Stability of the Intersection

Now we explain why the intersection of the Voronoi decomposition with the triangle $A_0$ is stable. For the choice of $p$ we have plotted, we see that all the triple points lie inside the interior of $A_0$. Our intersection remains stable as long as the triple points remain this way. In other words, we just have to check that there is no choice of $p \in T^o$ which leads to one or more triple point lying on the boundary of $A_0$.

This check can be done with geometric arguments, but they are rather elaborate. We give algebraic arguments because they work the same way
in all cases, and because for later calculations we want formulas for the triple points. We find these formulas by computing the various bisectors for our points \( p_0, \ldots, p_5 \) and then finding the intersection of these bisectors. For instance, \((012)\) is the intersection of the bisector \((01)\) with the bisector \((02)\).

Here are the formulas.

\[
t_1 = (012) = \frac{3\sqrt{3}x^2-6yx-11\sqrt{3}x+21y+5y^2\sqrt{3}+8\sqrt{3}}{2(\sqrt{3}x^2-3\sqrt{3}x+3y+y^2\sqrt{3}+2\sqrt{3})} + i\frac{3x^2-2\sqrt{3}yx-15x-3y^2-\sqrt{3}y+12}{2(\sqrt{3}x^2-3\sqrt{3}x+3y+y^2\sqrt{3}+2\sqrt{3})}
\]

\[
t_2 = (025) = \frac{2\sqrt{3}y^2+2xy-3y+3x\sqrt{3}-8\sqrt{3}}{2(y+x\sqrt{3}-2\sqrt{3})} + i\frac{2y^2-2\sqrt{3}xy+3\sqrt{3}y+3x-12}{2(y+x\sqrt{3}-2\sqrt{3})}
\]

\[
t_3 = (235) = \frac{\sqrt{3}y^2+xy+3y+3x\sqrt{3}+2\sqrt{3}}{y+x\sqrt{3}+\sqrt{3}} + i\frac{y^2-\sqrt{3}xy+6x+3}{y+x\sqrt{3}+\sqrt{3}}
\]

\[
t_4 = (345) = \frac{3\sqrt{3}x^2+8\sqrt{3}x-3y+y^2\sqrt{3}+4\sqrt{3}}{\sqrt{3}x^2+3\sqrt{3}x-3y+y^2\sqrt{3}+2\sqrt{3}} + i\frac{6x^2+2\sqrt{3}x+15x+6y^2-2\sqrt{3}y+6}{\sqrt{3}x^2+3\sqrt{3}x-3y+y^2\sqrt{3}+2\sqrt{3}}
\]

The vertices of \( A_0 \) are

\[
v_1 = 1 + i\sqrt{3}, \quad v_2 = \frac{5}{2} + \frac{3i\sqrt{3}}{2}, \quad v_2 = \frac{5}{2} + \frac{i\sqrt{3}}{2}
\]

Here \( i \) stands for \( \sqrt{-1} \).

For any complex numbers \( z_1, z_2, z_3 \), the expression

\[
\psi(z_1, z_2, z_3) = \text{Im}((z_1 - z_2)(\bar{z}_1 - \bar{z}_3))
\]

vanishes if and only if the three points are collinear. We consider the 12 functions

\[
\psi(v_i, v_{i+1}, t_j), \quad i = 1, 2, 3, \quad j = 1, 2, 3, 4,
\]

with indices taken mod 4 in the first argument. We make the change of variables in Equation 14, and then use Mathematica \([\text{Wo}]\) to factor the functions. When we divide the functions we get by \( \sqrt{3}/2 \) we get the following \( 3 \times 4 \) array of functions.

\[
\begin{pmatrix}
ba + 2a - b + 4 & ba - 4a - b + 10 & 2(-ba + a + b - 4) \\
ba - 4a - b + 10 & 2(-ba + a + b + 2) & ba + 2a - b - 8 \\
2(-ba + a + b + 5) & ba + 2a - b - 8 & (a - 1)(b - 4) \\
ba + 2a - b + 4 & ba - 4a - b - 8 & 2(-ba + a + b + 5)
\end{pmatrix}
\]

These functions all belong to \( \mathcal{U} \).
5 The Vertex Competition

5.1 Main Calculation

We keep the notation from the previous chapter. In this first section we make 3 calculations concerning the triple points of the Voronoi decomposition studied in the last chapter. Once again, we use Mathematica [Wo] for these calculations.

