The Topological Traveling Salesman Problem

Richard Evan Schwartz *

July 14, 2025

1 Introduction

This note is about a problem I call the *topological traveling salesman problem*, or *TTSP* for short. I came to the TTSP as an outgrowth of Chapter 2 of my recent children's book [**S**]. However, it seems very possible that this problem is in the literature already, under a different name, and that everything said below is well known. I would appreciate comments and references.

The TTSP is a variant of the famous traveling salesman problem (TSP) and seemingly related to the minimum spanning tree problem (MST). See e.g. [**W**] for information about these classic problems. The TTSP is also related to *meanders*, a subject with a large literature. See e.g. [**DGG**] and the references therein.

Given N *ordered* points in the plane, and any $\epsilon > 0$, the TTSP asks for an *embedded* path in the plane that connects the points in the given order, such that no other embedded path with this property is shorter by more than ϵ. It would be simpler just to ask for the shortest embedded path in the plane that connects the points in the given order, but the limit of a sequence minimizers might not be embedded.

Figure 1 shows an example. The path on the left is the shortest path connecting the given points in order and the path on the right is somewhat longer but embedded. The embedded example is roughly within a factor of 2 of the shortest.

* Supported by N.S.F. Grant DMS-2102802 and a Simons Sabbatical Fellowship

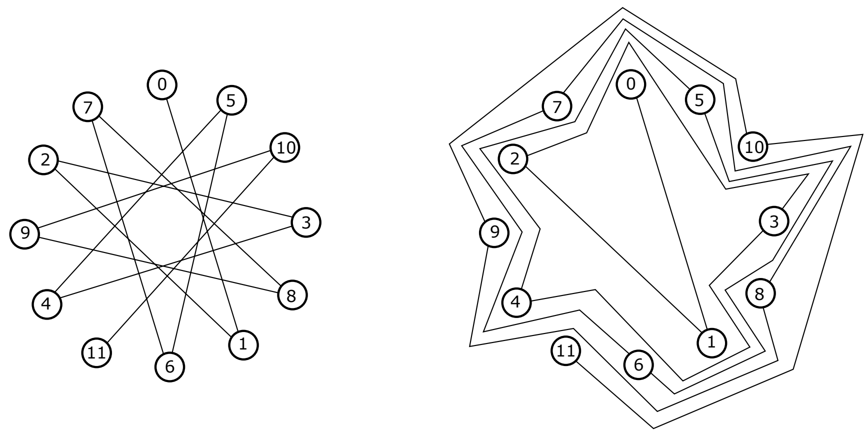

Figure 1.1: Two different connecting paths

To discuss quantitative features of this problem, we normalize our points to lie in the unit square $[0,1]^2$ and take $N \geq 2$. Let τ_N denote the infimal number such that every ordered collection of N points in $[0,1]^2$ can be connected in the given order by an embedded path of length less than τ_N. We will prove the following result.

Theorem 1.1 $\tau_N \leq 2N^{3/2}$.

By nearly alternating between two opposite corners of $[0,1]^2$, we can choose N points in $[0,1]^2$ such that the shortest path connecting them has length greater than $\sqrt{2}(N-1) - \epsilon$. So we have bounds like this:

$$\sqrt{2}\ (N-1) \leq \tau_N \leq 2N^{3/2}.$$

I wonder which bound is closer to reality. Here are two conjectures, the first much more ambitious.

Conjecture 1.2 *$\tau_N > KN^{3/2}$ for some $K > 0$.*

Conjecture 1.3 *For any K we have $\tau_N > KN$ for N sufficiently large.*

The rest of this note is organized as follows. In §2 I will prove Theorem 1.1target*.1. In §3 I will define the *meander number of a permutation*, a topological invariant of a permutation which somehow measures its "planarity". I will then reduce Conjecture 1.2target*.2 to a conjecture about meander numbers of permutations.

I would like to thank Brienne Brown, Elizabeth Denne, Slava Krushkal, and Katie Mann, who all played around with this at my instigation.

2 The Upper Bound

2.1 Main Construction

Figure 2.1 illustrates our main construction. It is quite similar to the construction illustrated in [**DEY**, Figure 11] but used for a different purpose. In this toy example, $N = 4$, and our path is very far from being the shortest. Let $\ell(\cdot)$ denote arc length.

