The Optimal Paper Moebius Band

Richard Evan Schwartz *

August 24, 2023

Abstract

In this paper we prove that a smooth embedded paper Moebius band must have aspect ratio greater than $\sqrt{3}$. We also prove that any sequence of smooth embedded paper Moebius bands whose aspect ratio converges to $\sqrt{3}$ must converge, up to isometry, to an equilateral triangle of semi-perimeter $\sqrt{3}$. These results resolve the optimal paper Moebius band conjecture of Halpern and Weaver from 1977.

1 Introduction

1.1 Informal Discussion

Here is how to make a paper Moebius band: Cut out a strip of paper, give it one twist, then tape the ends together. For the sake of discussion suppose the width of your strip is always the same. If you start with a long strip, you can do the task easily. If you start with a shorter strip, you will find the task more difficult. How short can you go?

![Figure 1.1: The optimal paper Moebius band?](image)

*Supported by N.S.F. Grant DMS-2102802
Figure 1.1 shows a particularly beautiful example you can try to make. If you start with a $1 \times \sqrt{3}$ strip you can fold and (try to) tape it as indicated in Figure 1.1. The tape runs along the dotted line in the “inside” of the little triangular “wallet” you are making. I’ve never tried the taping, but in my mind it is taped.

Ideally speaking, the result is a perfectly flat Moebius band made out of paper. The final rotation in Figure 1.1 is not necessary bit it highlights a kind of “T-pattern” you see in the picture, made from the top edge and dotted line. (This kind of T-pattern will be the star of our show.) Even if you stop reading this article now, you should fold up a strip as in Figure 1 and ponder it. It is a thing of great beauty and elegance.

The question is: Can you do better than this example? Informally speaking, can you turn a $1 \times \lambda$ strip into a paper Moebius band if $\lambda < \sqrt{3}$? That is what this paper is about.

1.2 Formal Definitions

In order to answer this kind of question we have to be more formal about what we are doing. Formally speaking, an embedded paper Moebius band of aspect ratio $\lambda$ is a smooth isometric embedding $I : M_\lambda \to \mathbb{R}^3$, where $M_\lambda$ is the flat Mobius band

$$M_\lambda = ([0, 1] \times [0, \lambda]) / \sim, \quad (x, 0) \sim (1 - x, \lambda)$$ (1)

An isometric mapping is a map whose differential is an isometry. The map is an embedding if it is injective. The map need not be an isometry with respect to the ambient metric on $\mathbb{R}^3$. It is, however, always distance non-increasing. The image need not lie flat in general.

Technically, the example in Figure 1.1 does not count as a paper Moebius band according to our definition because the map is not smooth. However, it is the limit of smooth examples. That is, you can approximate it as closely as you like by maps which are paper Moebius bands in the formal sense above. Why bother with smooth maps? Well, if you just look at ways of folding paper up to make a Moebius band you can get all kinds of weird examples that do not count. For instance, you could take a square, fold it like an accordion into a thin strip, twist, then tape. This monster is not approximable by smooth examples. The smooth formalism rules out this kind of garbage.
I learned about these objects from the beautiful expository article by Dmitry Fuchs and Sergei Tabachnikov. See [FT]. That article (a chapter in their book) is a great introduction to this paper. One interesting thing you can learn about in [FT] is that you can indeed make shorter examples if you allow for non-injective maps. These things, however, are not really physical. You can’t make them out of a strip in the real word, not even in an approximate sense. I will not consider these immersed examples in this paper.

Let \( \lambda_0 \) be the infimal value of \( \lambda \) such that there exists a smooth embedded paper Moebius band of aspect ratio \( \lambda \). Halpern and Weaver [HW] prove that \( \lambda_0 \in [\pi/2, \sqrt{3}] \). Halpern and Weaver conjecture that in fact \( \lambda_0 = \sqrt{3} \). In other words, you can’t do better than the example in Figure 1.1. The lower bound of \( \pi/2 \) also works for the immersed examples, and indeed the trouble with the Halpern-Weaver conjecture is that people have not really figured out how to use the embedding property to their advantage.

1.3 Main Results

In this paper we will prove the Halpern-Weaver Conjecture.

**Theorem 1.1 (Main)** A smooth embedded paper Moebius band has aspect ratio greater than \( \sqrt{3} \).

Our method easily characterizes the example in Figure 1.1 as the only “best limit”. That is,

**Theorem 1.2 (Triangular Limit)** Let \( \{M_n\} \) be a sequence of embedded paper Moebius bands whose aspect ratio converges to \( \sqrt{3} \). Then, up to isometry, \( M_n \) converges in the Hausdorff metric to an equilateral triangle of semiperimeter \( \sqrt{3} \).

For simplicity (and out of laziness) I am establishing a pretty weak kind of convergence – Hausdorff convergence. See §3.3 for a definition. The interested reader can easily extract something much stronger.

