Billiards Obtuse and Irrational

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1   Introduction

In *polygonal billiards*, one studys the trajectories made by a frictionless point mass as it rolls across a polygon P and bounces off the edges according to the law of geometric optics: The angle of incidence equals the angle of reflection. When the angles of P are rational multiples of π, one sees a wealth of structure and theory. For instance, Masur's theorem [**M**] says that the number of periodic orbits on P grows quadratically. Improving on other work of Masur, the paper [**BGKT**] proves that the set of pairs (start, direction) leading to periodic billiard paths is dense in the set of all such pairs. A *periodic billiard path* is one in which the billiard ball keeps retracing a closed path over and over again. In general, the study of rational billiards brings in Riemann surfaces, Teichmuller theory, hyperbolic geometry, and even algebraic geometry. There is a vast literature on rational polygonal billiards and you should consult e.g. [**G**], [**MT**] or [**T**] for an survey.

In this article we are going to completely ignore the beautiful structure of rational billiards and consider what happens when the angles are not rational multiples of π. In this case there is a great poverty of structure. For instance, it has been a problem for over 200 years to decide whether or not every triangle admits a periodic billiard path!

One case of this problem was settled by Fagnano in 1775. On an acute triangle, Fagnano showed that the small inscribed triangle connecting the altitudes is a billiard path. This is shown on the left hand side of Figure 1.
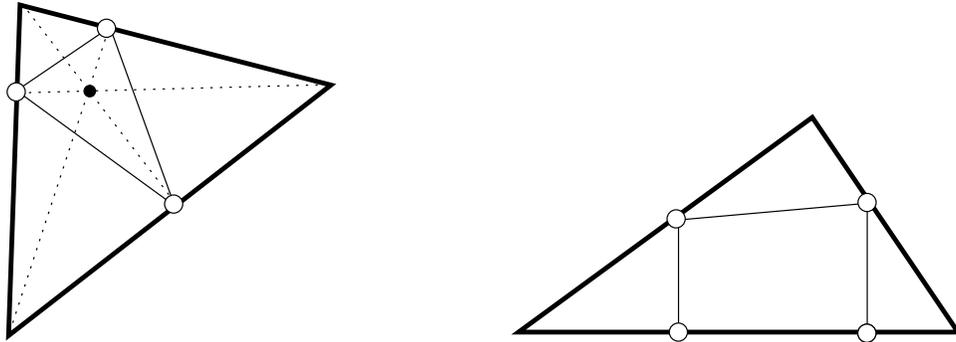
Figure 1.1

The right hand side of Figure 1 shows a length 6 periodic billiard path which works for any right triangle. The path always hits the hypotenuse at right angles and retraces the same 3 steps twice before closing up. See [**T**] and the references there for deeper theorems about right-angled billiards.

The papers [**GSV**] and [**HH**] give some examples of infinite families of combinatorial types of billiard path which describe periodic billiard trajectories on obtuse triangles. Here a *combinatorial type* is a sequence of digits in $\{1, 2, 3\}$ describing the sequence of edges hit by the billiard path. However, until recently it was not known if there was any $\epsilon > 0$ such that a triangle with all angles less than $90 + \epsilon$ has a periodic billiard path.

Pat Hooper and I wrote a computer program called McBilliards, which searches for periodic orbits in triangles, and we have discovered a number of things using this program. For example, I discovered (and eventually proved in [**S1**], [**S2**], [**S3**]) that a triangle has a periodic billiard path provided that the largest angle is at most 100 degrees.

We originally wrote McBilliards to try and solve the triangular billiards problem, and our early days using the program were filled with boundless optimism. After a year of working, our optimism has faded somewhat, but even so McBilliards reveals a zoo of new phenomena about triangular billiards. The purpose of this article is to describe McBilliards and some of the discoveries and questions which come out of it.

I hope that this article inspires you to download and install McBilliards. The program has a snappy graphical user interface and ultimately is fun to use. If you want to try a toy version of McBilliards on the web, check out my Java applet: **www.math.brown.edu/∼res/Java/App46/test1.html** This applet illustrates my 100 degree theorem, mentioned above.

2 Some Geometric Constructions

2.1 The Rational Case

Here we will sketch a nice proof, due to Boshernitsyn, that every rational polygon has a periodic billiard path. See [**MT**] for details.