Lemma 5.1 The following is true:

1. For any $p = p_0 \in T^o$, the point $(012)$ is closer to $p_0$ than is $(025)$.
2. For any $p = p_0 \in T^o$, the point $(345)$ is closer to $p_3$ than is $(235)$.
3. If $p \in T^o$ lies to the left of $J$ then $(025)$ is closer to $p_2$ than is $(235)$.
4. If $p \in T^o$ lies to the right of $J$ then $(235)$ is closer to $p_2$ than is $(025)$.
5. If $p \in J$ then $(025)$ and $(235)$ are equidistant to $p_2$.

Proof: We give computational proofs of these statements. To establish these results, we consider the following functions.

\[
F_1 = |(025) - p_0|^2 - |(012) - p_0|^2,
F_2 = |(235) - p_3|^2 - |(345) - p_3|^2,
F_3 = |(025) - p_3|^2 - |(235) - p_3|^2.
\] (19)

For $F_1$ and $F_2$ we make the change of variables from Equation 14 and we get two rational functions on $(0^2, 1)$. We find that the numerators of these two rational functions belong to $U$ and hence do not vanish in $(0,1)^2$. The factors from our list of 26 are

1. 7, 18, 21, 22.
2. 1, 1, 3, 7, 17, 20, 23.

We list 1 twice in Item 2 because $(a - 1)^2$ appears in the factorization.

The denominators of $F_1$ and $F_2$ cannot vanish on $(0,1)^2$ because then our functions would be unbounded. We know that this is not the case; the
functions are always less than 3. Hence $F_1$ and $F_2$ do not vanish on $(0, 1)^2$. We check that

$$F_1(1/2, 1/2) = \frac{896077}{179766} > 0, \quad F_2(1/2, 1/2) = \frac{2061}{1960} > 0.$$  

Hence $F_1$ and $F_2$ are positive on $(0, 1)^2$. This gives us Statements 1 and 2 above.

For the function $F_3$ we do not make the variable change in Equation 14. Rather, we compute the function in terms of $x + iy \in T^o$. We have

$$F_3 = \frac{18F_4}{(\sqrt{3}x + y - 2\sqrt{3})^2 (\sqrt{3}x + y + \sqrt{3})^2},$$

where

$$F_4 = \begin{pmatrix}
3x^5 - 6x^4 - 9x^3 + 15x^2 - 3x \\
3\sqrt{3}x^4 - 4\sqrt{3}x^3 - 6\sqrt{3}x^2 - 3\sqrt{3}x + \sqrt{3} \\
2x^3 - 6x^2 + 15x - 2 \\
2\sqrt{3}x^2 - 2\sqrt{3} \\
4 - x \\
-\sqrt{3}
\end{pmatrix} \cdot \begin{pmatrix}
y \\
y^2 \\
y^3 \\
y^4 \\
y^5
\end{pmatrix}.$$  

None of the factors of $F_3$ vanishes on $T^o$ except $F_4$. Using Mathematica to solve the equation $F_4 = 0$ for $y$ in terms of $x$, we find that the solutions are

$$y = \frac{x - 1}{\sqrt{3}}, \quad y = \sqrt{3}x,$$

$$y = \frac{1}{\sqrt{3}} \left( 1 - x - \omega^k((2 + x)(5 - 2x)(1 - 4x))^{1/3} \right). \quad (20)$$

Here $\omega = \exp(2\pi i/3)$ and $k = 0, 1, 2$. Of these solutions, only one of them intersects $T^o$, and that is the third solution in Equation 20 for $k = 0$. This is precisely the function in Equation 4, the one which defines the curve $J$ from the Main Theorem. Using this calculation, and one additional calculation to check which side of $J$ corresponds to (235) and which side corresponds to (025), we get Statements 3,4,5 above. ♠
5.2 The End of the Proof

Now we prove all parts of the Main Theorem. We have 4 cases to consider: the interior \( T^o \), the open horizontal edge of \( \partial T \), the open non-horizontal edges of \( \partial T \), and the vertices of \( T \).

**Interior:** Let \( p \in T^o \). Lemma 5.1 tells us the following.

- If \( p \) lies to the left of \( J \) then \( F_p = \Psi(025) \).
- If \( p \) lies to the right of \( J \) then \( \widehat{F}_p \Psi(235) \).
- If \( p \) lies on \( J \) then \( \widehat{F}_p = \Psi(025) \cup \Psi(235) \).

This tells us the map \( F \) on \( T^o \). It only remains to translate this knowledge into the terms of the Main Theorem.

