The Affine Shape of a Figure-Eight under the Curve Shortening Flow

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Abstract

We consider the curve shortening flow applied to a natural class of figure-eight curves, those with dihedral symmetry and some monotonicity assumptions on the curvature and its derivatives. We prove that when (non-conformal) linear transformations are applied to the solution so as to keep the bounding box the unit square, the renormalized limit converges to a quadrilateral ⋊ called a bowtie. Along the way we prove that suitably chosen arcs of our evolving curves, when suitably rescaled, converge to the Grim Reaper Soliton under the flow. Our grim reaper result is an analogue of a theorem of S. Angenent, which Angenent proved in the locally convex case.

1 Introduction

We say that a smooth family $C : S^1 \times [0,T) \rightarrow \mathbb{R}^2$ of closed immersed plane curves is evolving according to curve shortening flow (CSF) if and only if for any point $(u,t) \in S^1 \times [0,T)$ we have

$$\frac{\partial C}{\partial t} = kN$$

where $k$ is the curvature and $N$ is the unit normal vector of the immersed curve $u \rightarrow C(u,t)$. We often abbreviate this curve as $C(t)$. In all cases, there is some time $T > 0$, called the vanishing time, such that $C(t)$ is defined for all $t \in (0,T)$ but not at time $T$.

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Some powerful results are known about this PDE. In [9], M. Gage and R. Hamilton prove that when $C(0)$ is convex the following is true as $t \to T$:

- $C(t)$ shrinks to a point.
- There is a similarity $S_t$ such that $S_t(C(t))$ converges to the unit circle.

(When $C(0)$ is convex, $C(t)$ remains convex for all $t < T$.) See also [10] and [11]. In [12], M. Grayson proves that if $C(0)$ is embedded then there is some time $t \in (0, T)$ such that $C(t)$ is convex. Thus, the combination of these two results says informally that the curve-shortening flow shrinks embedded curves to round points.

In [3], S. Angenent proves that if $C(0)$ is immersed and with finitely many self-intersections, then the number $N_t$ of self-intersections is monotone non-increasing with $t$. More is known in the case of a “figure-8”. This is a smooth immersed closed loop with one self-intersection point and total curvature 0. One can think of the figure-8 as a union of two “lobes” joined at the intersection point. The figure-8 is called balanced if the two lobes bound equal areas, and unbalanced if not. In [13], M. Grayson proves two things:

- In the unbalanced case, the smaller lobe shrinks to a point before the diameter shrinks to 0, and then the flow can be continued through the singularity, turning into the embedded case.
- In the balanced case, the double point does not disappear before the vanishing time $T$, and the isoperimetric ratio of $C(t)$ tends to $\infty$ as $t \to T$.

Grayson conjectures [13] that every balanced figure-8 converges to a point under the curve-shortening flow, but this is as yet unresolved. In case $C(0)$ has 2-fold rotational symmetry, it follows from Corollary 2 of [6] that $C(t)$ does shrink to a point (the double point) as $t \to T$. In a related direction, the papers [1], [7], and [14] discuss self-similar solutions to the CSF. These shrink to a point and retain their shape.

We work with figure-8 curves that have convex lobes and 4-fold dihedral symmetry. We always normalize so that the coordinate axes are the symmetry axes and that the $x$-axis intersects the curve in 3 points. Thus, our curves look like $\infty$ symbols. We define the fundamental arc of such a figure-8 to be the minimal arc connecting the origin to the point on the positive $x$-axis and otherwise remaining in the positive quadrant. This arc is meant to be half-open; it does not contain the origin. The whole figure-8 (minus the origin) is made from 4 copies of the fundamental arc.

Let $C_+(t)$ denote the righthand lobe of $C(t)$. The lobe $C_+(t)$ is made of two copies of the fundamental arc. In [3] it is proved that if $C(0)$ has convex lobes then
so does $C(t)$ for all $t \in (0, T)$. We define $\kappa(\theta, t) > 0$ to be the curvature of $C_+(t)$ at the point where the tangent line makes an angle $\theta$ with the $x$-axis. We measure this angle in such a way that the fundamental arc is parametrized by $\theta \in (-\alpha(t), \pi/2]$, where $\alpha(t)$ is the tangent angle at the origin. Let $\kappa_\theta = \partial \kappa / \partial \theta$, etc. Computing the time evolution of $\kappa$, we have

$$\kappa_t = \kappa^2 (\kappa + \kappa_\theta\theta). \quad (1)$$


**Definition:** $C(0)$ is concinnous if

- $C(0)$ has 4-fold dihedral symmetry.
- $C(0)$ has convex lobes.
- $\kappa_\theta > 0$ on the interior of the fundamental arc.
- $\kappa_\theta\theta < 0$ on the fundamental arc.

This definition interacts well with the curve shortening flow. In §3.3 we prove that if $C(0)$ is concinnous then so is $C(t)$ for $t \in (0, T)$. The Lemniscate of Bernoulli is an example of a concinnous figure-8. In [2], S. Angenent makes similar curvature assumptions on the first derivative in some cases.

Define the bounding box of a compact set in the plane to be the smallest rectangle, with sides parallel to the coordinate axes, that contains the set. The main goal of the paper is to understand the limit of the sequence \{\(L_t(C(t))\)\} where $L_t$ is the diagonal matrix such that $L_t(C(t))$ has the square $[-1, 1]^2$ for a bounding box.

The Hausdorff distance between two compact subsets of the plane is the smallest $\epsilon$ such that each set is contained in the $\epsilon$-neighborhood of the other. This distance makes the set of compact planar subsets into a metric space, which has the appropriate topology for our purposes. The curves $L_t(C(t))$ now move around within a compact set of shapes, so it is natural to ask about a limit.

The bowtie is the quadrilateral whose vertices are

$$(-1, -1), \quad (1, 1), \quad (1, -1), \quad (-1, 1)$$

in this cyclic order. It is shaped like this: ⋊ ⋉.

**Theorem 1.1 (Bowtie).** Suppose that $C(0)$ is concinnous. As $t \to T$, the sequence $L_t(C(t))$ converges in the Hausdorff metric to the bowtie.
Figure 1 shows a picture of a numerical simulation of the curve shortening flow. The curve on the left is \( L_0(C(0)) \) where \( C(0) \) is the Leminscate of Bernoulli. The black curve on the right is \( L_t(C(t)) \) for some later time \( t \). The blue curve on the right is what we call \( \Gamma(t) \) in Equation 19 below; it is a rescaled version of \( C(t) \). The black and white dots correspond to where \( \theta = 0 \) and \( \theta = \pi \) respectively. Figure 1 shows some hints of the bowtie forming.

![Figure 1: Hints of the bowtie.](image)

A less restrictive version of the Bowtie Theorem is announced in [5], and some ideas in the proof are sketched there. Our proof incorporates some aspects of the sketch in [5] and also departs from it. We introduce the function

\[
F(\theta, t) = \frac{\kappa(\theta, t)}{\kappa(\pi/2, t)}.
\]

By concinnity, \( F \) is a concave increasing function such that \( F(\pi/2, t) = 1 \). In §2 we deduce the Bowtie Theorem from the statement that

\[
\liminf_{t \to T} F_\theta(0, t) \geq 1.
\]

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\liminf_{t \to T} F_\theta(0, t) \geq 1.
\]

In fact, the argument we give in §2.3 shows that the inequality in Equation 3 is an equality. See the remark at the end of §2.3. We think that \( \lim_{t \to T} F_\theta(0, t) = 1 \) but we do not know yet how to prove this.

Following the deduction of the Bowtie Theorem, the rest of the paper is devoted to establishing Equation 3. Because \( F_\theta(\ast, t) \) is monotone decreasing for each \( t \), Equation 3 is an immediate consequence of the following result we prove.
**Theorem 1.2** (Grim Reaper). Let $\theta \in (0, \pi)$ be arbitrary. As $t \to T$ the values $F(\theta, t)$ and $F_\theta(\theta, t)$ respectively converge to $\sin(\theta)$ and $\cos(\theta)$.

This Grim Reaper Theorem is the analogue of Theorem D in [2]. The reason for the name is that the result implies that a suitably rescaled copy of the arc of $C(t)$ corresponding to $\theta \in (0, \pi)$ converges to the Grim Reaper curve. The arc in question is the one between the two dots in Figure 1. Up to rotations and scaling, the Grim Reaper is characterized by the equation $\kappa(\theta) = \sin(\theta)$. Our proof departs from that in [2] because we are not working with locally convex curves as in [2]. See the discussion in §4.5.