Let P be a rational polygon and suppose that some initial direction v_1 on P is chosen. We think of v_1 as a unit vector. If a billiard ball starts rolling somewhere on P, in the direction of v_1 then the angle conditions guarantee that there is some minimal list $v_1, ..., v_n$ of directions such that our ball, no matter where it starts, can only travel in the directions $v_1, ..., v_n$.

Let $P_1, ..., P_n$ be n copies of P, with P_i "labelled" by v_i. Suppose e_i is an edge of P_i and e_j is the corresponding edge of P_j. Both edges correspond to the same edge e on P. We glue e_i to e_j iff a billiard path travelling on P in the direction of v_i bounces off e and then travels in the direction of v_j. Each edge gets glued to one other and we produce a compact surface Σ_P, called *the Katok surface* or *the translation surface*. Except for finitely many singular points, Σ_P is locally isometric to the Euclidean plane.
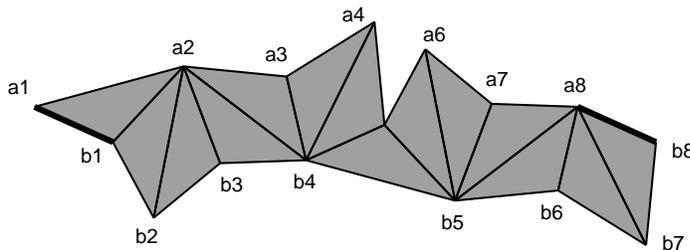
On each P_j we can draw the family of oriented line segments parallel to v_j. These line segments piece together on Σ_P to partition Σ_P into oriented curves. Given $p \in \Sigma_P$ we define $\phi(p)$ to be the point obtained by moving p one unit along the oriented curve on which it lies. ϕ is known as the *time one map* for the geodesic flow. Technically, ϕ is only defined almost everywhere, because we are not allowed to map into or out of singularities.

We choose v_1 to be perpendicular to one of the sides e of P and imagine a billiard path which starts at some point in the middle of P and goes in the direction of v_1 (pointing towards e.) Let $x \in \Sigma_P$ be the corresponding point. We claim that, for any $\epsilon > 0$ there is some k such that $\phi^k(x)$ and x are within ϵ of each other. Otherwise, the disk Δ of radius ϵ about p is such that $\phi^k(\Delta) \cap \Delta = \emptyset$ for all k. By symmetry we have $\phi^i(\Delta) \cap \phi^j(\Delta) \cap \emptyset$ for all $i \neq j$. But then we are filling up a finite area surface with infinitely many disks. (We are proving an easy version of the *Poincare Recurrence Theorem*.)

So, we can pick k such that $\phi^k(x)$ is very close to x. Back on P, this means that our billiard path at time k is so close, both in position and direction, to the original starting position that it *must be travelling in the same direction.* We are using the fact that there are only finitely many possible directions. But then the path must hit the side e at some time after k. Since our billiard path hits e perpendicularly twice, it is periodic.

2.2 Unfoldings

Now we will remove the rationality constraint on the angles, and describe a general construction, reminiscent of the one above, which works for any polygon. We restrict our attention to triangles for ease of exposition. The reader can see this construction in action either on my applet, or else using the **unfold** window in McBilliards.

Figure 2.1

Let T be a triangle and let $W = w_1, ..., w_{2k}$ be a word with digits in $\{1, 2, 3\}$. We always take W to be even length and we do not allow repeated digits. We define a sequence $T_1, ..., T_{2k}$ of triangles, by the rule that T_{j-1} and T_j are related by reflection across the w_jth edge of T_j. Here $j = 2, 3, ..., 2k$. The set $U(W, T) = \{T_j\}_{j=1}^{2k}$ is known as the *unfolding* of the pair (W, T). This is a well known construction; see [**T**]. Figure 2.1 shows an example, where $W = (1232313)^2$. (Note that we have left off the label of a_5 only because it is difficult to fit into the picture.)

We label the top vertices of $U(W, T)$ as $a_1, a_2, ...,$ from left to right. We label the bottom vertices of $U(W, T)$ as $b_1, b_2, ...,$ from left to right. This is shown in Figure 2.1.

W represents a periodic billiard path in T iff the first and last sides of $U(W, T)$ are parallel and the interior of $U(W, T)$ contains a line segment L, called a *centerline*, such that L intersects the first and last sides at corresponding points. We always rotate the picture so that the first and last sides are related by a horizontal translation. In particular, any centerline of $U(W, T)$ is a horizontal line segment. The unfolding in Figure 2.1 does have a centerline, though it is not drawn. To show that a certain triangle has W as a periodic billiard path we just have to consider the unfolding. After we check that the first and last sides are parallel, and rotate the picture as above, we just have to show that each a vertex lies (strictly) above each b vertex.