Let $p_1, ..., p_N \in [0,1]^2$. In §2.2subsection.2.2 we show that there is a polygonal tree $T \subset [0,1]^2$ such that $\ell = \ell(T) < 2\sqrt{N}$ and $p_k \in T$ for all k.

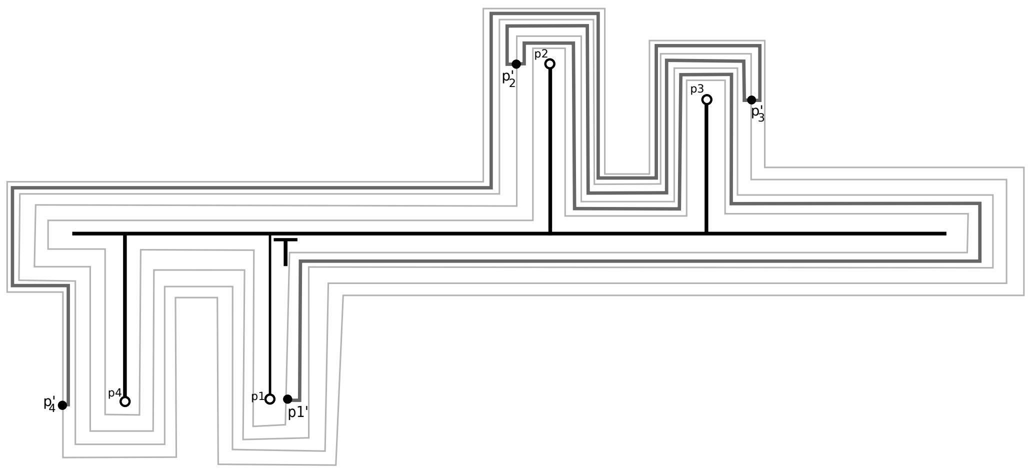

Figure 2.1 Constructing the path

Choose $\delta > 0$ small. Define a family $A_1, ..., A_N$ of nested annular neighborhoods which surround T and remain within δ of T. Let p'_k be a point within δ of p_k and on the inner boundary of A_k. Note that p'_{k+1} is also on the outer boundary of A_k. Choose a path $L'_k \subset A_k$ which joins p'_k to p'_{k+1} and only intersects ∂A_k at the endpoints. We can make all choices so that $\ell(L'_k) < 2\sqrt{N}$ for all k. The embedded path $L' = L'_1 \cup ... \cup L'_N$ connects the points $p'_1, ..., p'_N$ in order and has $\ell(L') < N\ell(T)$.

A map is K-*Lipschitz* if it expands distances by at most a factor of K. Choose ϵ so that $(1 + \epsilon) \times \ell(L') < N\ell(T)$. If δ is sufficiently small, then we have a $(1 + \epsilon)$-Lipschitz homeomorphism h of $\boldsymbol{R}^2$ which maps p'_k to p_k. See §2.3subsection.2.3 for a proof. The image $L = h(L')$ is an embedded path with $\ell(L) < N\ell(T) < 2N^{3/2}$ which connects the points of P in order.

2.2 Short Spanning Tree

Here we justify our claim about the tree T used in our main contruction. Let $P = \{p_1, ..., p_N\}$ be a collection of N disjoint points in $[0,1]$.

Lemma 2.1 *P is contained in a polygonal tree T of length less than $2\sqrt{N}$.*

Proof: Figure 2.2 illustrates our construction in the case $N = 19$.

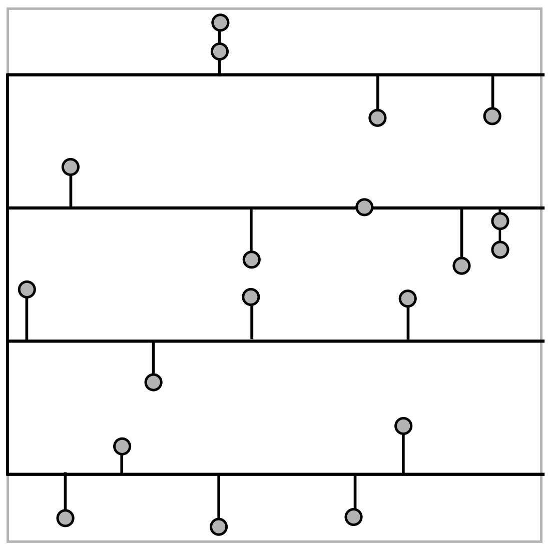

Figure 2.2: The tree T for the case $N = 19$.