We think that some of the side conditions used in our proofs are not necessary but we currently do not know how to remove them. The monotonicity condition on the curvature allows us to avail ourselves, often, of the well-known Tait-Kneser Theorem.

**Theorem 1.3** (Tait-Kneser). Suppose $\gamma$ is a curve of strictly monotone increasing curvature. Then the osculating disks of $\gamma$ are strictly nested. The largest one is at the initial endpoint and the smallest one is at the final endpoint.

The assumption on $\kappa_\theta$ allows us to deduce Equation 3 from the Grim Reaper Theorem. We also use it in Lemma 2.9 and Equation 49.

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2 The Bowtie Theorem

2.1 The Evolution of the Bounding Box

Let $[-X(t), X(t)] \times [-Y(t), Y(t)]$ be the bounding box of $C(t)$.

**Lemma 2.1 (Bounding Box).** $\lim_{t \to T} Y(t)/X(t) = 0$.

*Proof.* The perimeter of $C(t)$ and the area of the region bounded by $C(t)$ are respectively within a factor of 2 of the perimeter and area of the bounding box of $C(t)$. Thus, Grayson’s isoperimetric result tells us that the aspect ratio of the bounding box tends to 0. This means that either $Y(t)/X(t) \to 0$ or $Y(t)/X(t) \to \infty$ as $t \to T$.

We establish in §3.3 that $C(t)$ is concinnous for all $t \in (0, T)$. We assume that result here. Since $X_t(t) = -\kappa(\pi/2, t)$ and $Y_t(t) = -\kappa(0, t)$, we have

$$Y(t) = \int_t^T \kappa(0, u) \, du < \int_t^T \kappa(\pi/2, u) \, du = X(t).$$

This rules out the second option above. \qed

2.2 Two Integral Formulas

In this section we derive two integral formulas which we will use in the next section. The first formula is essentially the same as the one that appears in the proof of Lemma 8.2 in [2].

$$\kappa(\theta, t) = \kappa(\pi/2, t) \sin(\theta) + \int_{\theta}^{\pi/2} \sin(\phi - \theta) \left( \kappa_{\theta\theta}(\phi, t) + \kappa(\phi, t) \right) d\phi. \quad (4)$$

For the second formula, let $(x^*(t), Y(t))$ denote the top point on $C_+(t)$.

$$\frac{x^*(t)}{X(t)} = 1 - \frac{W(t)}{X(t)}, \quad W(t) = \int_0^{\pi/2} \frac{\cos(\phi)}{\kappa(\phi, t)} \, d\phi \quad (5)$$

A similar formula appears on p 630 (and elsewhere) in [2]. See also Lemma 10 in [5].

**Derivation of Equation 4:** We set $c(\cdot) = \cos(\cdot)$ and $s(\cdot) = \sin(\cdot)$. We have:

$$\frac{\partial}{\partial \theta} \left( s(\theta) \kappa_{\theta}(\theta, t) - c(\theta) \kappa(\theta, t) \right) = s(\theta) \left( \kappa_{\theta\theta}(\theta, t) + \kappa(\theta, t) \right).$$
\[
\frac{\partial}{\partial \theta} \left( c(\theta) \kappa_\theta(\theta, t) + s(\theta) \kappa(\theta, t) \right) = c(\theta) \left( \kappa_\theta(\theta, t) + \kappa(\theta, t) \right).
\]

Setting \( \psi = \pi/2 \) and integrating these identities from \( \theta \) to \( \psi \), we compute:

\[
\int_\theta^\psi s(\phi - \theta) (\kappa_\theta(\phi, t) + \kappa(\phi, t)) d\phi =
\]

\[
c(\theta) \int_\theta^\psi s(\phi) (\kappa_\theta(\phi, t) + \kappa(\phi, t)) d\phi - s(\theta) \int_\theta^\psi c(\phi) (\kappa_\theta(\phi, t) + \kappa(\phi, t)) d\phi =
\]

\[
c(\theta) s(\psi) \kappa_\theta(\psi, t) \quad c(\theta) c(\psi) \kappa(\psi, t) - c(\theta) s(\theta) \kappa_\theta(\theta, t) + c(\theta) c(\theta) \kappa(\theta, t) - s(\theta) c(\psi) \kappa(\psi, t) - s(\theta) s(\psi) \kappa(\psi, t) + s(\theta) c(\theta) \kappa_\theta(\theta, t) + s(\theta) s(\theta) \kappa(\theta, t) = \]

\[
\kappa(\theta, t) - s(\theta) \kappa(\theta, t).
\]

We repeatedly use \( \cos(\psi) = 0 \) and \( \sin(\psi) = 1 \). The last equality uses \( \kappa_\theta(\psi, t) = 0 \).

**Derivation of Equation 5:** For this derivation \( s \) denotes the arc length as a function of the angle \( \theta \). We normalize so that \( s(-\alpha(t)) = 0 \). The function \( s \) is a diffeomorphism from \((-\alpha(t), \pi + \alpha(t))\) to \((0, S_t/2)\) where \( S_t \) is the total length of \( C(t) \). The value of \( S_t \) is not important to us. Since \( s \) is a diffeomorphism we can also write \( \theta \) as a function of \( s \). The two functions \( s \) and \( \theta \) are inverses of each other.

In angular coordinates the points \((0, t)\) and \((\pi/2, t)\) respectively denote the points whose \( x \)-coordinates are \( x^*(t) \) and \( X(t) \). We compute the horizontal distance between these two points as follows:

\[
X(t) - x^*(t) = \int_{s(0)}^{s(\pi/2)} \cos(\theta(s)) ds = \int_{s(0)}^{s(\pi/2)} g(s) ds.
\]  \hspace{1cm} (6)

Using the change-of-variables formula from calculus, we have

\[
X(t) - x^*(t) = \int_0^{\pi/2} \left( \cos(\theta) \times \frac{ds}{d\theta} \right) d\theta.
\]  \hspace{1cm} (7)

Since the curvature is the rate of change of the tangent angle with respect to arc length, the inverse function theorem gives us:

\[
\frac{ds}{d\theta} = \frac{1}{\kappa(\theta, s)}.
\]  \hspace{1cm} (8)

Substituting this into Equation 7 we get

\[
X(t) - x^*(t) = \int_0^{\pi/2} \frac{\cos(\theta)}{\kappa(\theta, t)} d\theta = W(t).
\]

Dividing through by \( X(t) \) and rearranging gives the desired result.
2.3 The Migration of the Top Point

Let \( W, X, x^* \) be as in Equation 5. In this section we prove

\[
\lim_{t \to T} W(t) = 0, \quad \lim_{t \to T} x^*(t) = 1.
\]

(9)

In view of Equation 5, the first equation implies the second.

Let us first explain the geometric consequence of this result. We write

\[
C_+(t) = D(t) \cup V(t) \cup \overline{D}(t),
\]

where

- \( D(t) \) connects \((0, 0)\) to \((x^*(t), Y(t))\).
- \( V(t) \) connects \((x^*(t), Y(t))\) to \((x^*(t), -Y(t))\).
- \( \overline{D}(t) \) connects \((x^*(t), -Y(t))\) to \((0, 0)\).

Figure 2 shows this decomposition.

Figure 2: Decomposition of the right lobe.

The arc \( L_t(V(t)) \) is convex, and has endpoints which, by Equation 9, converge to (1, 1) and \((-1, 1)\). Moreover \((1, 0)\) is the midpoint of \( L_t(V(t)) \). From this structure, we see that \( L_t(V(t)) \) converges in the Hausdorff metric to the line segment connecting \((1, 1)\) to \((-1, 1)\). This is one of the sides of the bowtie. We can finish the proof of the Bowtie Theorem by showing that \( L_t(D(t)) \) converges to the line segment connecting \((0, 0)\) to \((1, 1)\). We take care of this after finishing the present section.

Now we turn to the proof of Equation 9.