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2.3   Stability

The word W is *stable* if the first and last sides of $U(W, T)$ are parallel for any triangle T. There is a well-known combinatorial criterion for stability:

Lemma 2.1 *Let $W = w_1, ..., w_{2n}$. Let $n_{d,j}$ denote the number of solutions to the equation $w_i = d$ with i congruent to j mod 2. Let $n_d = n_{d,0} - n_{d,1}$. Then W is stable iff $n_d(W)$ is independent of d.*

Proof: (Sketch) Let the angles of T be $\theta_1, \theta_2, \theta_3$. Going from T_0 to T_{2n} (and keeping track of the even indices) the integer n_j represents the total net number of times we rotate counterclockwise about the jth vertex. Each time we do such a rotation it is by $2\theta_j$. Thus, to get from T_0 to T_{2n} we (translate and) rotate by $2(n_1\theta_1 + n_2\theta_2 + n_3\theta_3) = 2n_1(\theta_1 + \theta_2 + \theta_2) = 2\pi n_1$. Hence these triangles are parallel. ♠

When W is stable, the existence of a periodic billiard path on T, described by W, comes down to question of whether or not there is a centerline which divides all the a vertices from all the b vertices. This is an open condition: if W describes a periodic billiard path on T then W also describes periodic orbits on all triangles T' which are sufficiently close to T.

The **word** window in McBilliards draws a useful graphical interpretation of the word. Let $\mathcal{H}$ denote the union of edges of the planar hexagonal tiling. There are 3 families of parallel line segments in $\mathcal{H}$. We label each edge in $\mathcal{H}$ by either 1, 2, or 3 depending on the family containing it. Given our word W we can form a path $P(W)$ in $\mathcal{H}$ simply by taking the edges of $\mathcal{H}$, in succession, according to the digits of W. Then W is stable if and only if the path $P(W)$ is closed. Figure 2.2 shows the path corresponding the word in §2.1, namely $W = (1232313)^2$.
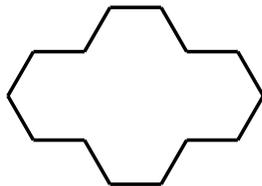

Figure 2.2

2.4   Orbit Tiles

We shall denote the parameter space of triangles by Δ. We think of Δ as the unit square, with the point (x, y) representing the triangle, two of whose angles are $\pi x/2$ and $\pi y/2$. (Sometimes we will wish to talk about points in $\partial\Delta$ and these correspond to "degenerate triangles", where one or more of the angles is zero.) Given a stable word W we define $O(W) \subset \Delta$ to be the union of points which correspond to triangles for which W describes a periodic billiard path. To save words we will say simply that W *works for* the given parameter point or triangle. As we have remarked above, W works for an open set of parameter points, and so $O(W)$ is an open set.

As we remarked above, the condition that $p \in O(W)$ is the same as the condition that the vertex a_i of the (correctly rotated) unfolding lies above the vertex b_j for every pair (i, j). Each one of these conditions is given by some analytic function and there are finitely many such functions. (The **tile analyzer** window in McBilliards actually gives you the formulas.) So, one would expect the boundary of W to be a piecewise analytic set, with the vertices coming from the points where several of the a vertices are at the same height as several of the b-vertices.

McBilliards exploits this structure to plot $O(W)$ given W. The program first finds the vertices of $O(W)$ using Newton's method, and then plots the edges connecting the vertices, again using Newton's method to keep from wandering off the edge. Since the method only traces out the boundary, it assumes that $O(W)$ is simply connected. In practice, we find that this is always the case.

2.5   Searching

In addition to plotting, McBilliards will search for periodic billiard paths: Given a point $p \in \Delta$ and a (smallish) number N, McBilliards finds all the stable words of length at most N which work for p. Here *smallish* means *roughly less than* 1000. Equipped with the plotting and searching capabilities, you can go around Δ trying to cover the space with orbit tiles. If you manage to find a covering, you win: You have shown that every triangle has a periodic billiard trajectory.

Here I will describe briefly how McBilliards *used to* search for periodic billiard paths. What McBilliards does these days is much faster and more sophisticated, thanks to some great programming and insights of Pat's. In

both the crude version I describe here and the refined version currently in use, McBilliards essentially does a depth-first search through a tree of words, pruning out branches which will not lead to success.