1.4 Context

The Moebius band question in a sense goes back a long time, and it is related to many topics. The early paper [Sa] proves rigorously that smooth paper
Moebius bands exist. (See [HF] for a modern translation to english.) The paper [CF] gives a general framework for the differential geometry of developable surfaces. Some authors have discussed optimal shapes for Moebius bands from other perspectives, e.g. algebraic or physical. See, e.g. [MK]. The Moebius band question has connections to origami. See e.g. the beautiful examples of isometrically embedded flat tori [AHL]. It is also related to the main optimization question from geometric knot theory about rope knot length. See e.g. [CKS].

Now comes the embarrassing part of the story. In my paper [S1] I improved the lower bound from $\pi/2$ to a somewhat larger and complicated number $\lambda_1 \in (\pi/2, \sqrt{3})$. In [S1] I first prove a geometric-topological result about objects I call $T$-patterns. The result I am talking about is formulated in this paper as Lemma T in §2. I deduce easily from Lemma T that $\lambda_1 > \phi$, the golden ratio. Then I give a more subtle argument, based on an optimization problem, that $\lambda_1$ is somewhat larger than $\phi$. I realized recently that I made a subtle error in setting up my optimization problem. I will explain the error in terms of a little quiz:

**Q:** Suppose you cut $M_\lambda$ open along an embedded line segment which joins two points in $\partial M_\lambda$ (and otherwise lies in the interior.) What kind of shape is it?

A. parallelogram
B. trapezoid?

The answer is B. In [S1] I mistakenly based an argument on the A answer. Somehow I made the mistake early on and then developed a blind spot. My stupid mistake did not affect the proof of Lemma T. It only affected the implementation of the optimization problem. My final bound in [S1], the conclusion of the erroneous optimization problem, is not justified. I also wrote (but didn’t publish) sequel papers [S2] and [S3]. The same mistake affects [S2] but not the very speculative [S3].

The good news is that when I re-ran the optimization problem correctly, using the trapezoid answer, I got $\sqrt{3}$ on the nose! The new proof is quick and clean, and so robust that the Triangular Limit Theorem follows easily. In this paper I re-prove Lemma T and combine this with the corrected optimization calculation (Lemma A in §2) to prove the conjectures. This paper subsumes every thing I have done on the paper Moebius band.

Dear reader, I hope that the discussion so far has not disqualified me, in your eyes, from trying to prove the Halpern-Weaver Conjecture. You might
even be wondering, as am I, how I managed to get through my high school geometry class. All I can say is that I hope to have this paper looked at very carefully. Since I am claiming a proof of a long-standing conjecture, I wrote this paper in a modular way that makes it possible to check in small pieces. The Main Theorem is a quick consequence of Lemmas A and T. Lemma T is a quick consequence of Lemmas B and C. And so on. The lemmas at the end of this directed logic graph have short proofs. The proof of the Triangular Limit Theorem amounts to inspecting the proof of the Main Theorem for a minimizing sequence.

1.5 The Logic Graph

Here is a list of all the lemmas in the paper and how they imply the Main Theorem. The brackets indicate where things are proved, or at least reduced to smaller lemmas.

![Logical Dependence Graph]

Figure 1.2: The logical dependence graph for the Main Theorem

1.6 Acknowledgements

I would like to thank Robert Connelly, Dan Cristofaro-Gardiner, Dmitry Fuchs, Javier Gomez-Serrano, Jeremy Kahn, Stephen D. Miller, and Sergei Tabachnikov for helpful discussions about this problem.
2 Main Theorem from Lemmas A and T

An embedded paper Moebius band $M = I(M_\lambda)$ is a developable (ruled) surface (with boundary). This means that

- $M_\lambda$ has a foliation by straight line segments which meet $\partial M_\lambda$ transversely at their endpoints. We call these segments the \textit{pre-bends}.

- The map $I$ is an isometry when restricted to each pre-bend. This means that the image of a bending line under $I$ is a line segment of the same length. We call these images the \textit{bends}.

The bends are sometimes called the ruling lines of $M$.

Every bend has an \textit{extending line}, namely the line that contains it. Given $M$ as above, we say that a \textit{T-pattern} is a pair of bends whose extending lines are coplanar and perpendicular. Figure 3.2 in §3 shows what this looks like. In §3 I will prove:

**Lemma 2.1 (A)** Suppose that $M = I(M_\lambda)$ is an embedded paper Moebius band with a \textit{T-pattern}. Then $\lambda > \sqrt{3}$.

In §4-7 I will (re)prove

**Lemma 2.2 (T)** Suppose that $M = I(M_\lambda)$ is an embedded paper Moebius band and $\lambda < 7\pi/12$. Then $M$ has a \textit{T-pattern}.