**Lemma 2.2.** We have \( \lim_{t \to T} W(t) = 0 \) and \( \lim_{t \to T} X(t) = 0 \).

**Proof.** Since \( C(t) \) shrinks to a point as \( t \to T \) we have \( \lim_{t \to T} X(t) = 0 \). Since \( x^*(t)/X(t) \in [0, 1] \) we see that \( W(t) \in [0, X(t)] \). Hence \( \lim_{t \to T} W(t) = 0 \) as well. \( \square \)
Since $W(t) \to 0$ and $X(t) \to 0$ we have the familiar variant of L’Hopital’s rule:

$$\limsup_{t \to T} \frac{W(t)}{X(t)} \leq \limsup_{t \to T} \frac{W_t(t)}{X_t(t)}.$$  \hspace{1cm} (11)

Our next result makes crucial use of Equation 3.

**Lemma 2.3.**

$$\limsup_{t \to T} \frac{W_t(t)}{X_t(t)} \leq 0.$$  \hspace{1cm} (12)

**Proof.** We differentiate Equation 5 then use Equation 1 to get:

$$W_t(t) = -\int_0^{\pi/2} \cos(\phi) \frac{\kappa_t(\phi,t)}{\kappa^2(\phi,t)} d\phi = -\int_0^{\pi/2} \cos(\phi)(\kappa(\phi,t) + \kappa_{\phi\phi}(\phi,t)) d\phi.$$  \hspace{1cm} (13)

Differentiating Equation 4 with respect to $\theta$ and taking $\phi = 0$ we have

$$\kappa_\theta(0,t) = \kappa(\pi/2,t) - \int_0^{\pi/2} \cos(\phi)(\kappa(\phi,t) + \kappa_{\phi\phi}(\phi,t)) d\phi = \kappa(\pi/2,t) + W_t(t).$$  \hspace{1cm} (14)

Given the definition of the curve shortening flow, we have

$$X_t(t) = -\kappa(\pi/2,t).$$  \hspace{1cm} (15)

Combining Equations 14 and 15, we get

$$\frac{W_t(t)}{X_t(t)} = 1 - \frac{\kappa_\theta(0,t)}{\kappa(\pi/2,t)}.$$  \hspace{1cm} (16)

Therefore,

$$\limsup_{t \to T} \frac{W_t(t)}{X_t(t)} = 1 - \liminf_{t \to T} \frac{\kappa_\theta(0,t)}{\kappa(\pi/2,t)} \leq 0.$$  \hspace{1cm} (17)

The final inequality comes from Equation 3. This establishes Equation 12. \qed

Since $W$ and $X$ are positive, Equation 12 combines with Equation 11 to give

$$0 \leq \liminf_{t \to T} \frac{W(t)}{X(t)} \leq \limsup_{t \to T} \frac{W(t)}{X(t)} \leq \limsup_{t \to T} \frac{W_t(t)}{X_t(t)} \leq 0.$$ 

This gives Equation 9.

**Remark:** Incidentally, we have used Equation 3 to show that

$$\limsup_{t \to T} \frac{W_t(t)}{X_t(t)} = 0.$$ 

This combines with Equation 17 to show that $\liminf_{t \to T} F_\theta(0,t) = 1$. Thus, the inequality in Equation 3 is an equality.
2.4 The Slope Bound

The goal of this section is to prove the Slope Lemma, which is the basis for a bootstrap argument given in the next section. We just need some finite constant in the Slope Lemma; the choice of 3 is convenient.

**Lemma 2.4** (Slope). The slope of \( L_t(D(t)) \) at \((0,0)\) is less than 3 provided that \( t \) is sufficiently close to \( T \).

We say that a concinnous arc is the graph \( G \) of a concave increasing function \( g : [a,b] \to \mathbb{R} \) such that \( g'(b) = 0 \) and the unsigned curvature of \( G \) is strictly increasing going from left to right. To \( G \) we associate the unique disk \( E(G) \) such that \( \partial E(G) \) contains the endpoints of \( G \) and the endpoint \((b,g(b))\) is the topmost point of \( E(G) \). Figure 3 shows the construction.

![Figure 3: G and E(G).](image)

**Lemma 2.5.** Suppose \( G \) is a concinnous arc. Then \( G \subset E(G) \).

**Proof the Slope Lemma:**

Assume Lemma 2.5 for now. The arc \( D(t) \) is concinnous as in the previous section. Hence \( D(t) \subset E(t) \) where \( E(t) = E(D(t)) \) is the disk from Lemma 2.5. But then

\[
L_t(D(t)) \subset L_t(E(t)).
\] (18)

The set \( L_t(E(t)) \) is a solid ellipse whose boundary contains the points \((0,0)\) and whose top point is \((x^*(t)/X(t),1)\). The latter point converges to \((1,1)\) as \( t \to T \). Moreover, by Lemma 2.1 the eccentricity of \( L_t(E(t)) \) converges to \( \infty \) as \( t \to T \). The limit \( \lim_{t \to T} L_t(E(t)) \) must be the solid region under a downward pointing parabola whose boundary contains \((0,0)\) and whose top point is \((1,1)\). There is only one region with this description: It is given by the equation \( y \leq 2x - x^2 \).

Given this picture, the slope of \( \partial L_t(E(t)) \) at \((0,0)\) converges to 2. Moreover, \( L_t(D(t)) \) starts at \((0,0)\) and lies beneath this boundary. Hence the slope of \( L_t(D(t)) \) at \((0,0)\) is at most \( 2 + \epsilon(t) \), where \( \epsilon(t) \) is some function which tends to 0 as \( t \to T \). In particular, this slope is less than 3 for \( t \) sufficiently close to \( T \).\[\square\]
Proof of Lemma 2.5:

Proof. Let $E = E(G)$. We suppose this result is false and derive a contradiction. If this result is false then there is some arc $\delta$ of $G$ whose endpoints $p, q$ lie in $\partial E$ but which is otherwise disjoint from $E$. The arc $\delta$ lies above $E$. We order the points so that $p$ lies to the left of $q$. Let $E^*$ be the osculating disk of $G$ at $q$. Let $r$ be the right endpoint of $G$. Figures 4, 5, 6 show these objects. There are three cases.

Case 1: Suppose $q = r$. Since $G$ is concinnous, $G$ and $E$ are tangent at $q$. Since $G$ lies above $q$ sufficiently close to $q$ we see that the curvature of $G$ at $q$ is not more than the curvature of $E$. But then $E \subset E^*$. The Tait-Kneser Theorem tells us that $G - \{q\}$ is disjoint from $E^*$. This is impossible because $p \in E \subset E^*$.

Case 2: Suppose that $q \neq r$, and that $G$ is tangent to $E$ at $q$. There are two subcases. (A) If $E \subset E^*$ we get the same contradiction as in Case 1. (B) Otherwise $E^*$ is a proper subset of $E$ and the two disks are tangent at $q$. But then no point of $E^*$ to the right of $q$ intersects $\partial E$. But then the continuation of $G$ to the right of $q$, which lies in $E^*$ by the Tait-Kneser Theorem, cannot reach $r \in \partial E$.
Case 3: Suppose that \( q \neq r \) and that \( G \) and \( \partial E \) are transverse at \( q \). Since \( \delta \) lies above \( E \), we see that when we move rightward along \( \partial E^* \) we cross into the interior of \( E \). Since \( \delta \) lies above \( E \), there must be some other point \( q' \), between \( p \) and \( q \) on \( E \), where \( \partial E \) and \( G \) intersect. But two unequal circles can intersect at most twice. This means that all points of \( E^* \) to the right of \( q \) lie in the interior of \( E \). Figure 6 shows the situation. This gives the same contradiction as in Case 2B.

![Figure 6: Case 3 of the argument.](image)

This completes the proof.

2.5 Curvature Decay

Define

\[
\Gamma(t) = \frac{1}{X(t)} C(t), \quad \Delta(t) = \frac{1}{X(t)} D(t), \quad H(t) = \frac{Y(t)}{X(t)}. \tag{19}
\]

Here \( \Gamma(t) \) is a homothetic rescaling of \( C(t) \) and \( \Delta(t) \) is the arc of \( \Gamma(t) \) connecting the double point to the topmost point. The bounding box for \( \Gamma(t) \) is

\([-1, 1] \times [-H(t), H(t)]\).