Let D be the largest integer such that $D^2 \le N$. There is a unique set of D evenly spaced unit horizontal line segments in $[0,1]^2$ such that every point in $[0,1]^2$ is within $1/(2D)$ of their union. Let T' be the tree which is the union of these segments and the vertical line segment of length $1 - D^{-1}$ that joins their left endpoints. Let T be the tree we get by joining each point of P to T' by a vertical line segment of length at most $1/(2D)$. We have

$$\text{length}(T) \le D + (1 - D^{-1}) + N/(2D) < 2\sqrt{N}, \qquad (1)$$

and $P \subset T$. The last inequality in Equation 1equation.1 is clear for $N > 100$, and one can check the first 100 values by explicit (computer) calculation. ♠

Remark: For what it is worth, the expression in Equation 1equation.1 is actually less or equal to $(7/4)\sqrt{N}$. Asymptotically it equals $(3/2)\sqrt{N}$.

2.3 Lipschitz Homeomorphisms

Here justify our use above of a $(1+\epsilon)$-Lipschitz homeomorphism.

Lemma 2.2 *Let $\epsilon > 0$ be given. There is some $\delta > 0$ with the following property. If p, p' are within $\delta \times \mathrm{diam}(Q)$ from the center of Q then there is a $(1+\epsilon)$-Lipschitz homeomorphism of Q which is the identity on ∂Q and which maps p' to p.*

Proof: We take a cell division of Q as shown in Figure 2.2. The shaded squares R' and R have half the side length of Q and respectively are centered at p' and p. The trapezoids are all affinely equivalent *via* affine maps which are translations when restricted to their vertical or horizontal edges.

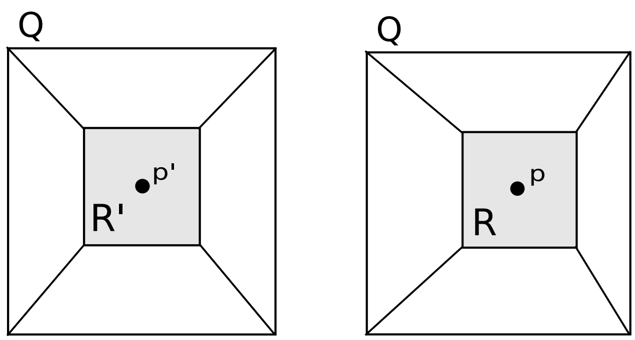

Figure 2.3: The subdivision into trapezoids and squares

Our homeomorphism h is the piecewise affine map which carries the left cell division to the right one in the obvious way. If δ small enough, all the trapezoids have nearly the same shape and h will be $(1+\epsilon)$-Lipschitz. ♠

Corollary 2.3 *Let $d > 0$. Suppose $p_1, ..., p_N \in \boldsymbol{R}^2$ are such that $\|p_i - p_j\| > d$ for all $i \neq j$. Suppose $p_1', ..., p_N'$ are such that $\|p_k' - p_k\| < \delta$ for all k. Then, if δ is sufficiently small there is a $(1+\epsilon)$-Lipschitz homeomorphism of $\boldsymbol{R}^2$ which maps p_k' to p_k for all k.*

Proof: Place disjoint squares $Q_1, ..., Q_N$ of side-length $d/4$ centered at $p_1, ..., p_N$, apply the previous result, and extend the homeomorphism to be the identity outside of the squares. ♠

3 The Lower Bound

The goal of this chapter is to discuss Conjecture 1.2target*.2 and relate it to meanders.

3.1 Cyclic Meanders

A *meander* is a finite disjoint union of embedded loops in $\boldsymbol{R}^2$, each of which has a nonzero finite number of transverse intersections with the X-axis in $\boldsymbol{R}^2$. A *cyclic meander* is a meander with just one loop. We only care about the cyclic meanders.

When we record the intersection points of a cyclic meander in order we get a cyclic permutation. We orient cyclic meanders so that they go counterclockwise around the region they bound. The permutation determines the meander up to isotopy. We consider two meanders equivalent if they are equivalent up to isotopy. Figure 3.1 shows an example of a cyclic meander.

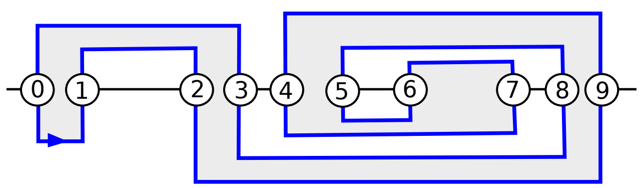

Figure 3.1: The cyclic meander corresponding to $(0, 1, 2, 9, 4, 7, 6, 5, 8)$.