First, here is how pruning works: Figure 2.2 shows a (roughly) drawn picture of an unfolding $U(W, T)$ where $W = 2313213$ and T is some triangle. Let $\widehat{W}$ be any word which contains W as a sub-word.
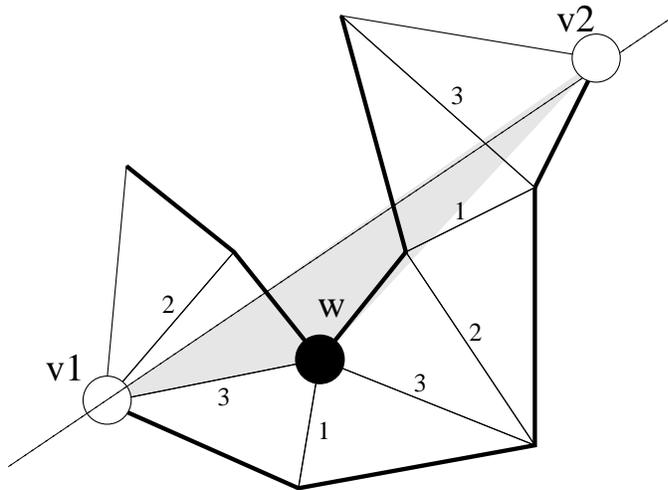

Figure 2.2

$U(\widehat{W}, T)$ does not contain a centerline because w lies on the wrong side of $\overline{v_1, v_2}$. Thus $T \notin O(\widehat{W})$. In case we find a triple of vertices in $U(W, T)$ as above we say that W *fails the pruning test*.

The search algorithm begins with two lists of words, A and B. Initially A consists of the empty word and B is the empty list. The algorithm proceeds until A is the empty list, then stops. At this point, B is the list of even length stable words of length less or equal to N which work for T.

1. If $A = \emptyset$ let W be the first word on A.

2. If W fails the pruning test, delete W from A and return to Step 1.

3. If W is nonempty, stable, and works for T, append W to B. Otherwise...

4. Let $L = \text{Length}(W)$. If $L \leq N - 2$ then delete W from A and prepend to A the 4 words W_1, W_2, W_3, W_4 which have length $L + 2$ and contain W as its initial word. Go to Step 1. If $A=\emptyset$ then stop.

3 Good and Bad Luck with Covering

When I was a kid I sometimes heard long drawn-out stories [1] which went like this: ...unluckily J— fell out of an airplane; luckily he had a parachute; unluckily it wasn't working; luckily he landed in a soft pile of hay; unluckily there was a pitchfork in the hay... The funny thing about our experience with the triangular billiards problem is that it goes sort of like the story above.

3.1 Good Luck near the Right Angled Line

The first thing I wanted to do with McBilliards was to show that every triangle having angles less than $90 + \epsilon$ had a periodic billiard path. I started searching and plotting near the right angled line and here is what I found:
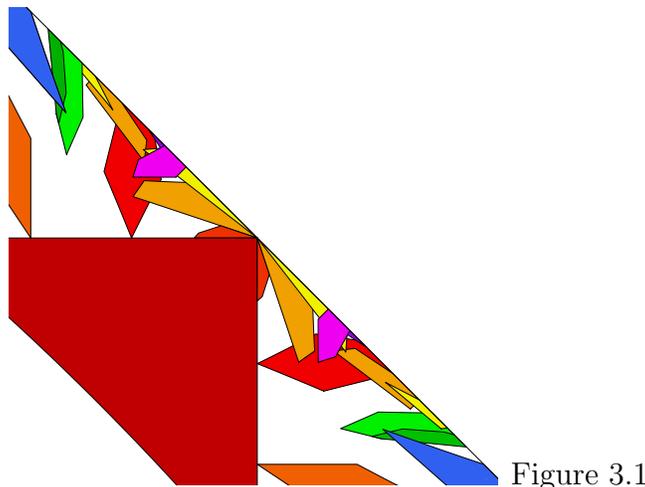

Figure 3.1

The central point in this figure represents the $45 - 45 - 90$ right triangle and the picture is symmetric about this point. The little gaps near the northwest and southeast corners of the picture correspond to the two points representing the $30 - 60 - 90$ triangle. The light blue tiles near these corners meet the diagonal precisely at these points and then stretch all the way off to the corners of the parameter space. The barely visible orange tiles near the edges also stretch backwards to the corners. (Compare Figure 3.5.) Looks promising, right?