The Main Theorem follows immediately from Lemma A and Lemma T once we verify that $\sqrt{3} < 7\pi/12$.

**Remarks:**

1. Our proof here of Lemma T is similar to the proof given in [S1] but the argument is more polished and the organization is different.
2. Call the \textit{T-pattern} \textit{embedded} if the two bends are disjoint. A \textit{T-pattern} would make sense even for an immersed paper Moebius band, though it might not be an embedded \textit{T-pattern}. For instance, the examples considered in [FT] all have non-embedded \textit{T-patterns}. Of course, every \textit{T-pattern} in the embedded case is itself embedded. Lemma A also holds for immersed paper Moebius bands with embedded \textit{T-patterns}. The proof is the same. Lemma T also holds for immersed paper Moebius bands. The proof is again the same.
3 Proof of Lemma A

3.1 A Result about Triangles

We begin with a helpful lemma. We consider triangles with horizontal base. The semi-perimeter \( s(\Delta) \) of a triangle \( \Delta \) is half of its perimeter. Let \( n(\Delta) \) denote the sum of the lengths of the non-horizontal edges of \( \Delta \).

Lemma 3.1 (A1) Suppose that \( \Delta \) is a triangle having base at least \( b \) and height \( h \geq 1 \). Then

\[
\begin{align*}
n(\Delta) &\geq \sqrt{b^2 + 4}, \\
s(\delta) &\geq \frac{b}{2} + \frac{\sqrt{b^2 + 4}}{2}.
\end{align*}
\]

Equality happens if and only if \( \Delta \) is isosceles and has base \( b \) and height 1.

Proof: This is an extremely well known result. See Figure 3.1.

Let \( b' \geq b \) be the base. The result for \( n(\Delta) \) implies the result for \( s(\Delta) \), so we only consider \( n(\Delta) \). Let \( v_1 \) and \( v_2 \) be the vertices of the base. Let \( L \) be the horizontal line through the third vertex \( v_3 \). Let \( v_2' \) be the point you get reflecting \( v_2 \) through \( L \). Then, by symmetry and the Pythagorean Theorem,

\[
n(\Delta) \geq \|v_1 - v_2'\| = \sqrt{(b')^2 + (2h)^2} \geq \sqrt{b^2 + 4},
\]

In the case of Equality, \( b' = b \) and \( h = 1 \) and \( v_1, v_3, v_2' \) are collinear, meaning that \( \Delta \) is as claimed. ♠
3.2 The Main Argument

We have \( I : M_\lambda \to M \). We often set \( X' = I(X) \) for any relevant set \( X \).

Let \( B' \) and \( T' \) be the pair of disjoint bends comprising a \( T \)-pattern of \( M \). Let \( B \) and \( T \) be the corresponding pre-bends in \( M_\lambda \). Either the line extending \( B' \) is disjoint from \( T' \) or the line extending \( T' \) is disjoint from \( B' \). (Or both). We label so that the line extending \( T' \) is disjoint from \( B' \). We rotate so that \( B' \) and \( T' \) are respectively vertical and horizontal segments in the \( XY \)-plane and \( B' \) is below the line extending \( T' \). It might or might not happen that the line extending \( B' \) intersects \( T' \). See Figure 3.2. Let \( \Delta \) be the triangle whose vertices are the vertices of \( T' \) and the lower vertex of \( B' \), as shown in Figure 3.2.

\[
\begin{align*}
\text{Figure 3.2: The } T \text{-pattern and the triangle } \Delta. \\
\end{align*}
\]

The base of \( \Delta \) is \( \ell(T) = \ell(T') \) and the height of \( \Delta \) is at least \( \ell(B) = \ell(B') \). Here and below \( \ell \) is the length function. Both these quantities are at least 1. The crucial application of the Lemma A1 is this:

\[
\ell(T) \geq \sqrt{1 + t^2} \implies n(\Delta) \geq \sqrt{5 + t^2}, \quad s(\Delta) \geq \frac{\sqrt{1 + t^2}}{2} + \frac{\sqrt{5 + t^2}}{2}. \tag{2}
\]

We also note that

\[
2\lambda = \ell(\partial M_\lambda) \geq \ell(I(\partial M_\lambda)) \geq 2s(\Delta). \tag{3}
\]

The starred inequality comes from the fact that \( I \) is distance non-increasing with respect to the ambient metric on \( R^3 \). In fact this is equality for isometric embeddings like \( I \), but writing an inequality requires less thought. The last inequality comes from the fact that \( I(\partial M_\lambda) \) contains all vertices of \( \Delta \).