There is a large literature on meanders and they appear across many disciplines – braid groups, statistical mechanics, protein folding, etc. See for example [**DGG**] and the references therein. Many papers in the literature deal with the enumeration of meanders. Here is an easy bound.

Lemma 3.1 *There are at most 4^{N-2} meanders having N intersections.*

Proof: Let $N = 2M$. If we look at the top half of the meander, we have a matching of the $2M$ points on the line. Each such matching is determined uniquely by a choice of M of the points, the lower of the two numbers paired. (In our example the subset $\{0, 1, 4, 5, 6\}$ gives the top half.) Note that 0 must appear in this list and $N-1$ does not. Thus there are at most $N-2$ choose $M-1$ top halves to a cyclic meander. This number is at most 2^{N-2}, by the Binomial Theorem. Hence there are at most 2^{N-2} top halves. Likewise, there are at most 2^{N-2} bottom halves. ♠

3.2 The Meander Number of a Permutation

We call a cyclic permutation π *meandric* if it comes from a cyclic meander. The number of cyclic permutations of N things is $(N-1)!$ and at most 4^N of these are meandric.

Let π be a cyclic permutation of $\{0,...,N-1\}$ and let $S \subset \{0,...,N-1\}$. The first return map of π to S is well defined and gives a cyclic permutation of S. For instance, the first return map of the permutation above to $\{0,3,7\}$ is the permutation $(0,7,3)$, meaning $0 \to 7 \to 3 \to 0$. It we write $0' = 0$ and $1' = 3$ and $2' = 7$ we get the permutation $(0',2',1')$. In this way, the permutation $\pi = (0,1,2,9,4,7,6,5,8)$ *induces* the permutation $\sigma = (0,2,1)$. We write $\pi \to \sigma$ in this case. In general, the construction is to relabel the elements of S as $0',1',2',...$, derive the new permutation from the first return map, and then forget the "primes".

Definition: If σ is a cyclic permutation, we let $\mu(\sigma)$ to be the size of the smallest cyclic meandric permutation π such that $\pi \to \sigma$. The quantity $\mu(\sigma)$ is a kind of topological invariant of σ.

Figure 3.2 shows an example drawn in an alternate way. You should imagine that this meander is drawn on the sphere, which we have projected into the plane. The red arcs lie in the upper hemisphere and the blue arcs lie in the lower hemisphere. The grey labels on the outside of the circle go in the usual order. The black labels are a guide to how the path moves around.

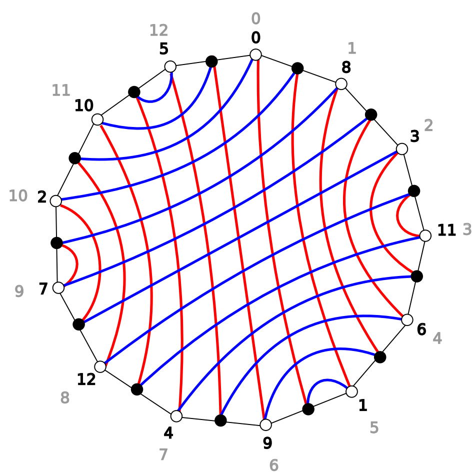

Figure 3.2: A 26-meander induces the permutation $k \to k+5$ mod 13.

The permutation σ is given by $\sigma(k) = k + 5 \mod 13$. This permutation is not meandric. However, Figure 3.2 shows that there is a meandric permutation $\pi \in S_{26}$ such that $\pi \to \sigma$. Hence $\mu(\sigma) \leq 26$. I would conjecture that in fact $\mu(\sigma) = 26$ for this example. One can do a similar things for other permutations based on arithmetic progressions in $\boldsymbol{Z}/N$.

Define

$$\mu_N = \max_{\sigma \in S_N} \mu(\sigma). \tag{2}$$

Here is a weak lower bound on the meander number, which just comes from counting.