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[1] Curt McMullen pointed out to me that the probable origin of these stories is a children's book called "Fortunately" by Remy Charlip.

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3.2 Bad Luck near the $30 - 60 - 90$ Triangle

Unluckily, there are these little gaps near the the points representing the $30 - 60 - 90$ triangle, between the green and blue tiles in the above picture. I tried to fill these gaps with a few more tiles and I eventually ran out of computing power. Then, inspired by my bad luck, I proved

Theorem 3.1 *Let $\epsilon > 0$ be given. Then there exists a triangle whose angles are all within ϵ of the $30 - 60 - 90$ triangle which has no periodic billiard path of length less than $1/\epsilon$.*

See [**S1**] for a proof. The bad triangles in Theorem 3.1 are all obtuse, since the Fagnano orbit works for all acute triangles and the order 6 orbit discussed in the introduction works for all right triangles.

Theorem 3.1 seems to put the brakes on the whole enterprise. Try as you like, you are not going to fill those little gaps just by searching for tiles and plotting them. You need an infinite number of orbit tiles.

3.3 Good Luck near the $30 - 60 - 90$ Triangle

Luckily, you look further and find the infinite family of words whose corresponding sequence of hexagonal paths is:

and so on. Below we show (in blue) the first 5 orbit tiles in the sequence. It looks like these tiles close down on the irritating gap, but unluckily the *tips* of the tiles pull in too quickly and converge to the $30 - 60 - 90$ point. This means that the union of the blue orbit tiles does not cover any other part of the gap. However, as it turns out, the part uncovered by the blue tiles is a very slim region, akin to the region between a straight line and a parabola tangent to that line.
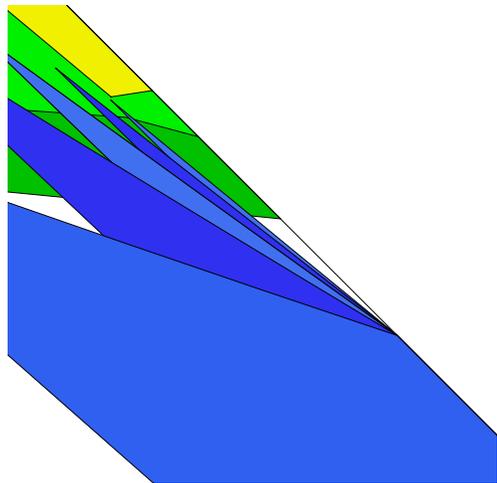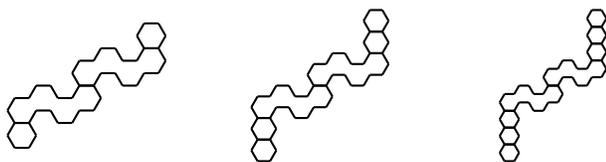
Figure 3.2

Luckily we can fill this slim region by an infinite family of green tiles, corresponding to the words whose hexagonal paths are

and so on. The blue and green families interlock and cover a neighborhood of the $30 - 60 - 90$ point, as suggested by the closeup below.
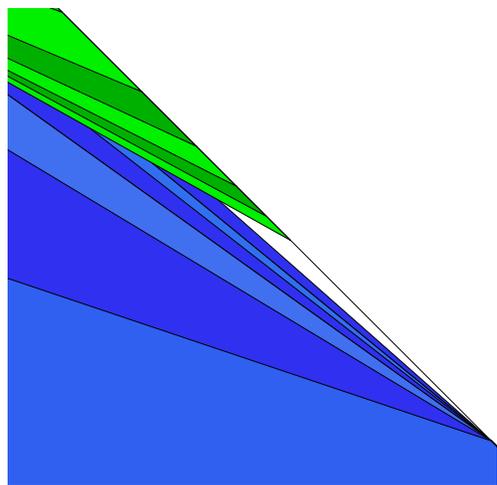

Figure 3.3

We give a proof in [**S1**] that these two families do indeed cover a neighborhood of the $30 - 60 - 90$ point.

It turns out that the $30 - 60 - 90$ points are the only trouble spots along the right-angle line, and we succeed in covering a neighborhood of the right angle line with orbit tiles. (Technically, we have just covered the obtuse side, but the rest is already taken care of.) Our covering result implies that every right triangle T has the following property: If T' is any other triangle sufficiently close to T then T' has a periodic billiard path.