**Key Remark:** In particular, if \( t \geq 1/\sqrt{3} \) then \( \lambda > \sqrt{3} \). We can’t get equality because in the case of equality \( t = 1/\sqrt{3} \) and \( \Delta \) is isosceles of height 1. This can’t happen because \( T' \) and \( B' \) are disjoint.
Now we cut $M_\lambda$ open along $B$. The result, $\tau$, is a bilaterally symmetric trapezoid. See the left side of Figure 3.3.

![Figure 3.3: The trapezoid $\tau$ (left) and the T-pattern (right).](image)

We normalize by isometries so that the edges of $\partial \tau$ coming from $\partial M_\lambda$ are vertical and so that the images of the vertices $u,v,w,x$ are as indicated. The red curves at the right are smooth space curves, but their endpoints lie in the XY-plane, the plane containing our T-pattern. The numbers $(-t)$ and $b$ are respectively the slopes of $T$ and the bottom choice of $B$. We are showing the case when $t,b \geq 0$. This case is easiest on the eyes of the reader who wants to geometrically see Equation 4, an equation that holds in all cases. We have

$$\ell(H_1) = \ell(D_1) - t - b, \quad \ell(H_2) = \ell(D_2) - t + b.$$  \hspace{1cm} (4)

Note that $\ell(T) = \sqrt{1 + t^2}$. Hence

$$\lambda = \frac{\ell(D_1) + \ell(D_2) + \ell(H_1) + \ell(H_2)}{2} = \ell(D_1) + \ell(D_2) - t \geq^*$$

$$\ell(D'_1) + \ell(D'_2) - t \geq n(\Delta) - t \geq \sqrt{5 + t^2} - t.$$  \hspace{1cm} (5)

The starred inequality works as in Equation 3. The last inequality comes from Equation 2. If $\lambda \leq \sqrt{3}$ then we get the following chain of implications:

$$\sqrt{5 + t^2} - t \leq \lambda \leq \sqrt{3}, \quad \Rightarrow \quad \sqrt{5 + t^2} \leq \sqrt{3} + t \quad \Rightarrow \quad t \geq 1/\sqrt{3}.$$

You get the last implication by squaring both sides of the previous implication and simplifying. But then, by the Key Remark, $\lambda > \sqrt{3}$. This is a contradiction. Hence $\lambda > \sqrt{3}$. This completes the proof of Lemma A.
3.3 Proof of the Triangular Limit Theorem

The *Hausdorff distance* between two compact subsets of $\mathbb{R}^3$ is defined to be the infimal $\epsilon$ such that each is contained in the $\epsilon$-tubular neighborhood of the other. This is a convenient metric to use when speaking about the convergence of subsets.

Suppose we have a sequence $\{M_n\}$ of embedded paper Moebius bands with $\lambda_n \to \sqrt{3}$. We run the constructions above for each one. Looking at Equation 2 and using the uniqueness of the minimum in the Lemma A1, we have $t_n \to 1/\sqrt{3}$ and $b_n \to 0$ and $\Delta_n$ converges up to isometry to an equilateral triangle $\Delta$ of semi-perimeter $\sqrt{3}$. Finally, the “slack” in Equation 3 is tending to 0. Hence $\ell(\partial M_n) \to 2s(\Delta)$. Hence $\partial M \to \partial \Delta$ up to isometry. But then $M_n \to \Delta$ up to isometry.

This completes the proof.
4 Lemma T from Lemmas B and C

4.1 Gentle Paper Moebius Bands

Let $M$ be a smooth paper Moebius band. We say that a bend of $M$ is gentle if it makes an angle of less than $\pi/4$ with the horizontal planes. We call $M$ gentle if all its bends are gentle. We say that $M$ is in horizontal position if it has a bend parallel to the $X$-axis and another one parallel to the $Y$-axis. Certainly we can rotate $M$ so that it has this property.

**Lemma 4.1 (B)** If $M$ has aspect ratio less than $7\pi/12$ and $M$ is in horizontal position then $M$ is gentle.

4.2 Polygonal Moebius Bands

As in [S1] we find it easier to work in the polygonal category and for added flexibility we allow affine maps which are not isometries.

We represent $M_\lambda$ as the quotient of a bilaterally symmetric trapezoid $\tau$, as shown in Figure 4.1. A transverse triangle in $M_\lambda$ is one having 1 edge $\partial M_\lambda$ and 2 edges with their vertices in $\partial M_\lambda$ as shown in Figure 4.1. We call the edge in $\partial M_\lambda$ the ridge. We call the vertex opposite the ridge the apex. We define a pre-bend of a transverse triangle to be a segment joining the apex to the ridge.

![Figure 4.1: Transverse triangulation and pre-bend foliation](image-url)

A transverse triangulation of $M_\lambda$ is a partition of $\tau$ into transverse triangles. Each transverse triangle has a foliation by pre-bends, and these piece together to give the pre-bend foliation of $\tau$. Say that polygonal Moebius band is a continuous map $I : M_\lambda \rightarrow \mathbb{R}^3$ that is piecewise affine with respect to
some transverse triangulation of \( M_\lambda \). We always cut open along a pre-bend to get the kind of trapezoid representation shown in Figure 4.1. The map \( I \) should be injective on each transverse triangle but not necessarily globally injective.