Let \( K \) be the curvature of \( \Gamma \). Let \( S_t \) be the arc length of \( \Delta(t) \). We find it convenient to let the interval

\( I_t = (-S_t, 0] \tag{20} \)

parametrize \( \Delta(t) \) by arc length. The point \( s = 0 \) corresponds to the top point of \( \Delta(t) \). We note that \( S_t > 1' \) once \( t \) is sufficiently close to \( T \). Here \( 1' \) is any constant less than 1. In this section we prove the following result.
Lemma 2.6 (Decay). For any $s \in (-1, 0)$ we have $\lim_{t \to T} K(s, t)/H(t) = 0$.

We prove this result through a series of lemmas. Let us first bring in the input from the Slope Lemma.

Lemma 2.7. For $s \in (-1, 0)$ we have $K(s, t)/H(t) < 3/|s|$ provided that $t$ is sufficiently close to $T$.

Proof. This is the same as showing that $K(s, t) < 3H(t)/|s|$. The slope of $\Delta(t)$ at $(0, 0)$ is $H(t)$ times the slope of $L_i(D(t))$ at $(0, 0)$. The Slope Lemma therefore implies that

$$3H(t) > \text{slope of } \Delta(t) \text{ at } (0, 0) = \tan(\alpha(t)) > \alpha(t). \quad (21)$$

for $t$ sufficiently near $T$.

Now we argue by contradiction. If this lemma is false, then the monotonicity of the curvature gives us $K(\ast, t) \geq 3H(t)/|s|$ on all of $[s, 0)$. The length of the corresponding arc of $\Delta(t)$ is at least $|s|$. Hence, the variation of the tangent angle along $\Delta(t)$ is at least $3H(t) > \alpha(t)$. But the variation of the tangent angle on $\Delta(t)$ is $\alpha(t)$. This is a contradiction. \qed

The idea of our proof is use Lemma 2.7 as the basis of a bootstrap argument.

Lemma 2.8. $\kappa(\pi/2, t)Y(t) \geq 1$.

Proof. Let $s_0$ and $s_1$ respectively denote the arc-length parameters that correspond to $\theta_0 = 0$ and $\theta_1 = \pi/2$. On the level of 1-forms:

$$dy = -ds \sin \theta, \quad \kappa(\theta, t)ds = d\theta.$$

(The minus sign appears because $y$ decreases as $s$ increases.)

$$Y(t) = \int_0^{Y(t)} dy = -\int_{s_0}^{s_1} \sin(\theta)ds = \int_{s_0}^{s_1} \sin(\theta)ds = \int_0^{\pi/2} \frac{\sin(\theta)}{\kappa(\theta, t)}d\theta.$$

Multiplying through by $\kappa(\pi/2, t)$, we have

$$Y(t)\kappa(\pi/2, t) = \int_0^{\pi/2} \left(\frac{\kappa(\pi/2, t)}{\kappa(\theta, t)}\right) \sin(\theta)d\theta \geq \int_0^{\pi/2} \sin(\theta)d\theta = 1. \quad (22)$$

The last inequality comes from the concinnity assumption, which says in particular that $\kappa(\ast, t)$ is monotone increasing on $[0, \pi/2]$. \qed
In this next result we are thinking of $K$ as a function of arc length, on the interval $I_t$. Here is where we use the assumption on $\kappa_{\theta\theta}$.

**Corollary 2.9.** $\frac{d}{ds}\log K(s)$ tends uniformly to $\infty$ on $I_t$.

**Proof.** By Equation 3 we have $F_\theta(0,t) > 1/2$ for all $t$ sufficiently close to $t$. This means that $\kappa_{\theta}(0,t) > (1/2)\kappa(\pi/2,t)$. For $s \in I_t$ we have

$$\frac{d}{ds}\log K(s) = \frac{K_s(s,t)}{K(s,t)} = \kappa(\theta,t) = \kappa(\theta,t)X(t) >^* \kappa(\theta,0)X(t) > (1/2)\kappa(\pi/2,t)X(t) = \kappa(\pi/2,t)Y(t) \times \frac{X(t)}{2Y(t)} > \frac{X(t)}{2Y(t)}.$$ 

Here $\theta \in (-\alpha(t), 0)$ is such that the tangent angle at the arc-length parameter $s$ is $\theta$. The starred inequality uses $\kappa_{\theta\theta} < 0$. The right hand side tends to $\infty$ by Lemma 2.1. $\Box$

Now we give the bootstrap argument. Suppose for the sake of contradiction that there is some $\eta > 0$ and some $s_1 \in (-1, 0)$ with $K(s_1,t)/H(t) > \eta$ for some sequence of $t$ converging to $T$. Call the times involved in this the special times. Let $s_2 = s_1/2$. Corollary 2.9 says that we can take $t$ sufficiently close to $T$ so that

$$\frac{d}{ds}\log K(s) > N := \frac{\log(3/|\eta s_2|)}{|s_2|},$$

on all of $I_t$. Integrating this inequality over $[s_1, s_2]$ at a special time very close to $T$ we get

$$\frac{K(s_2,t)}{H(t)} > e^{N|s_2|}\frac{K(s_1,t)}{H(t)} > \eta e^{N|s_2|} > \frac{3}{|s_2|}.$$ 

This contradicts Lemma 2.7. This completes the proof of the Decay Lemma.

**2.6 Proof of the Bowtie Theorem**

We begin with a general principle that lets us easily convert between related geometric quantities. We apply this principle to $\Delta(t)$, a curve whose maximum tangent slope tends to $0$ as $t \to T$.

**Uniform Lipschitz Principle:** For curves whose maximum tangent slope is less than $1$, the tangent angle and the slope are uniformly Lipschitz functions of each
other, and the 1-forms $ds$ and $dx$ are uniformly bounded multiples of each other.

To finish the proof of the Bowtie Theorem we just need to prove that $L_t(D(t))$ converges to the segment connecting $(0, 0)$ to $(1, 1)$. The arc $L_t(D(t))$ is the graph of a function defined on some interval which converges to $[0, 1]$. So, for any $x_0 \in (0, 1)$ it eventually make sense to speak of the arc $A(t, x_0)$ of $L_t(D(t))$ consisting of points $(x, y)$ where $x \in (0, x_0]$. It suffices to prove for any $x_0 \in (0, 1)$ that the difference in slopes at the endpoints of $A(x_0, t)$ converges to 0 as $t \to T$. If this is the case, then the arcs $A(x_0, t)$ converge to straight line segments $B(x_0)$ as $t \to T$, and the segments $B(x_0)$ converge to the segment connecting $(0, 0)$ to $(1, 1)$ as $x_0 \to 1$.

We get $A(x_0, t)$ by vertically stretching the arc of $\Delta(t)$ whose endpoints are $(0, 0)$ and some point $(x_0, \ast)$. The amount of vertical stretch is $1/H(t)$. Let $\sigma(x, t)$ denote the tangent slope of $\Delta(t)$ at the point whose first coordinate is $x$. As $t \to T$ the domain for $\sigma$ converges to $(0, 1)$. The Uniform Lipschitz Principle implies that the two quantities

$$K = \frac{d\theta}{ds}, \quad \frac{d\sigma}{dx}$$

are uniformly bounded multiples of each other. (Indeed, the multiple in question converges to 1 as $t \to T$.) Combining this fact with the Decay Lemma, we get:

$$\lim_{t \to T} \sup_{x \in (0, x_0]} \frac{d\sigma(x_0, t)}{H(t)} = 0. \quad (23)$$

Integrating Equation 23 we see that

$$\lim_{t \to T} \frac{|\sigma(x_0, t) - \sigma(0, t)|}{H(t)} = 0. \quad (24)$$

But then when we stretch vertically by $1/H(t)$ the difference in the tangent slopes at the endpoints of $A(x_0, t)$ is exactly the quantity on the left side of Equation 24. Hence, this difference converges to 0 as $t \to T$. This is what we wanted to prove.
3 Preservation of Concinnity

3.1 The Sturmian Principle

In this chapter we prove the following lemma.

**Lemma 3.1** (Concinnity). If \( C(0) \) is concinnous then so is \( C(t) \) for all \( t \in (0, T) \).

We begin with a discussion of the Sturmian Principle for strictly parabolic PDEs and then we give the main argument. We follow the notation in [8] and [2]. We first deal with the linear case and then with the nonlinear case.