3.4   Bad Luck near the Boundary

At first glance, it might seem to follow from what we have said above that there is some $\epsilon > 0$ such that a triangle has a periodic billiard path provided all its angles are less than $90 + \epsilon$. However, this stronger result requires not that we cover a neighborhood of the right angle line by orbit tiles, but actually that we cover a strip about the right angle line. Unluckily, we have the following result:

Lemma 3.2 (Boundary Demon) *Let $0 < t < 1$ be fixed and let $p_n \in \Delta$ be a sequence of points converging to $(0, t) \in \partial \Delta$. Then the length of the shortest periodic billiard path on the triangle T_{p_n} tends to infinity with n.*

Proof: (Sketch) The point $(0, t)$ represents the "degenerate triangle" whose angles are 0, $\pi t / 2$ and $\pi (1 - t) / 2$. Thus, if p_n is very close to $(0, t)$ then the triangle T_{p_n} is an obtuse triangle with the smaller angle very near 0 and the obtuse angle a definite amount greater than $\pi / 2$. Two sides of T are very long and the third side is short. Any billiard path in T must eventually hit the short side, but then at least one of its directions, forwards or backwards, must bounce down towards the far vertex, making a definite angle with the two long sides as it travels. This forces the billiard path to make a lot of bounces before returning. ♠
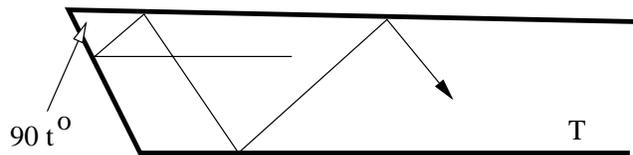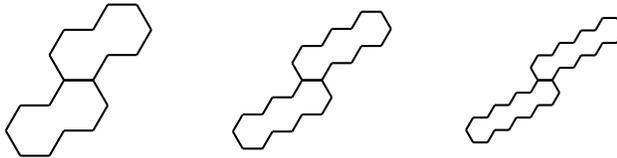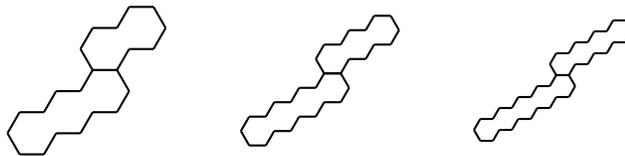

Figure 3.4

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3.5   Good Luck at the Boundary

Luckily, the orange tile creeping in around bottom edge of Figure 3.1 is part of an infinite family with paths:

and so on. Also, there is another helpful infinite family of tiles corresponding to the words with paths:

and so on. Figure 3.5 suggests how these families fit together to cover a neighborhood of the point $(0, 1)$ in Δ. The blue tile creeping in at the northwest corner is the same one as appeared in the southeast corner of Figure 3.1. You should try McBilliards, or else my applet, if you want to see how all the pictures fit together.
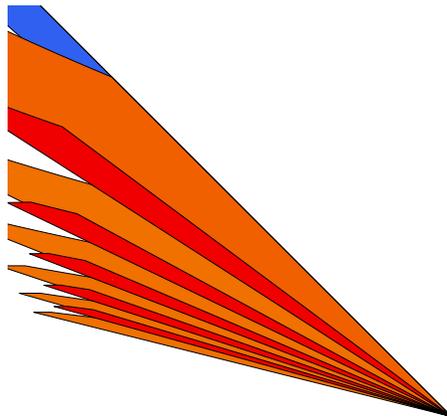
Figure 3.5

In [**S2**] we show that the union of the two families just described really go fill up a neighborhood (on the obtuse side) of the point $(0, 1)$. Thus, we solve the dilemma presented by the Boundary Demon Lemma for all values of

$(0, t)$ as long as t is sufficiently close to 1. That is we need an infinite family of tiles to cover a neighborhood of the point $(0, t)$ and the same family works as long as t is fairly close to 1.

It turns out that $t = 3/4$ is the cutoff. This corresponds to a (degenerated) triangle whose obtuse angle is

$$\pi - \left(\frac{\pi}{2} \times \frac{3}{4}\right) = \frac{5\pi}{8} \text{radians} = 112.5^o,$$

and whose small angle is 0 degrees. Once we take care of the trouble spots mentioned above, we can cover the rest of the strip S_{100} with about 400 orbit tiles. Here S_{100} is the strip consisting of points corresponding to obtuse triangles whose obtuse angle is at most 100 degrees. (If we use symmetry and just consider the left half of S_{100}, then we only use about 200 tiles.) We give a rigorous proof in [**S3**].