We define the bends to the images of the pre-bends under \( I \). As in the smooth case, a \( T \)-pattern in a polygonal Moebius bend is a pair of bends having perpendicular and coplanar extending lines. We say that a polygonal Moebius band is gentle if all its bends are gentle.

**Lemma 4.2 (C)** Any gentle polygonal Moebius band has a \( T \)-pattern.

### 4.3 The Main Argument

Suppose that \( I : M_\lambda \to M \) is a smooth paper Moebius band and \( \lambda < 7\pi/12 \). Then we put \( M \) in horizontal position. By Lemma B, \( M \) is gentle.

Take a finite list \( \beta_1, ..., \beta_n \) of pre-bends in \( M_\lambda \), with \( \beta_1 \) being the first bend and \( \beta_n \) being the last. Call this a mesh. These pre-bends divide \( M_\lambda \) into thin trapezoids. We add diagonals to get a transverse triangulation. We use the values of \( I \) on the vertices of the transverse triangles to define \( I' \). By construction \( I \) and \( I' \) agree on \( \beta_1, ..., \beta_n \). We call these bends the shared bends. Hence \( I' : M_\lambda \to M' \) is a polygonal Moebius band.

If we take our mesh fine enough then by compactness and continuity \( M' \) will be gentle. But then \( M' \) has a \( T \)-pattern \((u', v')\). Once we take our mesh fine enough we can make \( u' \) and \( v' \) as close metrically as we like to shared bends \( u \) and \( v \). Now we can say that \( M \) has a pair of bends whose extending lines are as close as we like to being perpendicular and coplanar. We can take a limit of these near \( T \)-patterns on \( M \) to get an actual \( T \)-pattern on \( M \). This completes the proof.
5 Proof of Lemma B

5.1 The Main Argument

In [FT], Tabachnikov and Fuchs explained a construction which proves the bound $\lambda \geq \pi/2$ for immersed smooth paper Moebius bands. We will use the same construction here, but with different inputs.

**Lemma 5.1 (B1)** Let $\gamma$ be a smooth curve which lies outside the open unit ball. Suppose $\gamma$ starts on the $X$-axis, goes to the $Y$-axis, then returns to the $X$-axis. Suppose also a point of $\gamma$ is a vector which makes an angle of $\pi/4$ with the $XY$-plane. Then $\gamma$ has length at least $7\pi/6$.

**Lemma 5.2 (B2)** Suppose $L_1$ and $L_2$ are two directed line segments in $\mathbb{R}^3$. We let $d_t$ be the distance between the tail points of $L_1$ and $L_2$ and let $d_h$ be the distance between the head points. Then we do not increase $d_t + d_h$ if we translate $L_1$ and $L_2$ so that their tail points are both at the origin.

**Proof:** We rotate to that $L_1$ starts at the origin and lies in the $X$-axis. If we move the tail of $L_2$ towards $L_1$ as directly as possible we decrease $d_h + d_t$, by the Pythagorean Theorem. So, we can assume that the tail of $L_2$ lies on $L_1$. If we now slide the tail of $L_2$ to the origin we decrease $d_h + d_t$ by the triangle inequality. ♠

Let $I : M_\lambda \to M$ be the smooth paper Moebius band with $\lambda < 7\pi/12$. Here $M$ is in horizontal position. Suppose that $M$ is not gentle. Then $M$ contains a bend which makes an angle of $\pi/4$ with the $XY$-planes. We arrange so that the initial bend of $M$ is parallel to the $X$ axis. Some other bend is parallel to the $Y$ axis. We orient the bends of $M$ so that they vary continuously away from the initial bend. When they go all the way around $M$ the orientation has switched. We produce a new curve $\gamma$ by translating all the bends so that their tails are at the origin. Here $\gamma$ connects a pair of antipodal points in $\mathbb{R}^3$ and stays outside the open unit ball. Moreover, $\gamma$ satisfies the hypotheses of Lemma B1.

By taking successively fine polygonal approximations and applying Lemma B2 we see that $\ell(\gamma) \leq \ell(\partial M)$. Hence, by Lemma B1,

$$2\lambda = \ell(\partial M_\lambda) = \ell(\partial M) \geq \ell(\gamma) \geq 7\pi/6.$$  \hspace{1cm} (6)

This contradiction completes the proof.
5.2 Proof of Lemma B1

A sphere curve is a piecewise smooth curve contained in the unit sphere.