Lemma 3.2

$$\mu_N \geq \left(\frac{\ln(N)}{\ln(8)} - 1\right) \times N.$$

Proof: Let $K = \mu_N/N$. There are less than 4^{KN} cyclic meandric permutations of size KN. For each of them, there are at most KN choose N induced permutations. But this number is at most 2^{KN}. So, all in all, there are at most $(1/2) \times 8^{KN}$ cyclic permutations $\sigma \in S_N$ with $\mu(\sigma) \leq KN$. But there are $(N-1)!$ cyclic permutations in S_N. Hence

$$(8^K)^N > (N-1)! > (N/8)^N.$$

The last inequality is an easy application of Stirling's formula. Taking natural logs and simplifying the expression we get the bound advertised in the lemma. ♠

Lemma 3.3 $\mu_N < N^2$

Proof: This is what we get from the method discussed in §2.1subsection.2.1. We put N points on the circle at the Nth roots of unity, then create the path from §2.1subsection.2.1, using the circle minus a tiny interval as a spanning tree. This gives us a meander which crosses the circle at most N^2 times – actually it is about $N^2/2$. By adjusting the labels, we can achieve any permutation in S_N we like this way. ♠

I conjecture that this method is often close to optimal.

Conjecture 3.4 (Meander Number) *There is some constant $K > 0$ such that $\mu_N > KN^2$ for all N.*

3.3 The Topological Salesman Again

In this section I will explain why the Meander Number Conjecture implies Conjecture 1.2target*.2. For simplicity we assume that N is an even perfect square. We place N points on the maximal evenly spaced $\sqrt{N} \times \sqrt{N}$ square grid of points in $[0, 1]^2$.

We choose a loop $L = L_N$ that contains all the vertices of the grid and has horizontal and vertical edges. Figure 3.3 shows the cases $N = 4, 6$. The edges of L_N have length $1/\sqrt{N - 1}$, which is greater than $1/\sqrt{N}$.

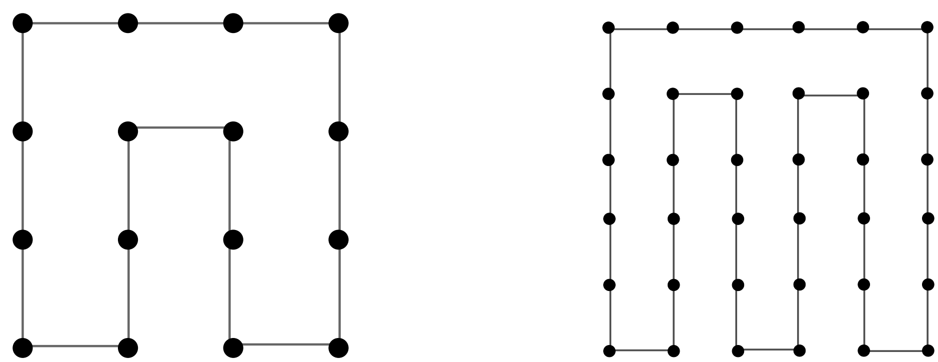

Figure 3.3: The loop L_N.

We choose some permutation σ such that $\mu(\sigma) > KN^2$ and then we order the points of the grid according to σ. If we can connect the points in order using a path of length less than ℓ then we can connect the points in order (and then cycle back to the beginning) using a loop of length less than 2ℓ. Let Λ be such a loop. We want to see that Λ has length on the order of $N^{3/2}$.

Consider the number of times Λ intersects L. By perturbing Λ an arbitrarily small amount we can guarantee that Λ intersects L transversely at each intersection point, meaning that Λ locally moves from one component of $\boldsymbol{R}^2 - L$ to the other. Figure 3.4 shows the kind of modifications we have in mind.

One kind of local modification, which we make at grid points, introduces a new crossing when it is used. The other kind of local modification, which we use at extra intersection points, removes an intersection. So, after adding at most N new crossings, we see that the pair (Λ, L) is homeomorphic to a pair (Λ', S^1), where Λ' is a meander and S^1 is the unit circle. So, Λ intersects L in $O(N^2)$ points.