It seems that we could get up to 112.5 degrees, or quite near it, with a lot of extra work. However, after 112.5 degrees we have no answer to the Boundary Demon Lemma.

3.6 Bad Luck in the Interior

Unluckily, the Boundary Demon Lemma is not the only source of trouble for us. Let $I(k)$ be the obtuse isosceles triangle with small angle $k \times \pi/2$.. Then $I(k)$ corresponds to the point (k, k) in parameter space.

In a still unwritten paper, Pat Hooper proved that $I(2^{-n})$ has no stable periodic billiard path for $n = 2, 3, 4...$. (Masur's theorem says that these points do have periodic billiard paths, though they are always unstable according to Pat's result.) Given the special instability at the point $(2^{-n}, 2^{-n})$ we might expect trouble trying to cover the neighborhoods of these points. Indeed, experimentally it seems *each* of the points $(2^{-n}, 2^{-n})$ for $n = 2, 3, 4...$ is like the $30 - 60 - 90$ point: No neighborhood can be covered by a finite union of orbit tiles. Thus, to answer the triangular billiards problem in the affirmative, we probably need to use infinitely many different combinatorial billiard paths around *each* of the infinitely many points just mentioned.

Luckily, there seem to be infinite families which come to our rescue, at least for the few values we can easily test−i.e. $n = 2, 3, 4$. We think that proving this is within our grasp, but we haven't even tried it yet.

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3.7   Good Luck along the Isosceles Line

Pat found, with proof, a doubly infinite family of words whose corresponding orbit tiles cover all points on the obtuse side of the isosceles line except those of the form $I(1/k)$ with $k = 3, 4, 5....$ (He has yet to write this up.) The beginning of Pat's covering is shown in Figure 3.6.
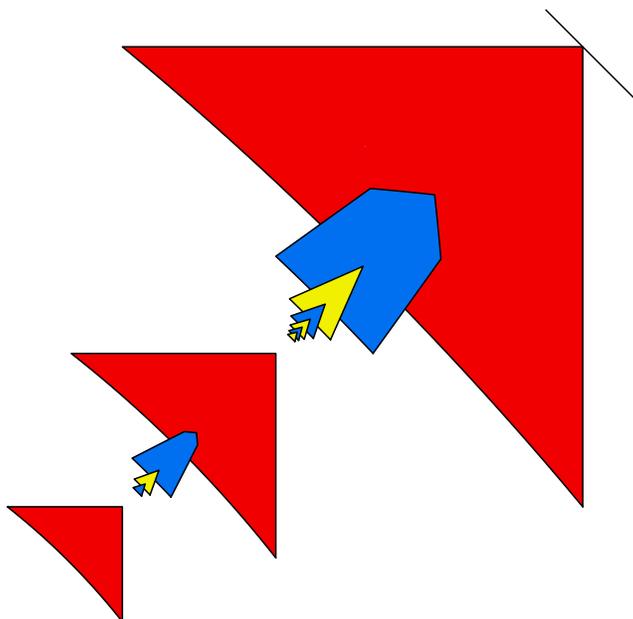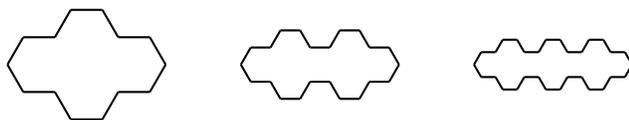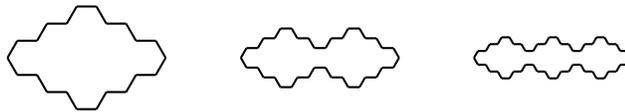
Figure 3.6

The series of red tiles was known to [**GSV**] and [**HH**] but not the rest. The first red, blue, yellow, blue, yellow, ... sequence limits to $(1/3, 1/3)$. The words are

and so on.

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The second series limits to the point $(1/4, 1/4)$—the first of the really troubling interior points mentioned in the previous section. The sequence of paths are:

and so on.

And so on. Thus, every isosceles triangle

$$T \notin \{T(1/k) |\ k = 3, 4, 5...\}$$

has the following property: If T' is sufficiently close to T then T' has a periodic billiard path.

Pat and I also managed to show that there are stable words which work for $T(1/k)$ as long as k is not a power of 2. The nature of the word seems to depend only on the odd part of k. Eerily, the number of words we find in our family which work for $T(1/k)$ is equal to the number which work for $T(1/2k)$. There seems to be a kind of renormalization going on here, but we have yet to figure it out.