**The Linear Case:** Let \( U \) be an open interval containing \([x_0, x_1]\). We suppose that \( u : U \times [0, \tau] \) satisfies the equation

\[
  u_t = a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u. \tag{25}
\]

This equation is called *strictly parabolic* if and only if \( a(x, t), b(x, t), \) and \( c(x, t) \) are smooth and \( a(x, t) > 0 \). We assume that \( u \) satisfies a strictly parabolic PDE. Here is the well-known Sturmian Principle;

**Theorem 3.2** (The Sturmian Principle). Suppose \( u \) is nonzero on \( \{x_0, x_1\} \times [0, \tau] \). Then the number \( N_t \) of times \( u(\ast, t) \) vanishes on \( (x_0, x_1) \) is non-increasing with time. Moreover, if \( u(\ast, t) \) vanishes to second order somewhere on \( (x_0, x_1) \) then \( N_{t'} < N_t \) for all \( t' \in (t, \tau) \).

C. Sturm discovered this principle in 1836. See [15]. The proof of the above version of the Sturmian Principle may be found in [3]. For more references about these theorems, see [8] or [3].

**The Nonlinear Case:** Now we observe how we may apply the Sturmian Principles to nonlinear strictly parabolic PDE. Let \( F(x, t, u, p, q) \) be a \( C^\infty \) function of five variables, and consider the nonlinear evolution equation

\[
  u_t = F(x, t, u, u_x, u_{xx}). \tag{26}
\]

We call the nonlinear PDE (26) strictly parabolic if and only if \( \frac{\partial F}{\partial q} > 0 \).

Let \( u, v \) be two \( C^\infty \) solutions of the evolution equation (26), and let \( w = v - u \) be the difference of the two solutions. The function \( w \) satisfies a *linear* PDE of the form (25), where

\[
  a(x, t) = \int_0^1 \frac{\partial F}{\partial q}(x, t, \tilde{w}(x, t, \phi), \tilde{w}_x(x, t, \phi), \tilde{w}_{xx}(x, t, \phi))d\phi > 0
\]

16
and where we have used the auxiliary function  \( \tilde{w}(x, t, \phi) = (1 - \phi)v + \phi u \). For details on deriving this associated linear PDE, see [2] or [3].

We can now apply the Sturmian Principle to the linear evolution equation for \( w \). Once again, we assume that \( u, v, w \) are defined on \( U \times [0, \tau] \) where \( U \) is an open set containing \([x_0, x_1] \).

**Corollary 3.3.** Suppose \( w \) is nonzero on \( \{x_0, x_1\} \times [0, \tau] \). Then the number \( N_t \) of zeroes for \( w(\ast, t) \) on \( (x_1, x_2) \) is finite and non-increasing. Moreover, at any time \( t \) when \( w(\ast, t) \) vanishes to second order, we have \( N_t < N_{t'} \) for all \( t' > (t, \tau] \).

In this picture the side \([x_0, x_1] \times \{0\}\) is the bottom side and the side \([x_0, x_1] \times \{\tau\}\) is the top. Here we are picturing time as running vertically and space as running horizontally.

**Curvilinear Domains:** Corollary 3.3 is too restrictive for one of our purposes. The same principle works when the rectangle in question is replaced by a piecewise analytic quadrilateral \( Q \) with the following two properties:

1. The top and bottom sides are line segments, with the bottom one corresponding to time 0 and the top one corresponding to time \( \tau \).

2. The function \( w \) does not vanish on the other two sides.

The other two sides play the role of \( \{x_0\} \times [0, \tau] \) and \( \{x_1\} \times [0, \tau] \). The main issue is that the non-horizontal sides prevent zeros from “leaking in or out”.

![Figure 7: The Curvilinear case](image)

Let us explain why the rectilinear principle implies the curvilinear principle. Suppose we have a situation where \( w \) has \( m \) zeros on the bottom of \( Q \) and \( n > m \) on the top of \( Q \). Let \( I \) be the set of times where \( w \) has more than \( m \) zeros. Let \( t = \inf I \). The zeros of \( w \) at times converging to \( t \) cannot converge to the non-horizontal sides of the domain. Hence at least two of them must coalesce. But then we can find
a small rectangle \( R \subset Q \) which surrounds these coalescing points. See the small shaded rectangle in Figure 7. (If more points coalesce, the picture would look more complicated.) This gives a contradiction to the rectilinear principle.

For what it is worth, the curvilinear domains we consider (in the proof of the Sine Lemma below) have only one non-straight side, and this side is the graph of a function \( x = g(y) \). See Figure 8.

### 3.2 Examples

There are 4 equations we need to consider:

1. The equation for \( \kappa \).
2. The equation for \( u = \kappa \theta \).
3. The equation for \( v = -\kappa \theta \).
4. The equation for the signed curvature as a function of the \( x \)-coordinate.

We give formulas for these equations. The equation for \( \kappa \) is given in Equation 1. Here it is again, written in a slightly different way:

\[
\kappa_t = \kappa^2 \kappa \theta \theta + \kappa^3. \tag{27}
\]

The equation for \( u \) is

\[
u_t = \kappa^2 u \theta \theta + 2 \kappa u u \theta + 3 \kappa^2 u \tag{28}\]

The equation for \( v \) is

\[
v_t = \kappa^2 v \theta \theta + 4 \kappa u v \theta - 2 \kappa v^2 + 2 u^2 v + 3 \kappa^2 v - 6 \kappa u^2. \tag{29}\]

These equations are all valid on the domain

\[
D = \bigcup_{t \in [0, t)} (-\alpha(t), \pi + \alpha(t)) \times \{t\}. \tag{30}\]

Now we get to the final equation. We let \( k \) denote the signed curvature, which we think of as a function of \( x \) and \( t \). Thus \( k(x, y) \) is the curvature at the point \( p = (x, y) \) at time \( t \). Note that the domain for \( x \) is shrinking to a point. Let \( y(x, t) \) be the evolution of the height of the (un-rescaled) curve \( C(t) \). The equation for \( k_x \) is

\[
(k_x)_t = \frac{(k_x)_{xx} - 2 y_x y_{xx}(k_x)_x}{1 + y_x^2} + 3 k^2 (k_x). \tag{31}\]
One can derive this equation by differentiating the evolution equation for $k$ (as a function of $x$). The evolution equation for $k$ is worked out in [9] and [12]. The equation for $k_x$ is valid away from places where our curve has vertical tangents. In particular on any time range $[0,t]$ for $t < T$ it is valid on each strand in a fixed neighborhood of the double point.

All equations are strictly parabolic because the coefficient of the leading order term in each equation is strictly (even uniformly) positive. We point out that we never need to know anything about the equations for $u,v,k_x$ except their strict parabolicity, and we only need to consider these 3 equations for the proof that the curve shortening flow preserves concinnity. The rest of the paper only uses Equation 1, the equation for $\kappa$.

### 3.3 Proof of the Concinnity Lemma

Let us first explain why it suffices to consider the case when $k_x(0,0) > 0$. If it happens that $k_x(0,0) = 0$, then $k_x(\ast,0)$ vanishes to order $o_0 \geq 2$ because it is an even function. But then we can apply the Sturmian principle to a small rectangle of the form $[-\epsilon,\epsilon] \times [0,\epsilon]$ to conclude that $k_x(0,\epsilon)$ vanishes to order at most $o_0 - 1$. (In fact we get $o_0 - 2$ because $k_x$ is an even function.) Iterating, we see that for any $\epsilon > 0$ we have $k_x(0,\epsilon) > 0$. In exactly the same way, it suffices to assume that $v(\pi/2,0) > 0$. Here $v = -\kappa_{\theta\theta}$.

**Lemma 3.4.** $k_x(0,t) > 0$ and $v(\pi/2,t) > 0$ for all $t \in T$.

*Proof.* We treat the case of $k_x$. The proof for $v$ is essentially the same. If the statement about $k_x$ fails, there is some first time $t$ such that $k_x(0,t) = 0$. But $k_x(\ast,t)$ is analytic for $t \in (0,T)$ and hence its zeros are isolated. We can then apply the Sturmian Principle to a rectangle of the form $[-\epsilon,\epsilon] \times [0,t]$ and we get a contradiction. \[ \square \]

**Lemma 3.5.** If $\kappa_{\theta}(\ast,0) > 0$ on $(-\alpha(0),\pi/2)$ then $\kappa_{\theta}(\ast,t) > 0$ on $(-\alpha(t),\pi/2)$.