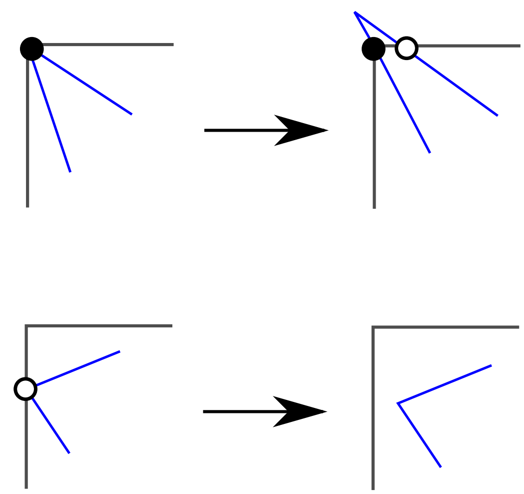

Figure 3.4: Two kinds of local modifications

If Λ intersects the interior of some edge e of L twice in a row, then we can find an arc A of Λ which has this property in a minimal way, in the sense the the union of A and the portion of e between the two intersection points bounds a disk that contains no other arcs of Λ. We can then shorten Λ by pushing A across the disk to a new arc A'. This removes two intersection points as well. Figure 3.5 shows the construction.

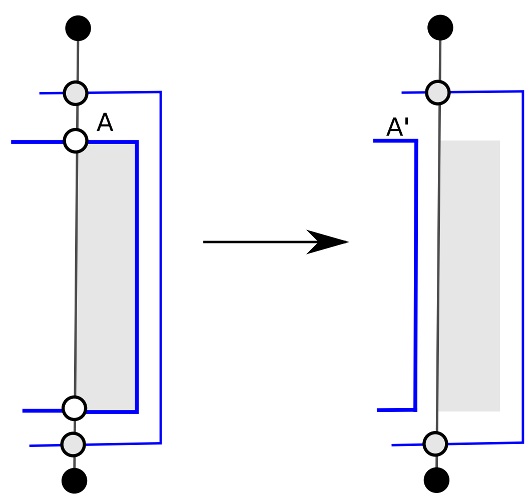

Figure 3.5: Pushing A across a disk

We claim that Λ cannot have the intersection pattern e_1^o, e_2^o, e_1^o, for 3 consecutive intersections. Here e_1 and e_2 are adjacent edges of L and e_j^o is the interior of e_j. Figure 3.6 shows how this pattern forces Λ to intersect the interior of one of e_1 or e_2 twice in a row.

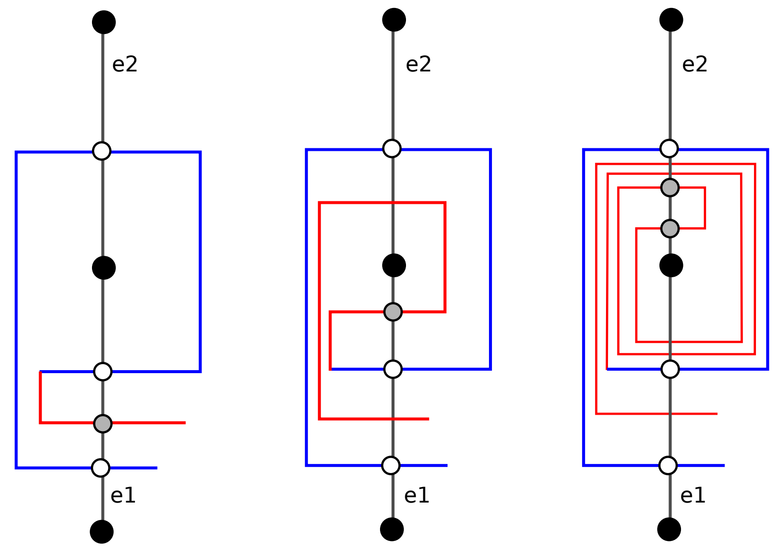

Figure 3.6 An alternating intersection forces a double intersection

In the cases depicted, the blue arc is the arc with the three crossings and then the red arc is a possible continuation. There are infinitely many cases here, but in all cases the blue arc sets up a trap from which the red arc must later escape, and it can only escape by creating a double edge intersection.

Now we give the final argument. Assuming the Meander Number Conjecture, we can say that Λ intersects L at least $O(N^2)$ times. But then we can find $O(N^2)$ disjoint arcs of Λ, each containing exactly 3 points where Λ crosses L at points interior to the edges of L. From what we have just finished arguing, each of these special arcs must intersect L in 3 distinct edges. But then each special arc has length greater than $1/\sqrt{N}$. Hence the total length of Λ is $O(N^{3/2})$. This is what we wanted to show.

The same argument shows that there are collections of N points in $[0,1]^2$ which cannot be connected by a path that is shorter than $O(\mu_N/\sqrt{N})$. This would only be interesting (for Conjecture 1.3target*.3) if we could show that μ_N grows faster than $N^{3/2}$.

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4 References

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