**Lemma 5.3 (B11)** A sphere curve which connects \((1,0,0)\) to \((0,1,0)\) and contains a point of the form \((u,v,\sqrt{2}/2)\) has length at least \(2\pi/3\).

**Proof:** Let \(A = (1,0,0)\) and \(B = (u,v,\sqrt{2}/2)\) and \(C = (0,1,0)\). The shortest path connecting \(A\) to \(B\) to \(C\) is a bigon has length at least

\[
\cos(A \cdot B) + \cos(B \cdot C) = \cos(u) + \cos(v) \geq 2\cos(1/2) = 2\pi/3.
\]

The starred inequality comes from the fact that the minimum, subject to the constraint \(u^2 + v^2 = 1/2\), occurs at \(u = v = 1/2\). ♠

**Lemma 5.4 (B12)** Let \(\gamma\) be a curve that lies outside the open unit ball. The radial projection of \(\gamma\) onto the unit sphere is no longer than \(\gamma\).

**Proof:** This is an extremely well known result. By polygonal approximation, it suffices to prove this when \(\gamma\) is a polygon. But then it suffices to prove this in the case when \(\gamma\) is a line segment. The extreme case is for line segments which are tangent to the unit sphere. In this case, the result comes from the fact that \(\theta < \tan(\theta)\) for \(\theta \in (0,\pi/2)\). ♠

Now we deduce Lemma B1. By Lemma B12 it suffices to prove this when \(\gamma\) is a sphere curve. In that case we can write \(\gamma = \gamma_1 \cup \gamma_2\) where \(\gamma_1\) joins \((\pm 1,0,0)\) to \((0,\pm 1,0)\) and \(\gamma_2\) is (up to signs) as in Lemma B1. But then \(\gamma_1\) has length at least \(\pi/2\) and by Lemma B11 \(\gamma_2\) has length at least \(2\pi/3\) and the sum of these is at least \(7\pi/6\). This completes the proof.
6 Lemma C from Lemmas C1 and C2

6.1 Reduction of Lemma C

Let $J : M_\lambda \to W$ be a polygonal Moebius band. A direction vector for a bend of $M$ is one which is parallel to the bend. We call two bends partners if their direction vectors are orthogonal. The bends themselves might lie on skew lines.

We identify the pre-bends of $M_\lambda$ with the circle $\mathbb{R}/\lambda \mathbb{Z}$ as follows: We map each pre-bend to its intersection with the centerline of $M_\lambda$, and this is a copy of $\mathbb{R}/\lambda \mathbb{Z}$. We call two points $r, s \in \mathbb{R}/\lambda \mathbb{Z}$ partners if the bends $J(\beta_r)$ and $J(\beta_s)$ are partners. We let $\Omega \subset (\mathbb{R}/\lambda \mathbb{Z})^2$ be the subset of partner points. We call $W$ nice if $\Omega$ is a piecewise smooth 1-manifold – i.e. a finite disjoint union of piecewise smooth embedded loops.

**Lemma 6.1 (C1)** If $W$ is a nice and gentle paper Moebius band then $W$ has a $T$-pattern.

Lemma C1 is really the heart of our overall proof. In §7 we prove

**Lemma 6.2 (C2)** Let $M'$ be a polygonal Moebius band. We can find a linear transformation $\phi$ as close as we like to the identity so that $\phi(M')$ is nice.

Now we deduce Lemma C. We know that $M'$ is gentle. If we choose any linear $\phi$ close enough to the identity then $W = \phi(M')$ is also gentle. This is just compactness. By Lemma C2 we can choose such a $\phi$ to that $W$ is both nice and gentle. So, we have a $T$-pattern on $W$ by Lemma C1. The image of this $T$-pattern under $\phi^{-1}$ is as close as we like to being a $T$-pattern on $M'$. Taking a limit we get a $T$-pattern on $M'$. This completes the proof.

6.2 Lemma C1 from Lemmas C11 and C12

Let $W$ be the gentle polygonal Moebius band. Suppose $u, v$ are are partner bends. There is a unique pair of parallel planes $U, V$ such that $u \subset U$ and $v \subset V$. We call $U$ or $V$ auxiliary planes. If $U = V$ then we have a $T$-pattern.