*Proof.* Recall that $u = \kappa_{\theta}$. All that is left to show is that $u(\theta,t) > 0$ on the domain $\mathcal{D}$. Suppose this fails. Let $I$ denote the set of times for which this happens. Let $t = \inf I$. There are several cases to insider.

Suppose first that $t \in I$. Then there is some $(\theta,t) \in \mathcal{D}$ such that $u(\theta,t) = 0$ but $u(\ast,t') > 0$ for all $t' \in [0,t)$. In this case we get a contradiction by applying the Sturmian principle to $u$ on a rectangle

$$[\theta - \epsilon,\theta + \epsilon] \times [t - \epsilon,t].$$
For sufficiently small $\epsilon$ this rectangle belongs to $D$. Since $u(\ast, t)$ is analytic we can further choose $\epsilon$ so that $u(\theta \pm \epsilon, t) > 0$. We now contradict the Sturmian Principle. Hence $t \not\in I$.

Let $(\theta_n, t_n)$ be a sequence of points in $D$ such that $u(\theta_n, t_n) = -\alpha(t)$ and $t_n \to t$. Since $t \not\in I$, we must have (after using symmetry and passing to a subsequence) either $\theta_n \to -\alpha(t)$ or $\theta_n \to \pi/2$. Intuitively, what we are saying is that the zeros must leak in from the left or the right boundary component. We consider the cases in turn.

- Suppose $\theta_n \to -\alpha(t)$. By the Chain rule, $k_x(x_n, t_n) = 0$ for a sequence $x_n \to 0$. But then $k_x(0, t) = 0$ by continuity. This contradicts the fact that $k_x(0, t) > 0$.

- Suppose $\theta_n \to \pi/2$. Since we are now in the interior of the domain $D$ and $u$ is a smooth function, we have

$$-v(\pi/2, t) = \lim_{n \to \infty} \frac{u(\pi/2, t_n) - u(\theta_n, t_n)}{\pi/2 - \theta_n} = 0,$$

which contradicts the fact that $v(\pi/2, t) > 0$.

This completes the proof.

Lemma 3.6. If $\kappa_{\theta \theta}(\ast, 0) < 0$ on $(-\alpha(0), \pi/2)$, then $\kappa_{\theta \theta}(\ast, t) < 0$ on $(-\alpha(t), \pi/2)$.

Proof. This time we consider $v = -\kappa_{\theta \theta}$. The proof is essentially the same as in the previous case, but we explicitly say what we do in the cases when $t \not\in I$. If $\theta_n \to \pi/2$, we can take a limit and get $v(\pi/2, t) = 0$, contradicting Lemma 3.4.

The case when $\theta_n \to -\alpha(t)$ is tricky. In this case we have $\kappa_{\theta \theta}(\theta_n, t_n) = 0$ for a sequence $\theta_n \to -\alpha(t)$. For this we treat curvature as a function of arc-length. We let $k(s, t)$ denote the curvature at the arc length parameter $s$ and at time $t$. We normalize so that $s = 0$ corresponds to the double point.

We have the change of variables formula derived in [9].

$$k_s = \kappa \kappa_{\theta} \quad \quad k_{ss} = \kappa \kappa_{\theta}^2 + \kappa^2 \kappa_{\theta \theta}. \quad (32)$$

Let $s_n$ be the arc length parameter corresponding to $\theta_n$. Note that $k_{ss}(0, t) = 0$ because $k$ is an odd function. From the second equation in Equation 32 we see therefore that

$$\lim_{n \to \infty} \kappa(\theta_n, t_n) \kappa_{\theta}^2(\theta_n, t_n) = 0 \quad (33)$$
By definition and by Equation 32 we have
\[ \kappa_\theta(\theta_n, t_n) = \frac{k_s(s_n, t_n)}{k(s_n, t_n)}. \]  
(34)

Combining this information with Equation 33 we have
\[ \lim_{n \to \infty} k_s^2(s_n, t_n) = 0. \]  
(35)

But we also have
\[ \lim_{n \to \infty} k_n(s_n, t_n) = k_s(0, t_n), \lim_{n \to \infty} k_s(s_n, t_n) = k_s(0, t) = k_s(0, t) \times \frac{dx}{ds} > 0. \]  
(36)

The last equality comes from Lemma 3.4. Equation 36 implies that the limit in Equation 35 is actually \( \infty \) rather than 0, and we have a contradiction.

These lemmas fit together to show that if \( C(0) \) is concinnous then so is \( C(t) \) for all \( t \in (0, T) \).
4 The Grim Reaper Theorem

In this chapter we prove the Grim Reaper Theorem. After collecting together three easy technical lemmas we get to the main point of the proof.

4.1 Some Asymptotics

The two easy lemmas in this section rely on Lemma 2.1 and the Tait-Kneser Theorem.

**Lemma 4.1.** \( \lim_{t \to \infty} \kappa(\theta, t) = \infty \) for any \( \theta \in (0, \pi/2] \).

*Proof.* Let \( \Gamma(t) \) be as in Equation 19. This is a rescaled version of \( C(t) \) whose bounding box has width 2. The height of the bounding box tends to 0 by Lemma 2.1. Let \( K(\theta, t) = X(t)K(\theta, t) \) be the curvature of \( \Gamma(t) \) at the point where the tangent angle is \( \theta \). Since \( \lim_{t \to T} X(t) = 0 \), it suffices to prove that there is some constant \( a = a_\theta > 0 \) such that \( K(\theta, t) > a \) for all \( t \) sufficiently close to \( T \).

Suppose that this result is false. Let \( D(\theta, t) \) be the osculating disk to \( \Gamma(t) \) at \( \theta \). There is a sequence of times on which \( D(\theta, t) \) converges to the half plane bounded by the line which makes an angle \( \theta \) with the \( x \)-axis. On the other hand, for one of these special times sufficiently close to \( T \), the disk \( D(\theta, t) \) contains the origin in its interior. But then the arc of \( \Gamma(t) \) starting at \( (\theta, t) \) and going leftward cannot reach the origin because, by the Tait-Kneser Theorem, it remains outside \( D(\theta, t) \). \( \square \)

**Lemma 4.2.** \( \lim_{t \to \infty} \alpha(t) = 0 \).

*Proof.* Let \( \Gamma(t) \) be as in Equation 19. Suppose that there is a sequence of times \( t_n \to T \) such that \( \alpha(t_n) > \delta > 0 \) for some constant \( \delta \). Let \( L \) be the line through the origin which makes an angle of \( \delta/2 \) with the \( x \)-axis. By Lemma 2.1, the height of the bounding box \( \beta(t) \) for \( \Gamma(t) \) tends to 0 as \( t \to T \). Hence, \( L \) hits the top of \( \beta(t) \) at a point whose distance to the origin tends to 0 as \( t \to T \).

By construction \( \Gamma(t_n) \) starts out from the origin lying to the left of \( L \). Since \( \Gamma(t_n) \subset \beta(t_n) \), we see that \( \Gamma(t_n) \) crosses \( L \) at some point \( p_n \) such that \( ||p_n|| \to 0 \). The total variation of the tangent angle of \( \Gamma(t) \) along the arc connecting \( (0, 0) \) to \( p_n \) is, by convexity, at least \( \delta/2 \). Since the length of this arc tends to 0, some point \( q_n \) on this arc has curvature at least 4. By construction \( ||q_n|| \to 0 \).

By the Tait-Kneser Theorem the arc of \( \Gamma(t) \) connecting \( q_n \) to \( (1, 0) \) is trapped in a disk of radius 1/4 which contains \( q_n \) in its boundary. This is a contradiction. \( \square \)
4.2 Counting Zeros

Our final lemma has nothing to do with the flow. A very similar principle is used in [2]. Let \( J \subset \mathbb{R} \) be some interval. Call a function \( g : J \to \mathbb{R} \) small if

\[
\sup_J g^2 + (g')^2 < 1.
\]  

(37)

Call \( J \) small if it has length at most \( \pi \). Every small interval is contained in a closed interval of length \( \pi \). Closed intervals of length \( \pi \) count as being small.

**Lemma 4.3.** If \( g \) is a small function and \( J \) is a small interval then the difference \( w(x) = g(x) - \sin(x) \) vanishes at most twice on \( J \), counting multiplicity.