Letting \( \omega = \exp(2\pi i/3) \), consider the map

\[
\alpha_0(z) = \overline{\omega}(2 - i\sqrt{3} - z). \tag{21}
\]

The map \( \alpha_0 \) has the property that \( \alpha_0(A_0) = \Delta_0 \) and \( \Psi \circ \alpha_0 = A \circ \Psi \). Here \( A \) is the antipodal map. Referring to Figure 2.1 the map \( \alpha_0 \) carries \( A_0 \), the upper right triangle labeled 7, to the central triangle \( \Delta_0 \) in a way the preserves the vertex coloring. We found the formula for \( \alpha_0 \) using these properties.

Let \( f = FA \) be the composition of the farpoint map \( F \) with the antipodal map \( A \). When \( p \) lies in \( T^o \) and to the left of \( J \), the point \( f(p) \) is the image of \( (025) \) under \( \alpha_0 \). This is exactly the map given in Equation 5. When \( p \) lies in \( T^o \) and to the right of \( J \), the point \( f(p) \) is the image of \( (235) \) under \( \alpha_0 \). This is exactly the map given in Equation 6.

**Open Horizontal Edge:** Suppose that \( p \) lies in the open horizontal edge of \( \partial T \). This case corresponds to the top left quadrant of Figure 5.1 below. The horizontal edge of \( \partial T \) is the bottom edge. This corresponds to the case \( b = 0 \) and \( a \in (0, 1) \). Given the boundary behavior listed at the end of §4.5 we see that the only interior vertices of the Voronoi decomposition are \((0125)\) and \((2345)\) as shown on the top quadrant of Figure 5.1 below. Moreover, these vertices are distinct and still lie in \( A_0 \). The polynomial \( F_3 \) extends to \( \partial T \) and so does the subvariety \( J \) where \( F_3 \) vanishes. So, the relative distances to \( p_3 \) are the same as for the interior points. Hence, the same formula for \( f = AF \) extends to the open horizontal edge of \( \partial T \).
Open Non-Horizontal Edges: Suppose $p$ lies in one of the open non-horizontal edges of $\partial T$. This corresponds to the bottom row of Figure 5.1. One edge corresponds to the case $b = 1$ and $a \in (0, 1)$ and the other edge corresponds to the case $b \in (0, 1)$ and $a = 0$. Given the boundary behavior listed at the end of §4.5 we see that in both cases the interior vertices are $(012)$ and $(345)$ and $(0235)$ as shown in the bottom row of Figure 5.1. These vertices again lie in $A_0$.

![Figure 5.1: Representative boundary cases.](image)
We have
\[ F_1(a, 1) = \frac{3}{2}(1 + a)^2, \quad F_2(a, 1) = \frac{3}{2}(1 - a)^2. \]
\[ F_1(0, b) = F_2(0, b) = \frac{b(224 + 40b - 24b^2 + 4b^3 - b^4)}{2(b^2 + 8)^2}. \]

These functions are all positive on the given open edges, so the same argument as above eliminates the sets 012 and 345. Hence \( F_p = \Psi(0235) \).

If \( \{p_n\} \in T^0 \) is a sequence converging to \( p \in \partial T \), then \( |f(p_n) - p_n| \to 0 \). This follows from Equations 5 and 6, which we have already established on \( T^0 \). This implies that \( F_p \) must contain the antipodal point \( A(p) \). Otherwise, there is some other point of \( X \) further from \( p \) than \( A(p) \), and eventually this point would be further from \( p_n \) than \( A(f(p_n)) = F(p_n) \), which is contradiction. So, we know that \( F_p \) contains \( A(p) \). Hence \( \Psi(0235) = A(p) \). Hence \( f \) is the identity on the union of the open non-horizontal edges of \( \partial T \).

**Vertices:** The right vertex of \( T \) is a cone point of \( X \), and we already know that \( f \) is the identity there. The other two vertex cases refer to Figure 5.2.

![Figure 5.2: The left and top vertex cases.](image)

The same limiting argument as above shows that \( F_p \) contains the antipodal point \( A(p) \). Looking at the pictures above we see that all the interior vertices of the Voronoi decomposition lie at the center of \( A_0 \) when \( p \) is the left vertex of \( T \) and all the interior vertices lie at the center of the vertical edge of \( A_0 \) when \( p \) is the top vertex. This shows that \( F_p \) is single-valued in these cases. Hence \( F_p = A(p) \) int both cases, so that \( f(p) = p \).
6 References