**Lemma 6.3 (C11)** No associated plane contains a vertical line.
Proof: We argue by contradiction. Let \((U, V)\) be a pair of associated planes which supposedly contain the vertical direction. Let \((u, v)\) be the corresponding partner bends. Let \(L_u\) and \(L_v\) be the lines parallel to \(u\) and \(v\) through the origin. Let \(L^*\) be the line through the origin perpendicular to both \(L_u\) and \(L_v\). Since \(L^*\) is perpendicular to all vectors in \(U\) and \(V\), we know that \(L^*\) is perpendicular to \((0, 0, 1)\). Hence \(L^*\) lies in the \(XY\)-plane. We might as well rotate about the \(Z\)-axis so that \(L^*\) is the \(Y\)-axis. But then \(L_u\) and \(L_v\) lie in the \(XZ\) plane. So we have 2 lines in the \(XZ\) plane which are perpendicular and both make an angle of less than \(\pi/4\) with the \(XY\) plane. This is impossible. ♠

Let \(\Omega\) be the partner set for \(W\). By hypothesis, \(\Omega\) is a piecewise smooth 1-manifold, a subset of the open cylinder

\[ \Upsilon = (\mathbb{R}/\lambda \mathbb{Z}) - \text{diagonal}. \]  

We call \(\Omega\) odd if \(\Omega\) represents the nontrivial element of the homology group \(H_1(\Upsilon; \mathbb{Z}/2) = \mathbb{Z}/2\).

Lemma 6.4 (C12) \(\Omega\) is odd.

Let us deduce Lemma C1 from these results. Consider \(f: \Upsilon \to \Upsilon\) given by \(f(a, b) = (b, a)\). This map is an involution and an isomorphism on \(H_1(\Upsilon; \mathbb{Z}/2)\). By construction \(f\) permutes the components of \(\Omega\). Since \(\Omega\) is odd, there must be some component \(\omega\) of \(\Omega\) such that \(f(\omega) = \omega\). This means that we can find a continuous path in \(\Omega\), namely a suitable arc of \(\omega\), such that \(f\) swaps the endpoints of our path. Call this the swapping path.

Let \((u_t, v_t)\) be the continuous path of pairs of partner bends in \(W\) corresponding to the swapping path. Let \((U_t, V_t)\) be the corresponding continuous path of auxiliary planes. Let \([U_t]\) and \([V_t]\) be the points where these planes intersect the \(Z\)-axis. These points are well-defined and vary continuously by Lemma C2 and compactness. By construction these intersection points switch places as we traverse the swapping path. Hence there is some parameter \(s\) for which \([U_s] = [V_s]\). But then, because \(U_s\) and \(V_s\) are parallel, \(U_s = V_s\). But then \(u_s\) and \(v_s\) make a \(T\)-pattern in \(W\).

6.3 Proof of Lemma C12

We use the notation from the previous section. We let \(\overline{\Upsilon}\) be the compactification of \(\Upsilon\) obtained by adding 2 boundary components. The point \((a, b)\)
lies near one boundary component if \( b \) lies just ahead of \( a \) in the cyclic order coming from \( \mathbb{R}/\lambda \mathbb{Z} \). The point \((a, b)\) lies near the other boundary component if \( b \) lies just behind of \( a \) in the cyclic order coming from \( \mathbb{R}/\lambda \mathbb{Z} \). We get a path \( \gamma \) which runs from one boundary component of \( \mathcal{Y} \) to the other by holding \( a \) fixed and varying \( b \) all the way around from ahead of \( a \) to just behind \( a \). Let \( \gamma \) be such a path. If we pick \( a \) generically then \( \gamma \) intersects \( \Omega \) transversely. In particular, \( \gamma \) intersects \( \Omega \) a finite number of times.

**Lemma 6.5 (C121)** If \( \gamma \) intersects \( \Omega \) an odd number of times then \( \Omega \) is odd.

**Proof:** If \( \omega \) is a component of \( \Omega \) then \( \gamma \) intersects \( \omega \) an even number or an odd number of times, depending respectively on whether \( \omega \) is trivial or nontrivial in \( H_1(\mathcal{Y}; \mathbb{Z}/2) \). Hence \( \Omega \) has an odd number of homologically non-trivial components. Hence \( \Omega \) is odd. ♠

**Lemma 6.6 (C122)** \( \gamma \) intersects \( \Omega \) an odd number of times.

**Proof:** We give an orientation to the pre-bend \( \beta_a \) corresponding to \( a \). This gives an orientation to the bend \( I(\beta_a) \). We attempt to give a continuous orientation to the bends \( I(\beta_b) \), knowing that this is impossible because we are on a Moebius band. But we can almost do this. When \( b \) is just ahead of \( a \) we orient \( I(\beta_b) \) so that it points almost in the same direction as \( I(\beta_a) \). After we have gone all the way along \( \gamma \) until \( b \) is just behind \( a \), the bend \( I(\beta_b) \) points almost in the opposite direction as \( I(\beta_a) \). This means that the bends are partners an odd number of times along the path. Hence \( \gamma \) intersects \( \Omega \) an odd number of times. ♠

Lemma C12 is an immediate consequence of Lemmas C121 and C122.
7 Proof of Lemma C2

7.1 Lemma C2 from Lemma C21

Say that an anchored line in $\mathbb{R}^3$ is a line through the origin. Say that an anchored plane is a plane in $\mathbb{R}^3$ through the origin. Let $\Pi_1$ and $\Pi_2$ be anchored planes. A sector of the plane $\Pi_j$ is a set linearly equivalent to the union of the $(++)$ and $(- -)$ quadrants in $\mathbb{R}^2$. Let $\Sigma_j \subset \Pi_j$ be a sector. The boundary $\partial \Sigma_j$ is a union of two anchored lines crossing at the origin.