**Proof.** Let \( f(x) = \sin(x) \). We note the crucial property that

\[
f^2 + (f')^2 = 1 > g^2 + (g')^2.
\]

Let \( F \) and \( G \) respectively denote the graphs of \( F \) and \( G \). These graphs must be transverse wherever they intersect. Otherwise we would have \( g^2 + (g')^2 = 1 \) at an intersection point. This is impossible. We show that \( f = g \) at most twice. Given the transversality just mentioned, this is equivalent to the statement that \( w = g - f \) vanishes at most twice on \( J \), counting multiplicity.

As usual in calculus, say that \( x \in J \) is an extreme point if \( f'(x) = 0 \). The only way that \( J \) can contain two extreme points is if \( J \) has length \( \pi \), and the endpoints are the two extreme points, and \( |f| = 1 \) at these endpoints. In this case \( f \neq g \) at the endpoints because \( |g| < 1 \). So, even in this case, we can replace \( J \) by a smaller interval which contains all the points where \( f = g \). Thus, we can assume without loss of generality that \( J \) contains at most one extreme point.

Suppose first that \( J \) has no extreme points. Then \( f \) is either monotone increasing on \( J \) or monotone decreasing. Consider the case when \( f \) is monotone increasing. Suppose it happens that there are two consecutive points \( x_1, x_2 \in J \) where \( f \) and \( g \) agree. The portion of \( G \) between \((x_1, g(x_1))\) and \((x_2, g(x_2))\) either lies above \( F \) or below. In the first case we have \( g'(x_1) > f'(x_1) \), which is a contradiction. In the second case we have \( g'(x_2) > f'(x_2) \) and we have the same contradiction. Hence \( f(x) = g(x) \) for at most one point \( x \in J \). The same argument works when \( f \) is monotone decreasing on \( J \).

Now consider the case when \( J \) has exactly one extreme point. In this case we can write \( J = J_1 \cup J_2 \) where \( f \) is monotone on each \( J_i \). In this case, the same argument above, applied to each of these sub-intervals, shows that they each have at most one point where \( f = g \). Hence \( J \) has at most 2 such points. \( \Box \)
4.3 The Sine Lemma

Here is the crucial step in the proof of the Grim Reaper Theorem. This section is devoted to proving the following result.

**Lemma 4.4 (Sine).** Suppose $\theta \in (0, \pi/2)$ and $\epsilon > 0$ are given. If $t$ is sufficiently close to $T$ then

$$\left| \frac{\kappa_{\theta}(\theta, t)}{\kappa(\theta, t)} - \frac{\cos(\theta)}{\sin(\theta)} \right| < \epsilon.\tag{4.3}$$

Let $\theta_0 \in (0, \pi/2)$. We will assume for the sake of contradiction that there is a sequence of times $t_n \to T$ such that

$$\left| \frac{\kappa_{\theta}(\theta_0, t_n)}{\kappa(\theta_0, t_n)} - \frac{\cos(\theta_0)}{\sin(\theta_0)} \right| > \epsilon.\tag{4.4}$$

We choose a constant $\Sigma = \Sigma(\theta_0, \epsilon) > 0$ so that

$$\phi + \theta_0 \in (0, \pi/2) \text{ and } \left| \frac{\cos(\phi + \theta_0)}{\sin(\phi + \theta_0)} - \frac{\cos(\theta_0)}{\sin(\theta_0)} \right| > \epsilon \implies |\phi| > \Sigma.\tag{4.5}$$

Call the non-horizontal sides of our domains the *sidewalls*. Thanks to Lemma 4.2 we can omit the initial portion of our evolution and arrange that

$$\sup_{t \in [0, T)} \alpha(t) < 10^{-100} \Sigma.\tag{4.6}$$

We are making the horizontal displacement of the sidewalls of $\mathcal{D}$ extremely small in comparison to the other relevant quantities that arise below. We don’t need the factor of $10^{-100}$; we add it for emphasis.

Let

$$C = \sup_{\theta \in [0, \pi/2]} \kappa^2(\theta, 0) + \kappa^2_{\theta}(\theta, 0), \quad B_n = \kappa^2(\theta_0, t_n) + \kappa^2_{\theta}(\theta_0, t_n).\tag{4.7}$$

By Lemma 4.1 there is some $n$ such that $B_n > C$. Our motivation for taking $B_n > C$ is the following corollary of Lemma 4.3.

**Corollary 4.5.** Suppose $J$ is an interval of length at most $\pi$ on which $\kappa(\cdot, 0)$ is entirely defined. Suppose

$$\sup_{\theta \in J} \kappa^2(\theta, 0) + \kappa^2_{\theta}(\theta, 0) \leq C.$$

Let $S(\theta) = \sqrt{B} \sin(\phi + \theta)$ for any value $\phi$. If $B > C$ then $w(\cdot) = \kappa(\cdot, 0) - S(\cdot)$ vanishes at most twice on $J$, counting multiplicity.
Proof. This follows from Lemma 4.3 by symmetry and scaling.

We fix $n$ for which $B_n > C$. We set $B = B_n$ and $t = t_n$. There is a $\phi$ such that
\[
S(\theta) = \sqrt{B} \sin(\phi + \theta)
\] (42)

has the following property:
\[
S(\theta_0, t) = \kappa(\theta_0, t), \quad S_\theta(\theta_0, t) = \sqrt{B} \cos(\phi + \theta_0) = \kappa_\theta(\theta_0, t).
\] (43)

Let $I$ be the segment of length $\pi$ such that

- $\theta_0 \in I - \partial I$.
- $S > 0$ on $I - \partial I$,
- $S = 0$ on $\partial I$.

Let $\Omega = I \times [0, t]$. This is exactly the domain considered in [2], but now our proof departs from [2].

**Lemma 4.6.** **One sidewall of $\Omega$ is disjoint from the closure of $\mathcal{D}$ and the other sidewall of $\Omega$ lies in $\mathcal{D}$.**

**Proof.** The properties of $S$ imply the following:
\[
\left| \frac{\cos(\phi + \theta_0)}{\sin(\phi + \theta_0)} \right| - \frac{\cos(\theta_0)}{\sin(\theta_0)} \right| = \left| \frac{\kappa(\theta_0, t)}{\kappa_\theta(\theta_0, t)} - \frac{\cos(\theta_0)}{\sin(\theta_0)} \right| > \epsilon. \tag{44}
\]

To show that $\phi$ satisfies the left hand conditions in Equation 39, we just have to see that $\phi + \theta_0 \in (0, \pi/2)$. Since $\kappa(\theta_0, t)$ and $\kappa_\theta(\theta, t)$ are both positive, we deduce from Equation 43 that indeed $\phi + t_0 \in (0, \pi/2)$. Equation 39 now tells us that $|\phi| > \Sigma$.

If we had $\phi = 0$ we would have $I = [0, \pi]$. As it is, we have $|\phi| > \Sigma$. This shifts $I$ and $\Omega$ by at least $\Sigma$ to the left or to the right. Given our bound on the horizontal displacement of the sidewalls of $\mathcal{D}$, this shift causes one sidewall or the other to stick out completely. See Figure 8 below.

Since $\phi + \theta_0 \in (0, \pi/2)$ and $\theta_0 \in (0, \pi/2)$, we have $|\phi| < \pi/2$. The other sidewall lies in $\mathcal{D}$ because $|\phi| < \pi/2$ and $\mathcal{D}$ has width everywhere greater than $\pi$. If we are shifting to the left (respectively right), we would have to shift by more than $\pi$ to push the right (respectively left) sidewall outside $\mathcal{D}$. 

We now create a new domain $\mathcal{Q}$ by intersecting $\Omega$ with $\mathcal{D}$ and pushing in the curvilinear sidewall a bit. We treat the case when $\Omega$ sticks out on the left. The other case is essentially the same.

25
Figure 8: The new domain $Q$, shaded.

Define

$$w(\theta, t) = \kappa(\theta, t) - S(\theta).$$  \hspace{1cm} (45)

The function $S$ is a stationary solution to Equation 1, meaning that $S_t = 0$. This means that $w$ is exactly the sort of difference of solutions to which the Sturmian Principle applies. Let us examine the behavior of $w$ on the boundary of $Q$.

**Left:** Since $\kappa$ limits to 0 on the sidewalls of $D$ and $S > 0$ on the left sidewall of $D$, we can by compactness make the perturbation small enough so that $w < 0$ on the left sidewall of $Q$.