Now, if we can manage to deal with the neighborhoods of the points of the form $(2^{-n}, 2^{-n})$, then we will know that any triangle sufficiently close to isosceles has a periodic billiard path. This result seems within our grasp, but it would be a ton of work—unless we get some new ideas.

After that our luck with covering runs out.

4   Other Results and Questions

4.1   Stability

One of the early observations we made playing with McBilliards is that the right triangles never seem to have stable periodic billiard paths. This was a result of [**GSV**] in the case that the small angle is $\pi/2n$, and Pat Hooper [**H1**] recently proved it for all right triangles. It would be nice to know which triangles, even which rational triangles, have stable periodic orbits. (I already mentioned that Pat Hooper proved that no stable word works for the parameter point $(2^{-n}, 2^{-n})$ for $n = 2, 3, 4....$ This result is an offspring of his right-angled result.)

Related to the stability result for right triangles, Pat proved that a stable word cannot work for both an acute and an obtuse triangle. Put in the language from the previous chapter, an orbit tile (corresponding to a stable word) cannot intersect points on both side of the right angle line. We could say that the right angle line confines all the orbit tiles to one side or the other. This result is part of a general theory, due to Pat, which helps decide from combinatorial data when a given rational line confines a given orbit tile. (This work is not yet written.)

4.2   The Shapes of Orbit Tiles

If you look at the figures in §3 you might be mislead into thinking that the orbit tiles are always (or at least usually) convex polygons. However, this is not the case. The edges are described by setting various trigonometric sums equal to 0. These sums have the form $\sum A_k \sin(B_k x + C_k y)$, where all the constant terms are integers. (The **tile analyzer** window of McBilliards will show you the formula for any edge of interest to you.)

In spite of the complicated formulas, the orbit tiles seem nearly convex, and this leads me to conjecture at least:

Conjecture 4.1 *An orbit tile is connected and simply connected.*

Pat and I agree about the simply connected part of the conjecture but disagree about the connected part. I think that the orbit tiles are always connected but Pat isn't convinced.

4.3 Veech Triangles and Bad Points

Say that a point $x \in \Delta$ is *bad* if no neighborhood of x can be covered by a finite number of orbit tiles. We've already mentioned that the points $(2^{-n}, 2^{-n})$ for $n = 2, 3, 4...$ seem to be bad points. It would be nice to classify all the bad points. What follows is a conjectural answer to this question.

Here we will give a quick definition of a *Veech polygon*. See [**V1**] and [**MT**] for much more detail. Recall from §2.1 that each polygon P has associated to it the translation surface Σ_P. An *affine automorphism* of Σ_P is defined to be a homeomorphism of Σ_P which, away from the singular points, is given locally by an affine transformation. The group of these automorphisms is sometimes called the *Veech group* and here we denote it by $V(\Sigma_P)$.

It turns out that the linear part of an element of the Veech group is indepdenent of the point at which it is measured, and also of determinant 1. Thus we get a homomorphism $\Phi : V(\Sigma_P) \to SL_2(\boldsymbol{R})$. P is called *Veech* if $\Phi(V_G)$ is a lattice in $SL_2(\boldsymbol{R})$. This is to say that the coset space $SL_2(\boldsymbol{R})/V_G$ is a space of finite volume, when $SL_2(\boldsymbol{R})$ is equipped with a left invariant volume form.

Veech [**V**] showed that billiards is especially nice on Veech polygons. One can get exact asymptotic formulas for the number of periodic billiard paths, and actually can completely classify the directions on the surface taken by periodic billiard paths. Ironically, all the trouble points we described in §3 correspond to Veech triangles. This makes us suspect that, while billiards is nice on Veech triangles, *billiards is especially bad near Veech triangles*. Here is one formulation of this principle.

Conjecture 4.2 *An interior parameter point is bad only if it is Veech.*

4.4 The Boundary Demon

In order to make further progress on the triangular billiards problem we need to deal with the points on the boundary. The Boundary Demon Lemma tells us that we need infinitely many orbit tiles in the neighborhood of any one of these points. Beyond the $t = 3/4$ cutoff mentioned in §3 we have no clue how to proceed.

It seems clear to us that we need to investigate billiard paths on very degenerate triangles. McBilliards is not equipped to do this, but we are (boundlessly) optimistic that McBilliards II is going to do a fine job of it once we get it off the ground.

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