**Lemma 7.1 (C21)** Suppose $\Pi_1$ and $\Pi_2$ are not perpendicular and no line of $\partial \Sigma_1$ is perpendicular to a line of $\partial \Sigma_2$. Then the set of perpendicular pairs of anchored lines $(L_1, L_2)$ with $L_j \in \Sigma_j$ for $j = 1, 2$ is either empty or diffeomorphic to a closed line segment. If $(L_1, L_2)$ corresponds to an endpoint then exactly one of these lines lies in the boundary of its sector.

We deduce Lemma C2 from Lemma C21. Say that an image triangle of a polygonal Moebius band $M = I(M_\lambda)$ is the image under $I$ of one of the triangles in the transverse triangulation. Each image triangle $\mu$ defines a sector. The anchored plane containing the sector is parallel to the one containing $\mu$. The boundary of the sector is the union of the two anchored lines parallel to the apex-incident edges of $\mu$.

Now we consider an affine adjustment using a linear map $\phi$. Since we just need to destroy finitely many perpendicularity relations we can take $\phi$ as close as we like to the identity such that every pair of sectors associated to $\phi \circ I$ satisfies the hypotheses of Lemma C21. We also call the new polygonal Moebius band $M$ and we show that it is nice.

Let $\Omega$ be the partner set. The space $(\mathbb{R}/\lambda \mathbb{Z})^2$ is tiled by special rectangles corresponding to pairs of transverse triangles. By Lemma C21 any nontrivial intersection of $\Omega$ with a special rectangle is a special segment with endpoints in the relative interiors of edges of $\partial R$. Any two special segments have disjoint interiors because their interiors lie in different rectangle interiors. Let $s_1$ be any special segment, contained in a special rectangle $R_1$. Let $v$ be an endpoint of $s_1$. Let $R_2$ be the special rectangle adjacent to $R_1$ and sharing the edge containing $v$. Since $\Omega \cap R_2$ is nonempty, this intersection is another special segment $s_2$ which also contains $v$. In this manner, $s_1$ continues across $v$ to a unique special segment $s_2$.

These properties, disjoint interiors and continuance across vertices, show that $\Omega$ is an embedded piecewise smooth 1-manifold.
7.2 Proof of Lemma C21

Let Π₁ and Π₂ be anchored planes.

**Lemma 7.2 (C21)** Suppose that Π₁ and Π₂ are not perpendicular. The set of perpendicular anchored lines \((L_1, L_2)\) with \(L_j \in Π_j\) for \(j = 1, 2\) is diffeomorphic to a circle.

**Proof:** For each anchored line \(L_1 \in Π_1\) the line \(L_2 = L_1^\perp \cap Π_2\) is the unique choice anchored line in Π₂ which is perpendicular to \(L_1\). The line \(L_2\) is a smooth function of \(L_1\). So, the map \((L_1, L_2) \rightarrow L_1\) gives a diffeomorphism between the space of interest to us and a circle. ♠

Now we deduce Lemma C21. Let \(S^1\) denote the set of perpendicular pairs as in Lemma C211. Let \(X \subset S^1\) denote the set of those pairs with \(L_j \in Σ_j\). Let \(π_1\) and \(π_2\) be the two diffeomorphisms from Lemma C211. The set of anchored lines in Σᵢ is a line segment and hence so is its inverse image \(X_j \subset S^1\) under \(π_j\). We have \(X = X_1 \cap X_2\).

Suppose \(X\) is nonempty. Then some \(p \in X\) corresponds to a pair of lines \((L_1, L_2)\) with at most one \(L_j \in \partial Σ_j\). But then we can perturb \(p\) slightly, in at least one direction, so that the corresponding pair of lines remains in \(Σ_1 \times Σ_2\). This shows that \(X_1 \cap X_2\), if nonempty, contains more than one point. But then the only possibility, given that both \(X_1\) and \(X_2\) are segments, is that their intersection is also a segment.

If neither \(L_1\) nor \(L_2\) lies in the boundary of its sector then we can perturb in both directions. This implies that \(X\) contains the point corresponding to \((L_1, L_2)\) in its (relative) interior. Hence the endpoints of \(X\) correspond to pairs with at least one line in the boundary of a sector. Both lines cannot be in the sector boundary because, by assumption, a sector boundary line of one sector cannot be perpendicular to a sector boundary line of the other sector.

This completes the proof of Lemma C21.


8 References


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