**Right:** The right sidewall of $Q$ lies in $D$. Since $S = 0$ on the right sidewall of $Q$ and $\kappa > 0$ everywhere in $D$, we have $w > 0$ on the right sidewall of $Q$.

**Bottom:** Applying Corollary 4.5 to the bottom side $J$ of $Q$, we see that $w(*,0)$ vanishes at most twice on $J$ counting multiplicity. Since $w$ has opposite signs on the sidewalls of $Q$ the number of zeros of $w$ on $J$ is odd, counting multiplicity. Since this number is at most 2, it must be exactly 1. In short, $w$ vanishes exactly once on the bottom side of $Q$, counting multiplicity.

**Top:** On the top side $J'$ of $Q$ we have arranged that $w$ and $w_\theta$ vanish at $(\theta_0, t)$. This means that $w$ vanishes at least twice, counting multiplicity, on $J'$. We have shown this double point in Figure 8. Since $w$ has opposite signs on the sidewalls of $Q$ the number of zeros of $w$ on $J'$ is odd, counting multiplicity. Since this number is at least 2 it is actually at least 3. In short, $w$ vanishes at least 3 times on the top side of $Q$ counting multiplicity.

The above properties violate the Sturmian Principle for (Equation 1, $Q$, $w$). This completes the proof of the Sine Lemma.
4.4 The End of the Proof

In this section we prove the Grim Reaper Theorem. We first observe something about the proof of the Sine Lemma. All the constants may be chosen uniformly over any given compact sub-interval $J \subset (0, \pi/2)$. Thus we have the following uniform version of the Sine Lemma:

**Lemma 4.7.** Let $\epsilon > 0$ be given and let $J \subset (0, \pi/2)$ be any closed interval. For $t$ sufficiently close to $t$ we have

$$\sup_{\theta \in J} \left| \frac{F_\theta(\theta, t)}{F(\theta, t)} - \frac{\cos(\theta)}{\sin(\theta)} \right| < \epsilon,$$

for $t$ sufficiently close to $T$.

We can replace $\kappa$ by $F$ because for each time these functions are constant multiples of each other.

**Corollary 4.8.** Suppose $\theta_0 < \theta_1 \in (0, \pi/2)$. If $\{t_n\} \to T$ and $F(\theta_0, t_n) \to L \sin(\theta_0)$ then $F(\theta_1, t_n) \to L \sin(\theta_1)$.

**Proof.** Let $J = [\theta_0, \theta_1]$. Consider the new function

$$G(\theta, t) = \frac{F(\theta, t)}{\sin(\theta)}, \quad (46)$$

Using Lemma 4.7 we have the following result:

$$|G_\theta| = \left| \frac{F_\theta(\theta, t) \sin(\theta) - F(\theta, t) \cos(\theta)}{\sin^2(\theta)} \right| < \frac{\epsilon F(\theta, t) \sin(\theta)}{\sin^2(\theta)} = \epsilon G, \quad (47)$$

This holds for all $\theta \in J$ provided that we take $t$ sufficiently close to $t$.

Hence

$$\left| \frac{d}{d\theta} \log(G) \right| < \epsilon \quad (48)$$

on $J \times (t, T)$ provided that $t$ is sufficiently close to $T$. Hence $G$ converges to a constant function on $J$. \qed

**Lemma 4.9.** $\lim_{t \to T} F(\theta, t) = \sin(\theta)$ for any $\theta \in (0, \pi/2)$. 

27
Proof. If this is false, then there is some $\theta_0 \in (0, \pi/2)$ and some subsequence $\{t_n\}$ such that $F(\theta_0, t_n)$ does not converge to $\sin(\theta)$. Passing to a subsequence, we can assume that $F(\theta_0, t_n) \to L \sin(\theta)$ for some $L \neq 1$.

Since $\kappa_{\theta\theta}(\theta, t) < 0$ for $\theta \in (0, \pi/2]$, the function $F$ is concave. This combines with $F(\pi/2, t) = 1$ to imply that $F(\theta, t)$ converges to $F(\pi/2, t)$ in a uniform way:

$$|1 - F(\theta, t)| \leq (2/\pi) \times |\pi/2 - \theta|.$$  \hfill (49)

Therefore, for any $\epsilon$ we can choose $\theta_1$ so close to 1 that

$$\sin(t_1) > 1 - \epsilon, \quad 1 - \epsilon \leq \liminf_{n \to \infty} F(\theta_1, t_n) \leq \limsup_{n \to \infty} F(\theta_1, t_n) \leq 1.$$  

Combining this with Corollary 4.8 we have

$$1 - \epsilon \leq L \leq \frac{1}{1 - \epsilon}.$$  

But $\epsilon$ is arbitrary. Hence $L = 1$. This is a contradiction. \hfill \Box

Lemma 4.7 now says that $F_\theta(\theta, t)$ converges to $\cos(\theta)$ as $t \to T$. This proves the Grim Reaper Theorem for $\theta \in (0, \pi/2)$. The case $\theta \in (\pi/2, \pi)$ follows from symmetry. Finally $F(\pi/2, t) = 1$ and $F_\theta(\pi/2, t) = 0$ for all $t$. Hence the Grim Reaper Theorem also holds for $\theta = \pi/2$. The completes the proof of the Grim Reaper Theorem.

4.5 Discussion

It is worth comparing our proof above to Angenent’s proof in [2]. Let us consider the construction in the Sine Lemma without any assumptions about $\kappa_\theta/\kappa$. In this case, the rectangle $\Omega$ might interact differently with $D$. One possibility is that the sidewalls of $\Omega$ and the sidewalls of $D$ intersect. In this very special situation, a zero of $w$ “leaks in” from the side of $\Omega \cap D$ so to speak. Here we can draw no conclusions at all.

The other possibility, also very special in our setting, is that $\Omega \subset D$. In this case $w > 0$ on the sidewalls of $\Omega$, and $w$ vanishes twice on the bottom side of $\Omega$. If $\kappa_{\theta\theta}(\theta_0, t) + \kappa(\theta_0, t) < 0$ then $w < 0$ in a neighborhood of $(\theta_0, t)$ along the top side of $\Omega$. This would force $\omega$ to vanish at least 4 times along the top side of $\Omega$, a contradiction. We conclude from this that

$$\kappa(\theta_0, t) + \kappa_{\theta\theta}(\theta_0, t) \geq 0.$$  \hfill (50)

28
A similar argument shows strict positivity. This would give $\kappa_t(\theta_0, t) > 0$. Were we to consider the situation at $\theta_0 = \pi/2$ we would get this case. Thus (if we wanted) we could conclude that $\kappa(\pi/2, t)$ is monotone increasing for $t$ sufficiently close to $T$.

In [2], the domain $D$ is a cylinder of width at least $2\pi$ and so it always happens that $\Omega \subset D$. Once $\kappa(\theta_0, t)$ is sufficiently large, the hypotheses needed for Equation 50 hold, and then they hold for all future $t$. To illustrate some of the main ideas of the proof in [2], let us assume that Equation 50 holds for all $\theta \in (0, \pi/2)$ and all $t$, and that $\lim_{t \to T} F(0, t) = 0$. Dividing Equation 4 by $\kappa(\pi/2, t)$, we get

$$F(\theta, t) = \sin(\theta) + \int_{\theta}^{\pi/2} \sin(\phi - \theta)(F(\phi, t) + F_{\theta\theta}(\phi, t))d\phi = \sin(\theta) + I_{\theta,t}.$$ 

Here $I_{\theta,t}$ is the integral of a positive function, thanks to Equation 50. The positivity of the integrand, together with the specific form of the integrand, implies that $I_{\theta,t} \leq I_{0,t}$ for all $\theta \in (0, \pi/2)$. Just knowing that $F(0, t) \to 0$ one can then conclude that $I_{\theta,t} \to 0$ for all $\theta \in (0, \pi/2)$. This gives the convergence of $F(\theta, t)$ to $\sin(\theta)$. This sketch suppresses many technical details, but it gives some feel for what is going on.

One could say that both our proof and Angenent’s proof use the Sturmian Principle to compare an arbitrary solution with the Grim Reaper soliton, but Angenent’s proof uses the integral formula and our proof uses the geometry of the domain $D$. 

29
References


